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ABSTRACT

This is the teacher's edition of a text for the first year of a two-year high school geometry course. The course bases plane and solid geometry and trigonometry on the fact that the translations of a Euclidean space constitute a vector space which has an inner product. Volume 1 deals largely with affine geometry, and the notion of dimension is introduced only in the last chapter. The principal geometric topics of this volume are parallelism of lines and planes, and ratios. This makes possible a good deal of the geometry of triangles and quadrilaterals. This commentary contains answers to all problems, sample quizzes, chapter tests, suggestions on teaching the texts, and a great deal of mathematical and logical background material which has proved helpful in orienting teachers. (Author/MK)

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A Vector Approach to Euclidean Geometry

VECTOR SPACES AND AFFINE GEOMETRY

HERBERT E. VAUGHAN & STEVEN SZABO

Volume 1

TEACHER'S EDITION

In memory of Max Beberman, a colleague,
mathematics teacher, and above all a friend

University of Illinois Committee on School Mathematics
The Macmillan Company, New York, New York
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INTRODUCTION

Volume 1 of A Vector Approach to Euclidean Geometry is the text for the first year of a two-year high school mathematics course which bases plane and solid geometry and trigonometry on the fact that the translations of a euclidean space constitute a vector space which has an inner product. [Although the word 'vector' is not introduced in the text until Chapter 5, there is a discussion of various notions of vector at the beginning of the commentary for section 1.06.] Volume 1 deals largely with affine geometry, and the notion of dimension is introduced only in the last chapter. Up to this point, although students may think in terms of 3-dimensional space, the space under consideration might be of any dimension. The principal geometric topics of volume 1 are parallelism of lines and planes, and ratios. This makes possible a good deal of the geometry of triangles and quadrilaterals. Congruence, however, is reserved for volume 2. Volume 2 opens with an analysis of basic properties of perpendicularity and distance which leads to the introduction of an inner product of translations and to the development of euclidean geometry and trigonometry. The basic facts concerning volume measures of solids are dealt with in an appendix to volume 2. By the end of volume 2 students will have become acquainted with the theorems dealt with in standard American geometry and trigonometry texts and with numerous additional geometric theorems. They will also have a good foundation of knowledge of a subject of contemporary mathematical importance -- that of linear vector spaces.

The preparation of the present course began in 1963 and has been carried out with support from the National Science Foundation. Beginning with the 1963-64 school year, the course has been taught in an ever-growing number of schools throughout the country. It has also been the basis for teacher education courses in NSF institutes. In most schools using the course it has been taught in grades 10 and 11, but it has also been used successfully in grades 11 and 12 and in grades 9 and 10.

Many teachers and students have contributed ideas and provoked discussions which helped us to develop the present text and teachers commentary for the course. We wish to thank all of those people who helped us in this endeavor, and to single out for special thanks Dr. J. Richard Dennis and Dr. Hyman Gabai.

In common with other UICSM courses, this one lays stress on students' discovery of concepts through doing appropriate exercises. The course also stresses logic and proof, and students are aided in developing the ability to write proofs, preferably in the form of paragraphs. To this end, some of the beginning chapters deal at some length with rules of logic in the contexts of the proofs of early theorems.

By the nature of the course much of the development of geometry is algebraic in nature and the algebra resembles [and includes] that of the real numbers to such an extent that previously acquired algebraic skills are maintained and developed further. Lest the reader thinks that this algebraization leads to the loss of much of the beauty of conventional geometry, it bears pointing out that after a sufficient number of geometrical results have been established algebraically other theorems can be derived synthetically from them. It should also be noted that, beginning in Chapter 10 and to a greater extent in volume 2, it becomes easy to introduce coordinate methods and deal with geometric matters analytically. As becomes obvious, however, the algebra of analytic geometry is less efficient than is the algebra of points and translations which is developed in this course.

Ideally, a student's preparation for this course should include a good background concerning functions. [This is the prime advantage of teaching the course during years 11 and 12, preceded by a second course in algebra which treats the concept of function in some detail.] Since not all students have such a background, the first half of Chapter 1 is devoted to a brief discussion of functions. [The remainder of the chapter serves to develop geometric intuitions, particularly concerning translations.] Some attempt has been made in the BACKGROUND TOPICS, which close several chapters, to familiarize students with algebraic topics with which they may not be acquainted. Nevertheless, there are some standard topics of algebra which are covered neither in a first year algebra course nor in the present course. Consequently, the course needs to be supplemented, either in the tenth year [as suggested above] or in the twelfth year by a second algebra course in which students will learn about such things as radicals, exponents, logarithms, and polynomial functions.

This commentary contains answers to all problems, sample quizzes and chapter tests, suggestions on teaching the texts, and a great deal of mathematical and logical background material which has proved helpful in orienting teachers.

The following is a brief lesson guide for volume 1. It gives a suggested number of pages to aim at covering in each lesson together with some comments on the kinds of activities and assignments which are appropriate with those pages. Note that this guide suggests the administering of "sample quizzes" from time to time and allows full lessons for chapter tests. Some sample quizzes have been included in the teacher's commentary in the following sections:

TC26(2)
TC101(3)
TC156
TC232(2)
TC310(2)
TC412

TC55
TC123(1)
TC186(1)
TC235(3)
TC335(2)
TC428(3)

TC69
TC130(2)
TC193(2)
TC289(2)
TC360(1)
TC441(3)

TC92(2)
TC131
TC219
TC294(3)
TC363

Suggested Teaching Schedule for Volume 1

Lesson	Section	Possible activities
1, 2	Introduction	Make use of force table [from Physics lab], mirror, and rubber glove to illustrate points made in this introduction.
3	1.01	Do A, B, C in class; assign D1-5(c).
4	1.01-1.02	Do D5(d) - (h), E in class; study text preceding A; discuss new terms; assign A.
5	1.03-1.04	Discuss B, 1.03 in class; discuss terms preceding A; do A, B1 in class; assign B2-4.
6	1.04	Do B5, C in class; discuss proof that function composition is associative; do A1, B1 in class; assign A2, B2, C.
7	-	Review all results concerning functions; sample quiz.
8	1.05	Have text read and discussed in class; assign problem posed on page 46.
9	1.05	Finish reading; begin drawing exercises in class; all nine problems should be done, either as seat-work or as homework.
10	1.06	Introduce tracing sheets as in text preceding A; do A in class, assign B, C.
11	1.06	Do as much of D, E, F in class as time permits; assign rest of D, E, F, and G.
12	1.06	Do H in class; assign I, J. [Begin as seatwork, finish rest as homework.]
13	1.07	Study text preceding exercises; begin exercises in class; assign all exercises for homework; sample quiz.
14	1.08-1.09	Discuss section 1.08; review ideas discussed in Chapter 1; assign Chapter Test as written work to be handed in during next class meeting.
15	2.01	Study text preceding exercises; assign Exercises 1-3.
16	2.02	Introduce Postulates 1, 2; do ensuing exercises in class; discuss proof of part (a) of Theorem 2-1; assign proof of (b).
17	2.03	Discuss new terms preceding A; do A in class; sample quiz; assign B.

Lesson	Section	Possible activities
18	2.04	Discuss text preceding exercises; study rules in context of samples; assign 1-10.
19	2.05	Study new terms and rule preceding A; do A1-7, B1-5 in class; do C1 in class; assign A8-13, B6-10, C2-3.
20	2.06	Discuss text preceding A; do A in class; assign B.
21	2.06	Study text preceding A; note (☆) and rule; go over examples in A; assign A.
22	2.06	Do B1-4, C1-3, D1-3 in class; assign rest of B, C, D.
23	2.07	Discuss text preceding A; do A, B1 in class; sample quiz; assign B2-3, C.
24	2.08	Study text preceding A; do A1, B1-3 in class; assign A2, B4-8.
25	2.09	Discuss rules; do A, B1, C1-2 in class; assign B2, C3-6.
26	2.10	Sample quiz; do A, B, C1 in class; assign C2, D, E.
27	2.11	Study text preceding A; do A, B, C1 in class; assign C2-3, D.
28	2.12	Discuss text preceding A; do A, D in class; assign B, C.
29	2.13	Use Chapter Test as written hourly exam, or review ideas covered in Chapter 2 and assign Chapter Test as out-of-class work to be handed in.
30	3.01	Do A, B, C, D1-2 in class; assign D3, E, F. [Note commentary on these exercises.]
31	3.02	Discuss text preceding A; note new postulates; do A, B, in class; assign C. [See commentary.]
32	3.03	Sample quiz; answer questions in text; do A1-10 in class; assign A11-20, B.
33	3.04	Study text preceding A; do A1-8, B1-3 in class; assign A9-16, B4-6, C. [See commentary.]
34	3.05	Discuss notion of group; note (☆); do A, B, in class; assign C. [See commentary for a take-home exercise on commutative groups.]

Lesson	Section	Possible activities
35	3.06	Sample quiz; discuss text preceding A; do A, B1-4 in class; assign B5-7, C. [See note in commentary on B.]
36	3.07	Do Exercise 1 in class; assign 2-6.
37	3.07	Note rule; do A, B in class; assign C. [See note in commentary on building stick models to aid in discussions.]
38	3.08	Study text and clarify notation used; assign 1-3.
39	3.09	Use Chapter Test as written hourly exam or as basis for review of ideas discussed in the chapter.
40	4.01	Discuss text; do A1-2 in class; assign A3-4, B.
41	4.02	Discuss text; do A in class; assign B, C.
42	4.03	Discuss modus tollens, double denial [See commentary.]; do A in class; assign B.
43	4.03	Do C, D1 in class; assign D2, E, F; sample quiz.
44	4.04	Discuss order relations; do A1, B in class; assign A2-6, C.
45	4.05	Introduce new terms; note rules; do A1-3, B1-3, in class; assign A4-6, B4-5.
46	4.05	Note rules and details of text discussions; do C1-3, D in class; assign C4-5, E.
47	4.06	Discuss text preceding A; do A in class; assign B.
48	4.07	Use Chapter Test as take-home exam or as a basis for review of ideas in the chapter.
49	4.07	Background topic — systems of equations; do A, B1-3 in class; discuss determinant; assign B4-5.
50	5.01	Do Exploration Exercises in class; discuss ratio; do A1-2, B1 in class; assign A3-5, B2-4.
51	5.01	Do C, E1 in class; assign D, E2-4, F.
52	5.02	Discuss postulates, theorems preceding A; do A in class, assign B, C.

Lesson	Section	Possible activities
53	5.03	Discuss matters of logic related to conditionals; do A in class; assign B1-3, [See commentary on assignments here.]; sample quiz.
54	5.03	Do B4, C, D1 in class; assign D2-3, E; note true-false items in commentary which may be useful practice in recognition of theorems.
55	5.04	Study text preceding A; discuss Definition 5-1, 5-2; do A1-5 in class; assign A6-9, B.
56	5.05	Sample quiz; discuss directed trips; do some of A, B, C in class; assign rest.
57	5.05	Discuss velocity; do some of A, B, C in class; assign rest.
58	5.05	Discuss force; do some of A, B, C in class; assign rest.
59	5.06	Study text preceding exercises; assign 1-3.
60	5.07	Use Chapter Test as take-home exam or as written hourly exam.
61	6.01	Note new terms; do A1-4, B1-2 in class; assign A5-6, B3-6. [Note that C is as advertised.]
62	6.01	Do D1-3, E1-3 in class; assign D4, E4-8.
63	6.02	Discuss new terms; do A in class; assign B.
64	6.03	Sample quiz; note Definition 6-2; do A, B1 in class; assign B2-10.
65	6.03	Note theorems; do A, B1 in class; assign B2-5.
66	6.04	Discuss new terms, theorems; do A, B1 in class; assign B2-3, C.
67	6.05	Note new terms; do A1, 5, B1 in class; assign rest of A, B. [Alternately, see commentary.]
68	6.05	Do C1-2, D1, 4 in class; assign rest of C, D; use E as extra-credit.
69	6.05	Do F, H1-3 in class; assign G, H4-7.
70	6.06	Sample quiz; discuss Theorem 6-12; assign 2-3.
71	6.07	Do Exploration Exercises in class; discuss terms preceding the exercises; assign 1-3.

Lesson	Section	Possible activities
72	6.07	Sample quiz; discuss pattern sentences [See commentary for suggestions.]; do 1-4 of first exercise set in class; do 1 of second exercise set in class; assign 2-4.
73	6.08	Discuss use of quantifiers in terms of three theorems whose proofs are outlined in the text [See commentary for objective.]; go through A1, B1; assign A2, B2-4. [Alternately, see commentary for assignment list.]
74	6.08	Do C1 in class; assign C2-6.
75	-	Review Chapters 1-6
76	6.09	Use Chapter Test as written hourly or as take-home exam.
77	6.09	Study text preceding exercises in Background topic; assign 1-10.
78	7.01	Discuss terms; do 1-2 in class; study Definition 7-1 and text following it; assign 1-4.
79	7.02	Study text and Definition 7-2; do A in class; assign B.
80	7.02	Discuss Definition 7-3; do C1 in class; assign C2.
81	7.03	Discuss text preceding A, do A in class; assign B.
82	7.03	Do C, D in class; review Definition 6-4; assign E.
83	7.04	Sample quiz; discuss definition and theorems preceding exercises; do 1-3 in class; assign 4-8.
84	7.05	Discuss text preceding A; do A in class; assign B.
85	7.05	Do C1 (a)-(c), 2, D1 in class; assign rest of C, D.
86	7.05	Do E1-7 in class; discuss Definition 7-7 and Theorem 7-8 and 7-9; assign E8-9, F.
87	7.06	Discuss text preceding A; do A1 in class; assign A2-5.
88	7.06	Sample quiz; assign, B, C.
89	7.07	Discuss text preceding A; do A, B1 in class; assign rest of B. [Alternately, see commentary on suggested assignments.]

Lesson	Section	Possible activities
90	7.07	Do one of logic proofs in C in class; do D1-2 in class; assign rest of C, D.
91	7.07	Do E1 in class; assign rest of E.
92	7.08	Discuss terms; do 1-5 in class; assign 6-10.
93	7.08	Do A1, B in class; assign A2-5. [See note on assignments in commentary.]
94	7.08	Do C1-2, E1-3(a) in class; assign C3-5, D, E3(b)-4.
95	7.08	Do A1-4 in class; assign B, C1. [Alternately, see note in commentary.]
96	7.08	Sample quiz; do C1-3 in class; go over discussion in C4; assign C4-5; use C6 as extra-credit.
97	7.08	Do A1-2 in class; assign A3, B.
98	7.09	Do A1-3, B1 in class; assign A4-5, B2-4. [Alternately, see note on assignments in commentary.]
99	7.09	Do C, D in class; assign E, F.
100	7.10	Review Chapter 7; go over Background Topic.
101	7.10	Use Chapter Test as take-home exam or as written hourly exam.
102	8.01	Study text preceding A, do A, B in class; assign C. [Alternately, see note on assignments in commentary.]
103	8.01	Study text preceding D, assign D.
104	8.01	Do some of E in class; assign rest.
105	8.02	Discuss Definition 8-1, Theorem 8-4, 8-5; do A, C1(a) in class; assign B, C1(b)-3. [Alternately, see note in commentary.]
106	8.02	Do D, E1-3 in class; assign E4-8.
107	8.02-8.03	Do F in class; discuss Definition 8-2; do A in class; assign B.
108	8.03	Do C1-2 in class; assign C3-5.
109	8.04	Sample quiz; discuss text preceding A; do A1 in class; assign A2-4. [Alternately, see note on assignments in commentary.]

Lesson	Section	Possible activities
110	8.04	Study Theorem 8-9, 8-10; do B1-3 in class; assign B4, C.
111	8.04	Do D1-4, E1-3 in class; assign D5-6, E4-6.
112	8.04	Do F1, 2, 5, 6, 8 in class; assign F3, 4, 7, 9, 10.
113	8.05	Discuss text preceding A; do A in class; assign B. [Alternately, see note on assignments in commentary.]
114	8.05	Do C1-2, D1-3 in class; assign C3-4; D4-6.
115	8.05	Do E1-2(b), F1 in class; assign E2(c)-(d), F2-3.
116	8.05	Do G1, H1 in class; assign G2-3, H2.
117	8.06	Discuss text, definitions; do 1-3 in class; do A1, B1-2 in class; assign rest of A, B.
118	8.07	Discuss Definition 8-6; do A1-3, B1 in class; Assign A4, B2-3. [Alternately, see note on assignments in commentary.]
119	8.07	Assign C1-8.
120	8.07	Sample quiz; assign C9-12, D.
121	8.08	Discuss text preceding A; do A1, B1 in class; assign rest of A, B. [Alternately, see note on assignments in commentary.]
122	8.08	Sample quiz; do C1-2, D1-2 in class; assign rest of C, D.
123	8.08	Do E, F1-2 in class; assign rest of F.
124	8.08	Do H1-2 in class; assign G, H3-4.
125	8.09	Use Chapter Test as take-home exam or as written hourly exam.
126	9.01	Study text preceding Definition 9-1, doing 1-5; do Exercise 1 in second exercise set in class; assign 2-6.
127	9.02	Discuss text preceding A; do A1, B1 in class; assign rest of A, B.
128	9.02	Discuss Definition 9-3; assign C.
129	9.03	Discuss text, Lemma preceding A; do A1-3 in class; assign A4-8. [Alternately, see note on assignments in commentary.]

Lesson	Section	Possible activities
130	9.03	Study text preceding B; do B in class; discuss Theorem 9-7; do C1-2 in class; assign C3-8.
131	9.03	Do some parts of D in class; assign rest of D.
132	9.03	Do E1, F1-2 in class; assign E2-5, F3-4.
133	9.04	Study text preceding A; do A, B1, C1 in class; assign B2-3, C2-3.
134	9.05	Study text preceding A; do A1-2, B1 in class; assign rest of A, B. [Alternately, see note on assignments in commentary.]
135	9.06	Discuss definition and theorems preceding A; do A in class; assign B1.
136	9.06	Do B2-5 in class; assign B6-9.
137	9.06	Do C1-4 in class; assign C5.
138	9.07	Discuss text preceding exercises; assign 1-4.
139	9.08	Discuss text of Background Topic; assign all problems not done in class.
140	9.08	Use Chapter Test as written hourly exam or as take-home exam.
141	10.01	Do A in class; discuss Postulate 4_9 and Theorem 10-1; assign B.
142	10.01	Do C1-3, 5 in class; assign rest of C.
143	10.01	Discuss text preceding D; do D1 in class; assign D2-4.
144	10.01	Do E1-2 in class; assign E3, F.
145	10.02	Discuss text preceding exercises; prove some of theorems in class; assign all of exercises not done in class.
146	10.03	Do A1-5 in class; assign A6-11. [Alternately, see note on assignments in commentary.]
147	10.03	Discuss text preceding B; do B in class; assign C1-5.
148	10.03	Sample quiz; do D1-3 in class; assign rest of D.
149	10.04	Study text, do Exercises 1-2 preceding A; do A1 in class; assign rest of A, B.
150	10.05	Discuss text preceding A; assign A1-5.

Lesson	Section	Possible activities
151	10.05	Do A6, B1-2 in class; assign A7-9, B3-4.
152	10.05	Do C, D1-3 in class; assign D4-8.
153	10.06	Discuss text preceding A; do A1, 2, 4 in class; assign rest of A.
154	10.06	Do B1-5 in class; assign B6, 8, 9.
155	10.06	Do B7, C1 in class; assign C2-4
156	10.06	Do D1-2 in class; assign D3-4.
157	10.06	Do E1, 2 in class; assign rest of E.
158	10.07	Sample quiz; discuss text preceding A; do A1, 2 in class; assign A3-5.
159	10.07	Do B1 in class; assign B2, C.
160	10.08	Discuss terms in text; do A, B1, 2(a) in class; assign rest of B.
161	10.08	Discuss Theorem A and its corollary; assign C.
162	10.09	Do A1-5, B1-2 in class; assign A6-10, B3-4. [See note on assignments in the commentary.]
163	10.09	Discuss Theorem 10-14; do C(a)-(b), D1-2 in class; assign rest of C, D; sample quiz.
164	10.09	Discuss text leading to Theorem 10-15; do E1(a), F1, 3 in class; assign rest of E, F.
165	10.10	Discuss text [See commentary for suggestions.]; do as much of A in class as possible; assign rest of A.
166	10.10	Begin work on B in class; assign rest of B.
167	10.10	Discuss Theorem 10-16 and the examples in the text; do C1-2, D1 in class; assign rest of C, D.
168	10.10	Do E1, F1 in class; assign rest of E and F.
169	10.11	Discuss Theorem C; do A1, B1-2 in class; assign rest of A, B. [Alternately, see note on assignments in commentary.]
170	10.11	Do C1-2, D1 in class; assign rest of C, D.
171	10.12	Use Chapter Test as take-home exam.

TO THE STUDENTS:

This book is the text for the first year of a two-year mathematics course which contains what is usually studied in a one-year plane geometry course, much of what is studied as solid geometry, and probably more than is usually studied in trigonometry. The course also contains a great deal of material which is not usually studied in any of these courses. This extra material furnishes a foundation which ties the rest together. If you wish a name for this extra material, call it *linear algebra*.

An advantage of the approach to geometry which is adopted in this course is that it illustrates the fact that mathematics is all of one piece – the usual distinctions between algebra, geometry, and trigonometry are not valid ones. Another advantage is that the knowledge of linear algebra which you will gain from this course will be of considerable help should you take mathematics or physics courses in college.

Something which you may at first think of as a disadvantage is that your friends who are taking courses in plane geometry will be learning some of *this* subject much earlier than you do. Remember, however, that you are learning useful things which they may not learn until late in college, and that you will learn all that they are learning, and more, by the end of this course.

This course builds directly on the knowledge of the real (or: signed) numbers which you have gained by studying algebra. This knowledge is reviewed in Chapter 4. You will learn more about the real numbers (and some other things) in the "background topics" which are given at the ends of some chapters, especially in Volume 2.

One reason why this course takes two years is that you are given opportunities in the exercises to work out ideas for yourself. If you take advantage of these opportunities, you will gain confidence in your ability to understand and to use mathematics. Understanding mathematics requires more than attending class and doing the exercises assigned as homework. It also requires you to think about what you do. This book is written so that a good deal of your learning can come from reading the text and thinking about it and the exercises. You will probably find that you need to re-read portions of the text, coming back to them from time to time, before you *completely* understand all that is to be learned from them. This is in the nature of mathematics, and we have done our best to make it possible for you to learn in this way. Learning to learn in this way will pay off in all your future study.

Many students have studied this course in earlier, experimental, editions and most of them have found it to be fun. We hope you do, too.

Herbert E. Vaughan & Steven Szabo

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Introduction

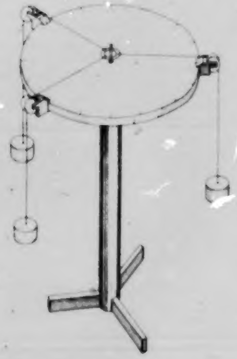
Before beginning a study of geometry we shall consider briefly some problems from physics. You may not be able to work these problems now, but later in this course you will see how a knowledge of geometry helps you to analyze and solve these problems.

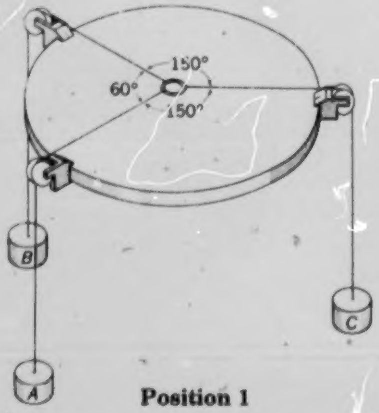
An Experiment with a Force Table

Below is a picture of a "force table". It is a device used in physics laboratories to perform experiments with forces.

As you can see in the picture, a pin at the center of the table is encircled by a small ring; the pin holds the ring near the center of the table. Strings which are tied to the ring pass over pulleys clamped to the edge of the table, and weights are attached to the ends of the strings. Each weight exerts force on the ring.

Let's consider an experiment in which just three strings are tied to the ring, as shown in the diagram below. (Position 1)





Position 1

The measures of the angles formed by the strings are indicated in the diagram. If the pin is pulled out of the table, do you think the ring will remain where it is? As a matter of fact, the weights *may* pull the ring away from the center of the table. [Describe conditions under which this will happen.] Now, do you think it is possible to select weights A, B, and C so that the ring will *not* be pulled away from the center of the table?

In this introduction, we wish to raise some questions which we feel are interesting and which have some relation to the kind of work that we will be doing in this course. We do not expect that the students will be able to answer the questions posed with any degree of precision. We would like to believe that the discussions in the Introduction will arouse the curiosity of the students and will begin to make them think about motions and about representing these motions by scale drawings.

One or, perhaps, two days should be sufficient time to spend in class on the Introduction. The discussion about mirrors seems to interest students a bit more than the discussions about the force table and directed trips. This may be the case because the questions were of more interest to the teachers involved or simply because mirrors are inherently interesting objects.

Arranging to have a force table in the classroom to demonstrate what is discussed in the text will stimulate the student's thinking about the notion of forces acting on a body [or, point]. The teachers who have borrowed a force table from the physics department for use with this introduction have found it very effective. Those who have tried to teach the lesson both with the apparatus and without it have expressed the opinion that the extra effort required to get the apparatus into the classroom is well worth it.

In connection with the discussion about reflections, it is most helpful to have a mirror and a pair of rubber gloves in the classroom. We have found that a flexible metal mirror works nicely here. As a supplement to the reading in the Introduction, the article "About left- and right-handedness, mirror images and kindred matters" by Martin Gardner (Scientific American, March, 1958) will give the students a chance to exercise their imaginations.

If the above devices are brought into the classroom, you will certainly spend two or three class periods with the material in the Introduction. Otherwise, you have just about enough material in the Introduction for one class period and one homework assignment.

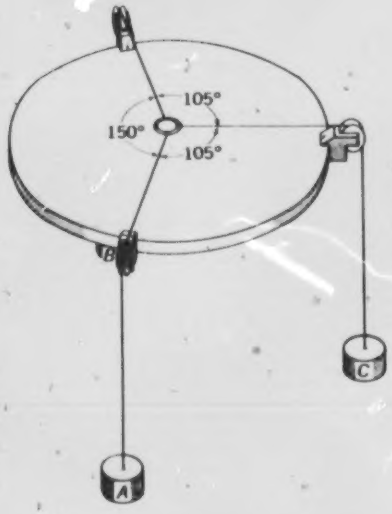
One of the weights can be made large enough to "overpower" the effects of the other two weights, pulling the ring away from the center of the table. If the weights are such that the system is in equilibrium, then very little additional weight at any one of the positions will cause the system to pull the ring away from the center. This is easily demonstrated if the apparatus is available in the classroom.

Given the apparatus, it is not difficult, using trial and error methods, to determine a system of three forces which is in equilibrium over the center of the force table.

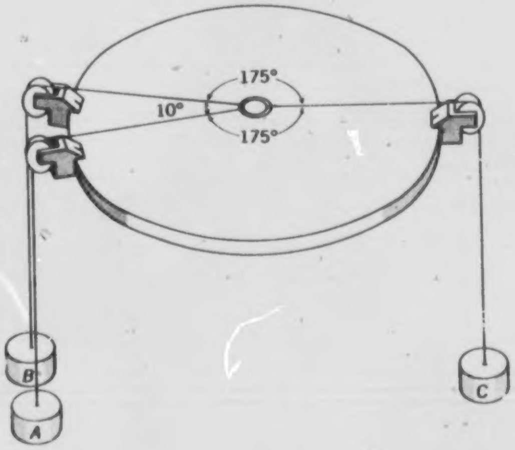
Suppose that *A* and *B* are each 20-gram weights. What should be the weight of *C* so that the ring will remain at the center of the table? Is a 40-gram weight what is needed at *C*? Is 40 grams too much or too little?

Now, let's imagine that one of the conditions of the experiment is changed. In particular, suppose that the weights of *A* and *B* are kept at 20 grams each, but that the pulleys are moved to the position shown in the diagram labeled 'Position 2'. Under these conditions, what should be the weight of *C* so that the ring will remain at the center of the table?

For a third case, let us again keep the weights of *A* and *B* unchanged, but suppose that the pulleys are moved to a third position as shown in the drawing below. To keep the ring at the center of the table, should the weight of *C* be more than it was for Position 2? Less? How would the weight of *C* for this situation compare with the weight of *C* needed for the situation when the weights were arranged in Position 1?



Position 2



Position 3

A Problem about Velocity

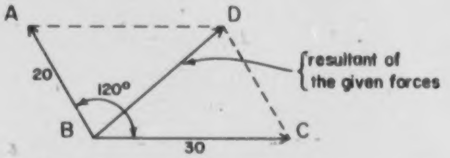
The captain of a ship wishes to sail due north. The ship's engines can move the ship in still water at a speed of 10 miles per hour. However, there is a strong current that moves the ship northeast at a speed

For the given system, a weight of approximately 34.6 grams is required at C in order to put the system in equilibrium. So, a 40-gram weight is not what is needed at C. 40 grams at C is too much.

* * *

The resultant of, say, two forces acting at a point in a body is the single force which, acting at the same point, would "produce the same effect." As shown by experiments with the force table, two forces acting at a point can be balanced by a single force. The resultant of the given forces is, then, the force with the same magnitude as the balancing force but with opposite sense. [In this course we distinguish between a direction, which parallel lines have in common, and the two opposite senses in a given direction.]

A force may be represented by an arrow whose length is proportional to the magnitude of the force and whose sense pictures that of the force. For example, forces of 20 lb. and 30 lb. which act on a point B in such a way that the angle formed by the rays along which they act is one of 120° may be represented by the arrows from B to A and from B to C, respectively, in the figure. Experiment shows that the resultant of these two forces is represented by the arrow from B to D



[ABCD is a parallelogram]. Quantities which, like forces, have both magnitude and direction, and which "add" according to the parallelogram-law, are called vector quantities. Other examples of vector quantities are displacements and velocities.

In the example pictured in the preceding paragraph, the magnitude of the resultant force can be estimated by measuring $\overline{BD}$ in the same scale as $\overline{BA}$ and $\overline{BC}$. Doing so shows the magnitude of the resultant to be about 26.5 lb. The sense of the resultant can be described by giving the size of, say, ∠DBC. This is about 41°. The magnitude of this resultant may be computed by making use of the law of cosines, and the size of ∠DBC may then be found by using the law of sines:

$$\begin{aligned}(BD)^2 &= (AB)^2 + (AD)^2 - 2(AB)(AD)(\cos A)\\ &= 20^2 + 30^2 - 2(20)(30)(\tfrac{1}{2})\\ &= 700\\ BD &= 10\sqrt{7}\end{aligned}$$

Hence, the resultant force is one of $10\sqrt{7}$ lbs.

$$\sin\angle DBC = \frac{DC \cdot \sin\angle DCB}{BD}$$
$$= \frac{20 \cdot \sin 60^\circ}{10\sqrt{7}}$$
$$\doteq .6547$$

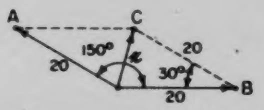
Hence, $\angle DBC$ is an angle of about 40° 54′.

It is not to be expected, of course, that students will be prepared at present to understand the preceding computations. Indeed, it is to be expected that they will learn the laws of cosines and sines at a much later point in this course. We give this solution solely for your information.

Given a force table and scales, the students should be able to get a reasonable rational approximation for the weight required at C to put the given system into equilibrium. To tenths of a gram, the answer is 10.4.

Under the conditions given, the weight needed at C is $10(\sqrt{6} - \sqrt{2})$ grams. One way to compute this is the following:

Let x be the weight, in grams, needed at C. Then, by the law of cosines,



$$x^2 = 20^2 + 20^2 - 2 \cdot 20 \cdot 20 \cdot \cos 30^\circ$$
$$= 2 \cdot 20^2 (1 - \cos 30^\circ)$$
$$= 400 (2 - \sqrt{3}).$$

So, $x = 20\sqrt{2 - \sqrt{3}} = 20\left(\frac{\sqrt{6} - \sqrt{2}}{2}\right) = 10(\sqrt{6} - \sqrt{2}).$

Of course, we do not expect that the students solve this problem by making use of the law of cosines. We give this solution solely for your information.

It is worth calling attention to the fact that once the system is in equilibrium, each of the forces has the property that it is the force that is required to "balance" the resultant of the other two forces.

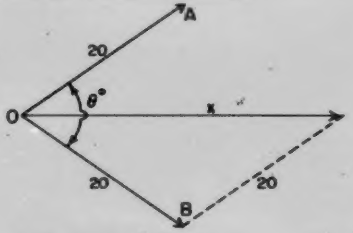
Under the given conditions, the weight at C should be more than it was for Position 2, and it should also be more than it was for Position 1. In fact, the weight, in grams, needed at C

for Position 1 is 40 cos 30°,

for Position 2 is 40 cos 75°,

and for Position 3 is 40 cos 5°.

In general, if the size of the angle between the strings for A and B is θ° then the weight at C is $40 \cos \frac{\theta^\circ}{2}$. We derive this as follows:



Let x be the weight, in grams, needed at C. Then, we have in turn:

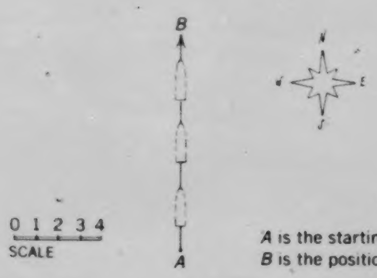
$$20^2 = x^2 + 20^2 - 2 \cdot x \cdot 20 \cos \frac{\theta^\circ}{2}$$
$$x^2 - 40x \cos \frac{\theta^\circ}{2} = 0$$
$$x\left(x - 40 \cos \frac{\theta^\circ}{2}\right) = 0$$

Since the weight needed at C is 0 only when $\theta = 180$, the weight needed at C is $40 \cos \frac{\theta^\circ}{2}$.

As before, we give this solution solely for your own information. Rational approximations for special positions may be obtained by making use of the apparatus.

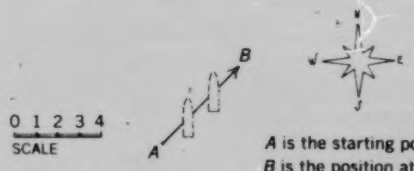
of 5 miles per hour. Will it be possible for the captain to steer the ship in such a way that, despite the current, it will sail due north?

Although you may not be able to answer this question convincingly now, you can think about some of its aspects. First, consider a situation in which there is *no* current. If the engines are operating and the captain points the ship north, the ship will move north at a speed of 10 miles per hour. At the end of the first hour it will be 10 miles north of the starting point. We can show the path of the ship during the first hour by a diagram. [In this diagram 1/8 inch represents 1 mile.]



A is the starting position.
B is the position at the end of 1 hour.

Second, consider a situation in which the water current is flowing northeast at a speed of 5 miles per hour, and the ship's engines are *not* operating. In this case, even though the captain is at the wheel keeping the ship pointed north, the ship will actually move in a northeasterly direction at a speed of 5 miles per hour. Hence, at the end of the first hour it will be 5 miles northeast of its starting point. The following diagram shows the path of the ship during the first hour. [Again, the scale is 1/8 inch to represent 1 mile.]



A is the starting position.
B is the position at the end of 1 hour.

Now, returning to the original problem, we see that we have to take into account:

(1) the condition which caused the ship to move in the first situation [i. e., the engines operating]

together with

There ought to be general agreement in the class that the ship can be steered in such a way that the resultant of the ship's velocity [10 mph in some direction] and the current's velocity [5 mph to the northeast] will be a velocity which has a due north heading.

Notice that we are using the term 'speed' in connection with a rate of motion which is undirected, and the term 'velocity' in connection with a directed rate of motion. This is consistent with the classical usage of these terms, in which speed is thought of as a scalar quantity and velocity as a vector quantity.

To give the students some basis for thinking about problems such as these, you might pose questions like the following:

> How far would the ship go in an hour if the captain steered the ship toward the northeast (i.e., directly with the current)?
>
> How far would the ship go in an hour if the captain steered the ship toward the southwest?
>
> What sorts of things might the captain do in order to keep his ship right at position A?

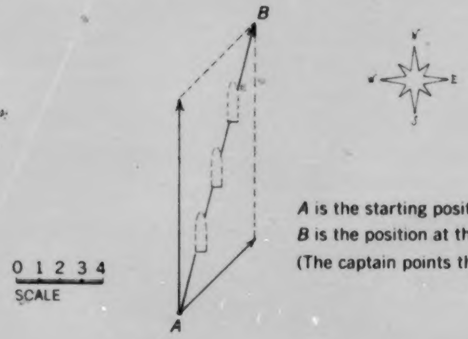
(2) the condition which caused the ship to move in the second situation [i. e., the water current].

Thus we must take into account the facts that

(1) the ship's engines are operating and can move the ship north at a speed of 10 miles per hour

and (2) there is a current which can move the ship northeast at a speed of 5 miles per hour.

Under these conditions, if the captain points his ship north, its path will be the path from *A* to *B* shown in the following diagram:



A is the starting position.
B is the position at the end of 1 hour.
(The captain points the ship north.)

In view of this last diagram, what do you conjecture as to the possibility of steering the ship so that it will travel due north? If you think it *is* possible, in which direction should the captain point the ship to accomplish this?

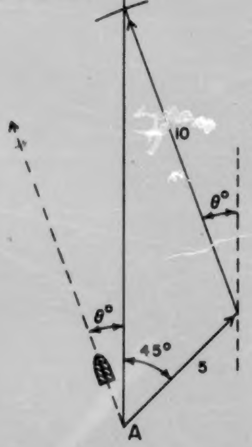
A Question about Mirrors

Suppose that you look in a full length mirror. As you face your image in the mirror, the reflection of your right hand seems to be the left hand of the image, and the reflection of your left hand seems to be the right hand of the image.

Try it. Face a mirror and hold a book in your right hand. The image facing you in the mirror is holding the book in its left hand. Now put down the book. Hold your right hand up to the mirror and observe the

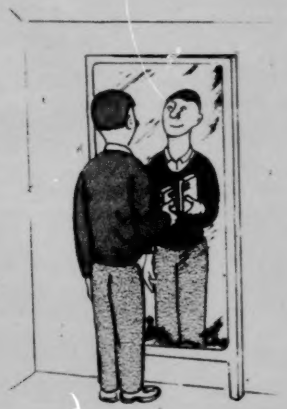
From the diagram, it should be reasonably clear that the captain must steer the ship somewhat toward the northwest in order that the resultant of the ship's velocity and the current's velocity have a heading which is due north. There may be a few students who can devise a scheme for making a scale diagram which will enable them to predict which heading the captain must take in order to get the required job done. All that is required here, however, is the notion that it is possible to accomplish the task.

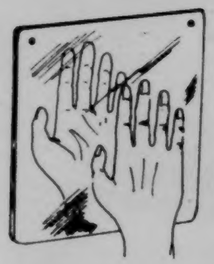
The following diagram and computations illustrate how one might solve the problem.



$$\sin \theta^\circ = \frac{5 \cdot \sin 45^\circ}{10}$$
$$= \frac{\sqrt{2}}{4}$$
$$\doteq .3535$$
$$\theta^\circ \doteq 20^\circ\ 42'$$

The captain should direct the ship on a heading of 20° 42′ west of north at 10 mph in order to have a resultant heading which is due north.





hand the image holds up to you. The image is holding up its left hand. Compare the hand held up by the image with your own left hand and you will see that they do look alike.

The reflection of your right hand seems to be the image's left hand. If, as it appears, the mirror "reverses right and left", why doesn't it also reverse top and bottom? If the mirror interchanges your right hand with your left hand, why doesn't it also interchange your head with your feet?

This naturally leads to a next question. Can one make a mirror which *does* interchange top and bottom? [Of course if you looked in such a mirror, you would see yourself standing on your head, so such a mirror might not be too useful around the house.]

Some Remarks about the Preceding Problems

The force problem. Many people think that C should be a 40-gram weight if A and B are each 20-gram weights. This is not correct.

Of course, the answer to this question can be found experimentally. However, through experimentation, physicists have found a way to answer this question [and similar questions] by using ideas and methods which we shall learn about in our study of geometry.

It turns out that if the pulleys are arranged as shown in Position 1, [page 1], C should be a weight of [about] 35 grams in order for the ring to remain at the center of the table. However, if the pulleys are arranged as shown in Position 2, C should be quite a bit less than 35 grams. For the pulley arrangement shown in Position 3, C should be more than 35 grams.

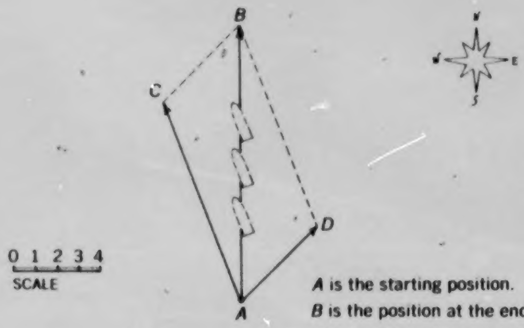
Do you think there is a position such that C should be 0 grams? Can a position be found such that the weight of C should be 40 grams? More than 40 grams?

These questions about mirrors are purely rhetorical. Thus, we do not expect that the student answer them. Of course, as you and the students will read in the remarks later in the Introduction, mirrors "reverse" front and back. In mathematical terms, mirrors reverse the orientation of space.

Some of the students may have seen mirrors in amusement parks that do all sorts of funny things. So, they will probably know that mirrors do exist that interchange top and bottom.

With a force table in the classroom, it can be verified experimentally that the larger the angle is between forces A and B, the smaller the weight is that is needed at C to put the system in equilibrium about the center of the force table. The limiting values are 0 grams when the angle between the strings for A and B is 180°, and 40 grams when the angle between the strings for A and B is 0°.

The ship problem. One way to find the direction in which the captain should point the ship in order to sail due north is to make a drawing something like the following:



A is the starting position.

B is the position at the end of 1 hour.

(The captain points the ship in the direction from *A* to *C*.)

The captain should point his ship in the direction from *A* to *C*, and during the first hour the ship will move from *A* to *B* [i.e., due north]. Compare your answer with this drawing.

The mirror problem. Of course, mirrors do not interchange top and bottom—and, in fact, they don't interchange right and left either.

It is a common misconception that a mirror interchanges left and right—and this misconception is partly due to the feeling one has that he could put himself in the apparent position of his image by going into the next room and turning around to face the wall on whose other side the mirror is hanging.

Actually to put yourself in the apparent position of your image, something more drastic is called for. To understand the difficulty involved, let us first consider an easier [but analogous] problem.

Imagine that you are holding a right-hand glove in front of you with the fingers pointing toward the mirror. What will its image look like? [Close your eyes and try to visualize it.] . . . Did you realize that the image of the glove will look like a left-hand glove with the fingers pointing toward you? [If you don't believe this, try it!] Can you make the glove you are holding appear to be the same as the image you saw in the mirror?

Though you may not think so, it is possible to accomplish this. How? By turning the glove inside out! That is, you must interchange the front [finger-tip end] of the glove with the back [cuff end]. You will then have made this glove into a left-hand glove with the fingers

Here is a ruler and compass construction which will enable you to determine the desired heading for the ship.

(1) Assume the ship is at A. Draw a north-south line through A. Then, draw an arrow $1\frac{1}{4}$ inches long (= 5 miles per hour) from A to the northeast to the point D.

(2) Using D as the center of a circle with radius $2\frac{1}{2}$ inches (= 10 mph) determine the location of B, the point directly north of A and 10 miles from D. The heading from D to B is the heading required to have the ship sail directly north. The length of $\overline{AB}$ in $\frac{1}{4}$-inches gives the resultant velocity to the north of the ship.

[As this construction shows, there is a triangle-rule for adding vector quantities. This is, of course, equivalent to the parallelogram-rule.]

The process of turning a glove inside out ought to work quite well with rubber gloves, and not very well (if at all) with ordinary gloves. It would no doubt be handy to have a mirror and a pair of rubber gloves in the classroom in order to demonstrate how a glove can be "made into" its mirror image.

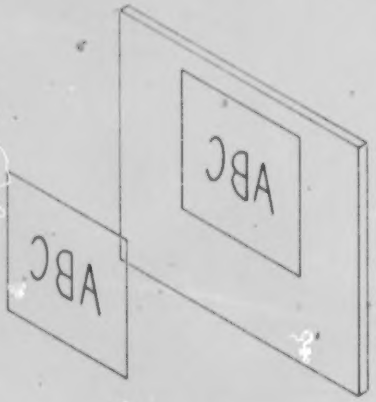
pointing toward you. The mirror did not interchange left and right, nor did it interchange top and bottom – it interchanged front and back.

To put *yourself* in the apparent position of your image, you must perform a contortion which interchanges your front and back. You must stand in the next room with your back toward the wall on whose other side the mirror is hanging. Then you must turn yourself front-side back [not around] by, say, exhaling so strongly as to pull your chest [and the rest of the front of your body] through your back!

We agree that it is impossible for a person to perform the above contortion, but, just as in the case of the glove, this is what would be needed to put yourself in the apparent position of your image!

Another easy way to see that a mirror interchanges front and back is to perform the following experiment. Write something on a thin sheet of paper and hold it between you and the mirror so that the side containing the writing is facing the mirror. What you see in the mirror is just what you see through the back of the paper.

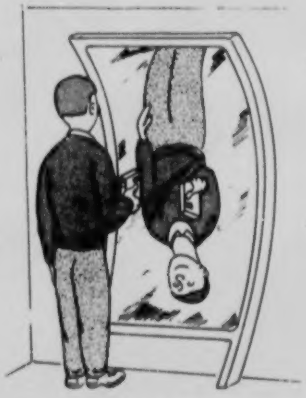
If the ink were pressed through the paper, from the front to the back, it would look like the image you now see in the mirror.



More about Mirrors

You can make a mirror in which your image will be standing on its head simply by bending any flexible reflecting surface slightly as shown in this diagram:

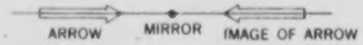
Let us now consider mirrors and reflections in some different, and perhaps unusual, situations.



Using a mirror and a piece of porous paper (e.g., newsprint) marked with a felt pen, the experiment that is described here can be performed quite easily in the classroom. Another ''experiment'' with which students will be acquainted is that of observing the image of the license plate of a following car in the rear-view mirror of the car in which one is riding. Here, again, one can understand why the license appears to be backward by imagining what the back of the license plate looks like when viewed directly.

It is interesting to note that figures which have an axis of symmetry can be held in front of the mirror in such a way that one cannot tell that the mirror ''interchanges'' left and right. Also, with such figures, when one places the mirror so that its face is along the axis of symmetry of the figure, one still has a view of the entire figure. The view of the figure is, of course, composed of the exposed ''half'' of the figure together with the image of this exposed half in the mirror. This notion serves as the basis for some very interesting exercises in visualization created by Miss Marion Walter in her Mirror Cards. [These are available through Educational Development Corporation, Watertown, Massachusetts.]

Suppose that you live in a straight-line, that is, in a one-dimensional world. In a one-dimensional world a mirror is represented by a dot. If you hold an arrow pointing toward a mirror, the image will be an arrow pointing toward you. The mirror has interchanged front and back.

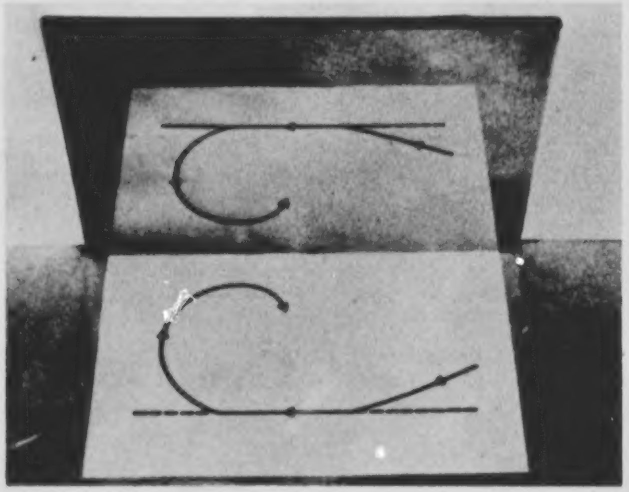


Notice that each arrow determines one of two *opposite senses on the line.* We may specify one of these senses to be *the positive sense on the line*, call the other *the negative sense on the line*, and then say we have *oriented* the line.

If you orient the line by specifying that *your arrow* represents the positive sense on the line, you have one orientation of the line. You may reverse this orientation of the line by specifying that the *image* of your arrow represents the positive sense on the line. [Of course, if you began by specifying that the *image* of your arrow represents the positive sense on the line, you may reverse this orientation of the line by specifying that the *arrow* represents the positive sense.]

We say that *reflection reverses orientation on a line.*

Next, imagine that you live in a plane, that is, in a two-dimensional world. In a two-dimensional world, a mirror is represented by a straight line. In such a world you can bend your arrow as shown in the diagram below and observe its image in a mirror:



The picture on page 8 is a reproduction of a photograph of a diagram and its reflection in a mirror placed on a table. It shows, quite effectively, that the orientation of the line is not altered, but that the orientation of the plane of the paper is reversed [from clockwise to counterclockwise when viewed from above].

Each arrow determines one of two *opposite senses of rotation on the plane*. We may specify one of these senses to be *the positive sense of rotation on the plane*, call the other *the negative sense of rotation on the plane*, and then say we have *oriented* the plane.

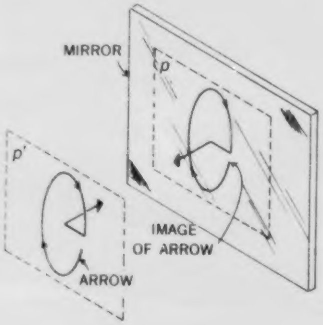
If you orient the plane by specifying that *your* arrow represents the positive sense of rotation on the plane, you have one orientation of the plane. You may reverse the orientation of the plane by specifying that the *image* of your arrow represents the positive sense of rotation on the plane.

We say that *reflection reverses orientation on a plane.*

The arrow contained in the dotted line indicates the orientation of that line, and the image of the arrow gives the orientation of the image of the dotted line. Notice that the orientation of the dotted line and its image is the same. [Reflection, however, did interchange front and back, so that the sense of rotation on the plane is reversed.]

A plane may be a rather uncomfortable place in which to live, so let us return to our three-dimensional world, which we refer to as "*space*".

In our three-dimensional world we can bend our arrow as shown in the diagram and observe its image in a mirror:



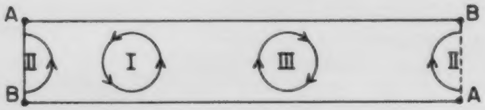
Each of the two arrows determines one of two *opposite senses of twist in space*. We may specify one of these senses to be *the positive sense of twist in space*, call the other *the negative sense of twist in space*, and then say we have *oriented* space.

If we orient space by specifying that our arrow represents the positive sense of twist in space, we have one orientation of space. We may reverse the orientation of space by specifying that the *image* of our arrow represents the positive sense of twist in space.

We say that *reflection reverses orientation in space.*

The two orientations of 3-dimensional space may be correlated with the two kinds of helices — right-handed helices and left-handed helices. [Models of these are easily constructed by wrapping pipe cleaners about a pencil. A mirror-image of a helix of either kind is a helix of the opposite kind.] Indeed, the arrow pictured on the preceding page is a somewhat deformed portion of a right-handed helix, while its mirror-image is a similarly deformed portion of a left-handed helix. Somewhat surprisingly, the helices themselves, without arrowheads, suffice to indicate the two orientations. On the basis of a general theory of orientation of Euclidean spaces of arbitrary dimension it is easy to see why arrowheads are required on pictures of segments and arcs to indicate orientation of one and two-dimensional spaces, but are not needed on pictures of helices to indicate orientations of three-dimensional spaces. It turns out that, for n-dimensional space, arrowheads are required or not according as $n(n + 1)/2$ is odd or even. [It would be inappropriate to develop here the general theory of orientation on which this result is based. For readers who are acquainted with the theory it will be sufficient to point out that $(a_n, \ldots, a_1, a_0)$ is an even permutation of $(a_0, a_1, \ldots, a_n)$ if and only if $n(n + 1)/2$ is even.]

It is worth remarking that the orientability of Euclidean planes depends on the fact that however a sensed circle is moved about in such a plane it will, if brought back to its original center, return with its original sense. Thus, having chosen a positive sense of rotation at one point, a positive sense of rotation is thereby determined at each point. A Moebius surface — models of which can be constructed by giving a half-twist to a strip of paper and joining the ends — is a counterexample to the generalization that all surfaces are orientable. As is easily seen, a sensed circle which is moved all the way "around" such a surface will return to its original position with its sense reversed. A non-Euclidean plane of the Riemannian or elliptic sort — equivalently, a projective plane — furnishes another example of a nonorientable surface.



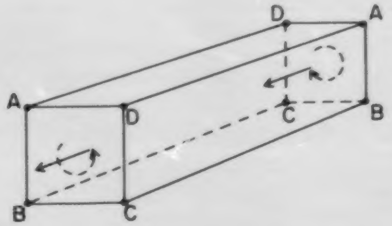
Imagine the circle moving around the surface from position I to II to III to I. Notice that its sense is reversed.

Similar remarks apply to Euclidean 3-space. However a helix is moved about in such a space it will, if brought back to its original axis, return with its original sense. There are, however, nonorientable 3-dimensional spaces in which a right-handed helix can be moved in such a way as to reverse its sense. Since such sense-reversal is impossible in Euclidean 3-space, nonorientable 3-spaces cannot be modeled in Euclidean 3-space. However, the same trick used in the

The arrow contained in the plane *p* [represented by dotted lines] indicates the orientation of *p*, and the image of the arrow [contained in the plane *p'*] gives the orientation of *p'*. Notice that the orientation of the planes is not reversed. [Reflection, however, did interchange front and back, so that the sense of twist in space is reversed.]

We shall have more to say about mirrors and orientation of lines, planes, and space later in the course.

preceding figure to obtain a plane picture of a Moebius surface can be resorted to to obtain a representation of an analogous nonorientable 3-space:



[The space in question is obtained by "identifying" corresponding points of the square ends of the bar according to the correspondence indicated by the lettering. Just as three dimensions are needed if one is to twist a strip of paper to form a Moebius strip, so four dimensions would be needed if one were to twist a bar of, say, plasticene into a model of the nonorientable 3-space described above.]

Chapter One
Mappings

1.01 Functions

In this section, and in the three following sections, we shall review some notions with which you are probably familiar from earlier courses. These are all related to the notion of a *mapping*. [Another word for 'mapping' is 'function'.]

Before recalling what a mapping is, let's recall what a mapping "does". A mapping of a set S into a set T gives a way of assigning to each member of S some corresponding member of T. For example, the squaring operation for real numbers gives a way of assigning to any real number a single real number, its square. This squaring operation is a mapping of the set of all real numbers into itself. We can picture it by drawing two pictures of the number line [that is, of the set of all real numbers] and drawing a few arrows to show, for each of a few real numbers, what number is assigned to it by this mapping. As the figure above shows, squaring *maps* both $-\frac{1}{2}$ and $\frac{1}{2}$ *on* $\frac{1}{4}$, maps 0 on 0, maps both 3 and -3 on 9, etc. We say that 4 is the *image of* -2 under the squaring mapping; that, under this mapping, -3 is not the image of anything, etc. When the word 'function' is used instead of 'mapping' it is customary to use the word 'value' instead of 'image'. We say, for example, that the *value* of the squaring function *at* 2 is 4, and that -3 is not a value of this function.

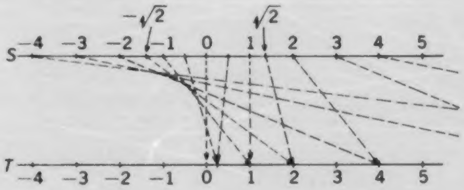


Fig. 1-1

Our purpose in this course is to develop Euclidean geometry on the basis of postulates concerning a certain kind of mapping of Euclidean space into itself. [Our "undefined terms" will be 'point' and 'translation'.] Consequently, it is important that students have such a background as will enable them to think, in some comfort, about mappings and to use, with some facility, the relevant notation. By and large, we assume that students of this course do have such a background. Although sections 1.01 through 1.04 contain the essential definitions and theorems, these sections will hardly serve, in themselves, as an adequate introduction to the concepts reviewed therein. Their purpose is to serve as a review of certain material, most [if not all] of which is familiar to the reader, and to direct attention especially to mappings of 1-dimensional space which are prototypes of the mappings of 3-dimensional space which we will deal with later. [Students who lack familiarity with this material can best supply it by studying the first two chapters of High School Mathematics, Course 3, by Beberman and Vaughan (D. C. Heath and Company, Boston, 1966).]

It is also assumed that readers of this text are familiar with the notion of a set and some of the simpler notions — for example, those of ordered pair, union, intersection, and relative complement — related to it, and that they are accustomed to the use of the usual notational devices in writing about sets.

As does any deductively organized study of geometry, this one requires of students some intuitive feeling for the geometry of physical space, as well as some familiarity with geometric terminology. We attempt to fulfill these needs in sections 1.05 and 1.06 and in other "concept-development sections" in later chapters. [The role of such sections is discussed in the first four paragraphs of the commentary for section 1.05.]

Finally, participation in a task of deductive organization — or, even, appreciation of the outcome of such a task — requires some knowledge of methods of proof. Since there is considerable evidence that neither the ability to argue rationally, nor the techniques of formulating arguments in some uniform and succinct fashion, are innate, space will be devoted to these matters both in the commentary and in the text.

*

Before reaching section 1.05, students should have familiarized themselves to some extent with the use of the parallel ruler. A good device is to issue these gadgets to students and tell them to take them home and see what they can draw with them. [Unlined paper and sharp pencils are appropriate.] Don't tell them that one use for parallel rulers is to draw parallel lines. [Parallelism of lines should come up near the end of their study of section 1.05.] Students should be inventive enough to think of various things to do, and the only purpose of this "exercise" is to develop a modicum of manual dexterity.

*

In section 1.01 we recall the notion of a function as a set of ordered pairs, no two of which have the same first component. We use 'mapping' and 'function' as synonyms. Other notions are sometimes — entirely legitimately — associated with these words, but we have no need for these notions. From a formal point of view it is largely a matter of taste whether one chooses to consider as primitive the notion of set and that of the membership relation, or the notion of mapping and that of the mapping relation. On the bases of pedagogical experience and of personal preference we elect for the former. To the argument — sometimes advanced by partisans of the latter — that a mapping "maps", but a set just "is", an adequate answer would seem to be that, in either case, the verb 'map' is a technical term and, as such, must either be taken as primitive or related to other terms which are taken as primitive. To put it more bluntly, 'map' does not have an intrinsic meaning which the definition 'A mapping f maps a on b if and only if $(a, b) \in f$' perverts. Finally, the fact that "a working mathematician" seldom thinks of a function as a set of ordered pairs is not so damning as it's sometimes claimed. We are accustomed to sit in chairs without — on each occasion — being conscious of the appropriate definition; and "a working zoologist" might well make use of the family pet on a collecting expedition without ever recalling the technical definition of 'dog'. Just so, in one's dealings with functions, one comes to think of them most of the time in terms of what they do — or, better, what he can do with them — rather than in terms of what they are. Because functions — unlike chairs and dogs — are abstract entities, learning what one can do with functions must proceed either from postulates which prescribe some of the things one can do with them or from a definition in terms of other abstract entities — say, sets. If, as in the case with most students now-a-days, one already has some acquaintance with these other entities, the latter method has very definite pedagogical advantages.

In speaking of mappings we adopt the customary distinction between 'into' and 'onto'. A mapping is said to map its domain into any set which contains the images of all members of its domain, and to map its domain onto the set which consists of just these images. Thus, for example, the squaring operation for real numbers maps the set of all real numbers into this same set, into the set of all real numbers greater than -1, and both into and onto the set of all nonnegative real numbers.

We shall tend to use the word 'operation' to refer to functions of a certain kind. Briefly, a singulary operation on a set S is a function which maps S into S, a binary operation on S is a function which maps $S \times S$ into S, etc. [$S \times S$ is, of course, the set of all ordered pairs both of whose components belong to S.] Sometimes such functions are called inner operations on S and 'operation on S' is used in referring to any function whose domain is either S or $S \times S$ or .... We shall occasionally use 'operation' in this more general sense.

At this point it is also appropriate to call attention to our use of 'component' and 'member'. A set has members and we use the word 'member' only in this sense. Now, however one may choose to construe the notion of ordered pairs of objects, the objects in question are not members, in this sense, of the ordered pair. Consequently, to speak of them as such is confusing, and another word is needed. The word 'component' seems to fill this need best.

The method illustrated in Fig. 1-1 for picturing a mapping of the set of all real numbers into itself has, for our purposes, advantages which other graphical representations lack. For one thing, it prepares students to make use of diagrams like those in Figures 1-5, 1-6, and 1-7 as conventional aids to explaining properties of arbitrary mappings. [In Fig. 1-1, the horizontal lines picture the set of all real numbers, and the arrows indicate how a particular mapping works. In the later figures, the loops and the dots inside them play the role of variables — the former ranging over sets, the latter over their members.] Of more immediate interest is the fact that Fig. 1-1 suggests another way of thinking about a mapping of the set of all real numbers into itself — a way of thought which is very worth acquiring. Picture the domain of the squaring operation as a calibrated elastic thread, and picture its range as the edge of a ruler. Assuming that the thread is so designed as to be thickest at the point labeled '0' and to become thinner in an appropriate manner toward its ends, one could show what the squaring operation does by folding the thread at its 0-point, holding the thread along the ruler with 0-points coinciding, and pulling with sufficient force on the other end.

The suggestion just made is, of course, for a "thought experiment", since appropriately fashioned threads are not readily available.

Knowing a mapping amounts just to knowing which objects it maps on which. If, for some mapping, we know which are the ordered pairs (a,b) such that this mapping maps a on b, then we know the mapping itself. For this reason, we shall say that a mapping is a set of ordered pairs. For example, the squaring function is $\{(x,y)\colon y = x^2\}$ [read as 'the set of all ordered pairs x,y, such that $y = x^2$']. To say that the squaring function maps 2 on 4, or that its value at 2 is 4, amounts exactly to saying that the ordered pair (2,4) belongs to the squaring function.

Of course, it is not the case that every set of ordered pairs is a mapping; an object which has an image under a given mapping has *just one* image. Note how this restriction is taken into account in the definition.

> A function is a set of ordered pairs no two of which have the same first component [and any such set is a function].

Exercises

Part A

Here are some sets of ordered pairs of real numbers. In each case tell whether the set is a function. If it is not, explain why.

1. $\{(1,2),(2,5),(3,1),(4,1),(-5,7)\}$
2. $\{(1,2),(3,4),(5,1),(3,4),(2,0)\}$
3. $\{(2,3),(7,4),(6,-2),(2,1)\}$
4. $\{(1,1),(2,2),(3,3)\}$
5. $\{(4,5)\}$
6. $\varnothing$ [the empty set]
7. $\{(x,y)\colon y = x + 1\}$
8. $\{(x,y)\colon x^2 + y^2 = 9\}$
9. $\{(x,y)\colon y = |x - 3|\}$
10. $\{(x,y)\colon y = x^2\}$
11. $\{(x,y)\colon x = y^2\}$
12. $\{(x,y)\colon y = x\}$

*

As we said earlier, the second components of the members of a function are called its *values*. The first components of the members of a function are called its *arguments*. The set of all arguments of a function is the *domain* of the function; the set of all values of a function is its *range*. For example, the domain of the squaring operation for real numbers is the set of all real numbers, and the range of this function is the set of all nonnegative real numbers. A function [equivalently: a mapping] is said to map its domain *onto* its range.

Part B

For each function given in Part A, describe the domain of the function and the range of the function.

Part C

The particular functions which we have mentioned up to now have had only real numbers as arguments and values. The definition of

Answers for Part A

The sets described in Exercises 1, 2, 4, 5, 6, 7, 9, 10, and 12 are functions. In particular, as to Exercise 2, the fact that the ordered pair (3, 4) happens to have been listed twice in describing the set in question is irrelevant; as to Exercises 5 and 6, both sets are functions because any set with fewer than two members certainly cannot have two members with the same first component. As to Exercise 6, it may also be well to point out that a set of, say, elephants is, by current linguistic conventions, one which has no members other than elephants. Consequently, $\varnothing$ is a set of elephants, and is also a set of ordered pairs of real numbers.

That each of the sets described in Exercises 3, 8, and 11 is not a function should be shown by giving two ordered pairs which belong to the set in question and do have the same first component.

Answers for Part B

1. $\{1, 2, 3, 4, -5\}$, $\{1, 2, 5, 7\}$
2. $\{1, 2, 3, 5\}$, $\{0, 1, 2, 4\}$
4. $\{1, 2, 3\}$, $\{1, 2, 3\}$
5. $\{4\}$, $\{5\}$
6. $\varnothing$, $\varnothing$
7. the set of all real numbers, the set of all real numbers
9. the set of all real numbers, the set of all nonnegative real numbers
10. [Answers are the same as for Exercise 9.]
12. [Answers are the same as for Exercise 7.]

TC 13 (1)

Answers for Part C

1. The sets described in parts (a), (c), and (f) are functions, the others are not. [As to part (c), distinguish between 'owns' and 'is part owner of'.]

2. (a) the set of all rooms, the set of all buildings
   (c) the set of all houses which are owned by individuals, the set of all individuals who own houses
   (f) the set of all points in your classroom, the set of all points which would be in your classroom if it were to be moved 2 feet northeast and one foot up

Some of your students may not be familiar with the existential quantifier '$\exists$'. For examples of how to interpret and work with '$\exists$' in, say, the given sentence:

$$\text{For any function } f,\ \mathcal{D}f = \{x\colon \exists_y\ (x, y) \in f\}$$

you might return to the functions given in Part A on page 18. In Exercise 1 on that page, 3 is in the domain of the function for there is a number y [namely 1] such that (3, y) is in the function. In Exercise 2, 4 is not in the domain of the given function for there is no number y such that (4, y) is in the function. In Exercise 10 on page 19, −2 is in the domain of the function for there is a number y [namely 4] such that (−2, y) is in the function. Similar work with discussing the ranges of these functions will provide any additional practice that may be needed.

'function' contains no such restriction and, later in this chapter and throughout this course, we shall have much to do with functions whose domains and ranges do not consist of real numbers. Which of the following sets of ordered pairs are functions?

1. The set of all ordered pairs each of which has
   - (a) as its first component, a room; as its second component, a building which contains this room
   - (b) as its first component, an automobile; as its second component, a person who drives this automobile
   - (c) a house; a person who owns this house
   - (d) a person; a house which this person owns
   - (e) a city; a state route which runs through this city
   - (f) a point in your classroom; a point 2 feet northeast of this point and 1 foot above it
2. For each function described in Exercise 1, describe its domain and its range.

Part D

We shall use 'D' as short for 'the domain of' and 'R' as short for 'the range of'. So, for any function f,

$$Df = \{x\colon \exists_y\, (x,y) \in f\} \text{ and } Rf = \{y\colon \exists_x\, (x,y) \in f\}.$$

[Read '$\exists_y$' as 'there exists a y such that' or as 'for some y'.]

In earlier courses you will have learned to use *function notation.* Recall that, for any function f, and any $a \in Df$,

$$b = f(a) \text{ if and only if } (a,b) \in f.$$

[If, in '$f(a)$', you replace 'a' by an expression which does not refer to any member of Df then the resulting expression is nonsense.]

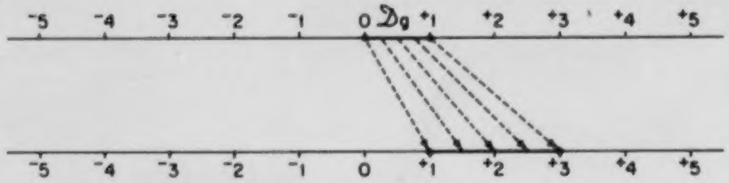
1. Let g be a function such that $Dg = \{x\colon 0 \le x \le 1\}$ and, for each $x \in Dg$, $g(x) = 2x + 1$.
   - (a) Is there such a function? Is there more than one such function?
   - (b) Draw a picture like Fig. 1-1 to describe g. [On your drawing thicken part of one line to indicate the domain of g, and label it.]
   - (c) What is Rg?
   - (d) Answer the following questions:
     $g(1/3) = ?$, $g(1/4) = ?$, $g(1/2) = ?$, $g(3/4) = ?$, $g(2) = ?$
     $g(?) = 3/2$, $g(?) = 2$, $g(?) = 8/3$, $g(?) = 1/2$
2. Let h be the function such that Dh is the set of all real numbers and, for each real number x, $h(x) = 2x + 1$.
   - (a) What is Rh?
   - (b) Draw a picture like Fig. 1-1 to describe h.
   - (c) Is there a real number which is mapped by h onto itself? Is there more than one such number?

Answers for Part D

Note. The bracketed comments following some answers are there to clarify those answers. Such comments are not part of the answers expected of the students.

1. (a) Yes.; $\{(x, y)\colon y = 2x + 1\}$ is one such function.
   No.; since a set is uniquely determined by its members it follows, by the definition of 'function', that a function is uniquely determined when its domain is specified and a rule is given which specifies its values at its various arguments.

   (b)

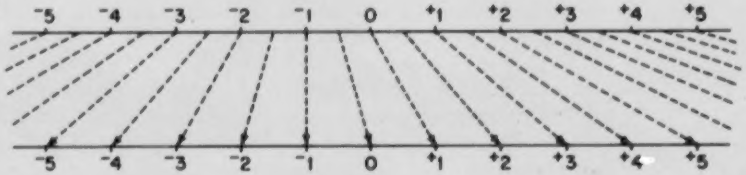


   (c) $\{x\colon 1 \le x \le 3\}$ [Since, for $0 \le a \le 1$, $1 = 2\cdot 0 + 1 \le 2a + 1 \le 2\cdot 1 + 1 = 3$ it follows that $\mathcal{R}g \subseteq \{x\colon 1 \le x \le 3\}$. Since, for any b, $2(\frac{b-1}{2}) + 1 = b$, and since, for $1 \le b \le 3$, $0 = \frac{1-1}{2} \le \frac{b-1}{2} \le \frac{3-1}{2} = 1$, it follows that $\{x\colon 1 \le x \le 3\} \subseteq \mathcal{R}g$. Consequently, $\mathcal{R}g = \{x\colon 1 \le x \le 3\}$.]

   (d) 5/3, 3/2, 2, 5/2, 'g(2)' is nonsense [$2 \notin \mathcal{D}g$], 1/4, 1/2, 5/6, 1/2 is not a value of g.

2. (a) The set of all real numbers. [For any real number a, 2a + 1 is a real number; for any real numbers a and b, 2a + 1 = b if a = (b - 1)/2.]

   (b)



   (c) Yes. [h(-1) = -1]; No.

3. Repeat Exercise 2 for each of the functions f_1, f_2, f_3, and f_4, each of which has the set of all real numbers for its domain and which are such that for each real number x,

$$f_1(x) = \frac{x - 1}{2},\ f_2(x) = -x - 2,\ f_3(x) = x + 3, \text{ and } f_4(x) = x.$$

4. Let h, f_1, f_2, f_3, and f_4 be the functions described in Exercises 2 and 3.

(a) Compute:

(1) $h(f_1(3))$ (2) $f_1(h(-2))$ (3) $h(f_2(4))$ (4) $f_2(h(4))$

(5) $h(f_3(-1))$ (6) $f_3(h(-1))$ (7) $h(f_4(10))$ (8) $f_4(h(10))$

(b) Let g_1, g_2, . . . , and g_8 be the functions whose domain is the set of all real numbers and which are such that, for each real number x,

$$g_1(x) = h(f_1(x)),\ g_2(x) = f_1(h(x)),\ g_3(x) = h(f_2(x)),$$
$$g_4(x) = f_2(h(x)),\ g_5(x) = h(f_3(x)),\ g_6(x) = f_3(h(x)),$$
$$g_7(x) = f_2(f_3(x)), \text{ and } g_8(x) = f_3(f_2(x)).$$

Express these eight g-functions in terms of 'x'. [For example, $g_1(x) = h(f_1(x)) = h\left(\frac{x - 1}{2}\right) = 2\left(\frac{x - 1}{2}\right) + 1 = x - 1 + 1 = x$. So, $g_1(x) = x$.]

5. Mappings like those you have studied in Exercises 2, 3, and 4 are said to be *linear*.

> A function f is linear if and only if
> (i) Df is the set of all real numbers, and
> (ii) for some nonzero real number m and some real number b,
> $f(x) = mx + b$ for each real number x.

(a) Is the function g of Exercise 1 a linear function?

(b) By definition, all linear functions have the same domain. Do all linear functions have the same range?

(c) The number m referred to in the definition of 'linear' is called the *slope* of the function. Is there any real number which is not the slope of some linear function?

(d) Complete:

If f is a linear function with slope m then, for each real number x and each real number y,

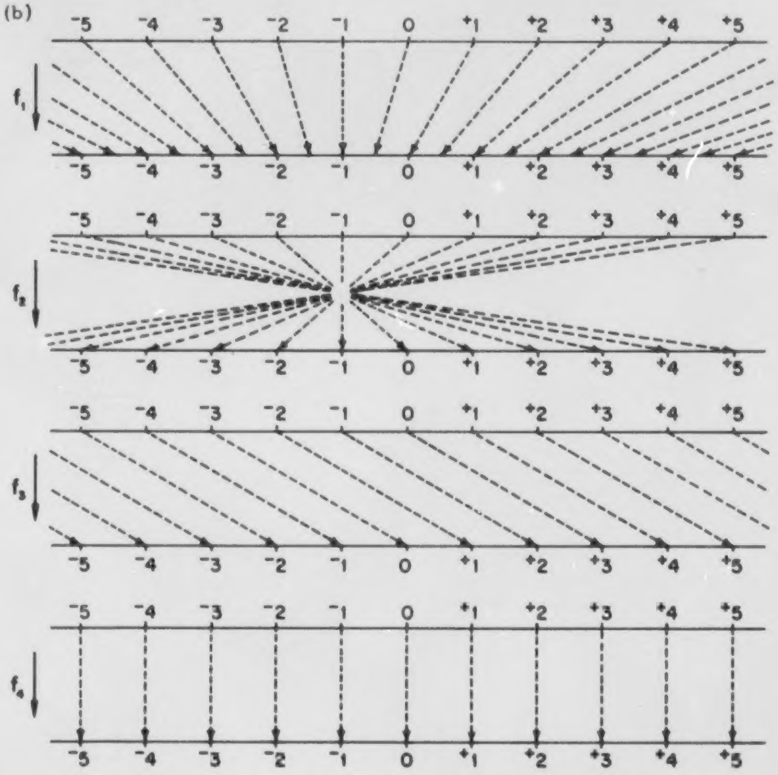
$$f(x) - f(y) = \_\_\_\_\_\_\_\_.$$

(e) Suppose that f is a linear mapping and that a and b are *two* real numbers. Can it happen that $f(a) = f(b)$?

(f) In working Exercises 2 and 3 you found that a given linear function may map some real number on itself, but that there are linear functions which map no real number on itself. Make

3. (a) The range of each of the functions f_1, f_2, f_3, and f_4 is the set of all real numbers.

(b)



(c) Both f_1 and f_2 map -1 on itself and neither maps any other number on itself. f_3 maps no number on itself. f_4 maps each number on itself. [For the present, these facts are sufficiently obvious from the figures. Students should, however, be able to justify their answer as to, say, f_1 by solving the equation '$(x - 1)/2 = x$'. For more on this question, see the discussion, below, of Exercise 5(g). You may wish to use the elastic thread gimmick as described there to foreshadow the result of that exercise.]

4. (a) (1) 3 (2) −2 (3) −11 (4) −11 (5) 5 (6) 2
(7) 21 (8) 21

[The answers for (1) and (2) illustrate the fact that f_1 and h are inverses of each other — inversion of functions is discussed in sections 1.03 and 1.04. The agreement between the answers for (3) and (4) is due to the fact that f_2 and h are linear functions each of which maps the same real number on itself — see the optional exercise 5(h), below.]

(b) $g_1(x) = x$, $g_2(x) = x$, $g_3(x) = -2x - 3$, $g_4(x) = -2x - 3$,
$g_5(x) = 2x + 7$, $g_6(x) = 2x + 4$, $g_7(x) = -x - 5$, $g_8(x) = -x + 1$

[The fact that each of g_1 and g_2 is f_4 is due, of course, to the fact that f_1 and h are inverses. That $g_3 = g_4$ is due to the fact that the linear mappings f_2 and h have the same fixed point.]

*

In preface to giving answers for Exercise 5, it is worth noting that the nature of linear mappings can be well brought out by "thought experiments" with elastic threads like the experiment described earlier in connection with the squaring mapping. This time, imagine the domain of such a function — say, the function h of Exercise 2 — represented by a calibrated thread of uniform elasticity. [Unfortunately, the elasticity of a rubber band is by no means uniform.] To aid your students' imaginations, write 'h(x) = 2x + 1' on the chalkboard, represent the range of the function by a calibrated horizontal chalk line, and pretend to hold the elastic thread against this line. Point out that you are holding it in such a way that the calibrations match. Now, move your arms apart, saying that you are stretching the thread to twice its former length, but are taking care that the 0-mark on the thread remains at the 0-mark on the line. Ask questions like: Where is the negative 2-mark on the thread? [Answer: At the negative 4-mark on the line.] ['Negative' is a more descriptive word to use when referring to negative numbers than is 'minus'.] Finally, move your hands a unit distance to the right [announcing that it is a unit distance] and repeat your questions. When your students seem to have grasped the relevance of your antics to the mapping h, move your hands a distance 3 units to the left and ask: What linear mapping does this represent — what are the values of 'm' and 'b'? [Answer: 2 and −2.] A few repetitions of this kind of experiment should give your students a good feeling for linear mappings. To act out a linear mapping with negative slope either reverse the ends of the thread before holding it against the chalk line or, after stretching it, make a shift to cross your hands without releasing the tension in the string. [This may result in a dislocated shoulder. If not, you will — in conjunction with the first technique — have illustrated the fact that, for any real number a, $-2a = 2 \cdot -a$. This is a fact that you should mention as justification for using the first technique.] To act out a linear mapping with slope between 0 and 1, pretend to hold a stretched thread against the chalk line and announce that you have previously marked it so that, as stretched, the calibrations now match up with those on the board. Then release the tension, etc.

In doing such experiments you are — preferably without mentioning it — illustrating the notion of function composition. [This notion is introduced in section 1.04.] For you are showing that a linear function may be thought of as the resultant of a stretching [or a shrinking] followed by a translation. [Of course, in particular cases, either the stretching or the translation may reduce to the identity mapping.] You can also, explicitly, bring out some of the properties of linear mappings. For example, a stretching moves each point other than 0 away from 0, whereupon a translation is bound to move exactly one point back to its original position. [This point will be one which is to the right of 0 if the translation is toward the left.] So, as your students should see, a linear function with slope greater than 1 maps exactly one real number on itself. A linear function with slope 1 — that is, a translation — either maps no real number on itself or, if it is the identity mapping, leaves each real number fixed. Linear mappings with slopes less than 1 can be investigated in a similar manner.

*

5. (a) No.; $\mathcal{R}g$ is not the set of all real numbers.

   (b) Yes.; for $m \neq 0$ and any b, the equation '$mx + b = c$' has a solution whatever value may be given to 'c'.

   (c) Yes.; the number 0. [The funtions which, were the definition different, would be linear functions with slope 0, are called constant functions, or, of course, constant mappings.]

   (d) $m(x - y)$

   (e) No.; by part (d), $f(a) - f(b) = m(a - b) \neq 0$ if $a \neq b$. [Note that in this text 'two' means two — that is, two numbers, or two points, or two whatever are, by virtue of being two, different. However, English is such that one needs sometimes to use plurals without implying difference. Thus, 'Suppose that a and b are real numbers' leaves open the possibility that a may be b.]

a conjecture as to which linear functions map exactly one real number on itself.

(g) The conjecture you made in part (f) could be stated in the form:

A linear mapping leaves exactly one real number fixed if and only if the slope of the mapping ______. [Complete.]

Show that this conjecture is correct. [*Hint*: Suppose that f is a linear mapping with slope m. It follows, by definition, that, for some real number b, $f(x) = mx + b$ for each real number x. So, f leaves the real number a fixed if and only if $ma + b = a$. What can you say about $\{x\colon mx + b = x\}$ if $m \neq 1$? If $m = 1$?]

(h) In Exercise 4 you dealt with a way of combining mappings. [We shall have more to say about this in another section.] A linear mapping f and a linear mapping g are said to be *permutable* if, for each real number x, $f(g(x)) = g(f(x))$. For example, the linear mapping f_4 of Exercise 3 is [obviously] permutable with every linear mapping. As your work in Exercise 4 shows, h and f_2 are permutable. Are h and f_3 permutable?

(i) Try to conjecture under what conditions linear functions f and g other than f_4 are permutable. [*Hint*: It is easy to show that if, for each real number x, $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$, then f and g are permutable if and only if $(m_1 - 1)b_2 = (m_2 - 1)b_1$. Try, however, to formulate your conjecture so that it does not refer to the slopes of f and g or to their *intercepts* (the latter are the numbers b_1 and b_2).]

Part E

1. Suppose that f is a function and that $S \subseteq f$. What can you say about the set S? [*Hint*: Is S a set of ordered pairs? Can there be two members of S which have the same first component?]
2. Suppose that f and g are functions and that $g \subseteq f$.
   (a) What may you say about Dg?
   (b) What information about Dg would convince you that $g = f$?

*

> Any subset of a function is a function.

> For any function f and any function g such that $g \subseteq f$,
> (a) $Dg \subseteq Df$, and
> (b) $g = f$ if and only if $Dg = Df$.

1.02 Translations of the Number Line

From some points of view, the simplest linear functions are those whose slope is 1. The function f_3, where $f_3(x) = x + 3$, is such a function. You have already pictured this mapping, as in Fig. 1-2, by drawing two pictures of the set of all real numbers and some arrows. Since,

(f) The correct conjecture is to the effect that the linear functions which have exactly one fixed point are precisely those whose slopes are not 1.

(g) differs from 1 [or: is not 1, etc.]
For any real numbers m and b, there is a unique number x such that $mx + b = x$ if and only if $m \neq 1$ [and, there is a number x such that $mx + b = x$ if and only if ($m \neq 1$ or $b = 0$)].

(h) No.

(i) Two linear mappings, neither of which is the identity mapping, f_4, are permutable if and only if they map the same real numbers on themselves [or: if and only if they leave the same real numbers fixed].

*

There are various ways of establishing the correctness of the conjecture in part (i). One method begins by noting that, as indicated in the hint, f and g are permutable if and only if $(m_1 - 1)b_2 = (m_2 - 1)b_1$. [For any real number a, $f(g(a)) = m_1(m_2a + b_2) + b_1$, $g(f(a)) = m_2(m_1a + b_1) + b_2$ and, so, $f(g(a)) = g(f(a))$ if and only if $m_1b_2 + b_1 = m_2b_1 + b_2$.] If one then recalls that, for $m_1 \neq 1$, the unique real number left fixed by f is $-b_1/(m_1 - 1)$, he sees that, in case the slopes of f and g both differ from 1 these functions are permutable if and only if they leave the same real number(s) fixed. In case m_1, say, is 1 it follows from the same condition that f and g are permutable if and only if $m_2 = 1$ or $b_1 = 0$. Since, for $m_1 = 1$ and $b_1 = 0$, f is the identity mapping it follows that, excluding this mapping from consideration, in case either f or g has slope 1, if f and g are permutable then both have slope 1 and so leave the same real numbers [to wit, none] fixed. On the other hand, if neither f nor g leaves any number fixed then, as we have previously learned, both have slope 1 and so, as the condition from the hint tells us, are permutable. Consequently, in any case [assuming that neither f nor g is the identity mapping] f and g are permutable if and only if both leave the same numbers fixed.

A somewhat more sophisticated argument goes as follows: We have seen that, for any real number a, $f(g(a)) = g(f(a))$ if and only if $(m_1 - 1)b_2 = (m_2 - 1)b_1$. It follows that if there exists an x such that $f(g(x)) = g(f(x))$ then $(m_1 - 1)b_2 = (m_2 - 1)b_1$ and, also, if $(m_1 - 1)b_2 = (m_2 - 1)b_1$ then, for each x, $f(g(x)) = g(f(x))$ — that is, then f and g are permutable. In particular if f and g both map a given number a on itself then, since, for this a, $f(g(a)) = a = g(f(a))$, f and g are permutable. So we have shown that if f and g are linear mappings both of which map some number on itself then f and g are permutable. From this we deduce that if f and g leave the same numbers fixed then f and g are permutable. For this follows from what has just been proved in case f, say, leaves some number fixed, while in case f leaves no number fixed then, since, by hypothesis, g behaves likewise, both f and g have, as we know, slope 1 and, in consequence are permutable. It remains to be shown that if f and g are permutable then either one of them is the identity or both map the same real numbers on themselves. As an acceptable alternative, we proceed to show that if neither f nor g is the identity mapping, and one of them maps some real number on itself which the other doesn't, then f and g are not permutable. To this end, suppose that neither f nor g is the identity mapping and that f, say, maps a number a on itself, but g doesn't. Then, $f(a) = a$ and $g(a) = b \neq a$. So, $f(g(a)) = f(b)$ and $g(f(a)) = b$. Now, since f is not the identity mapping, and since $f(a) = a$ and $b \neq a$, it follows that $f(b) \neq b$. Hence, $f(g(a)) \neq g(f(a))$ and it follows that f and g are not permutable.

As a final comment on part (i) we note that it is easy to show, by elastic thread experiments, "why" linear mappings f and g with slope greater than 1 are permutable if both map the same real number a on itself. To do so, label a point on your chalkboard line 'a' and indicate that you have done the same on the elastic thread. Then, announcing that you will show what a linear mapping f with slope $m_1 > 1$, which maps a on itself, does, hold the imaginary thread against the chalk line without stretching, so that the points marked 'a' are together. Now, by stretching the elastic thread appropriately — doubling its length if $m_1 = 2$ — all the while keeping the points marked 'a' together, you will show your students the action of f. Stretching, instead by the factor m_2 will, since g also maps a on itself, show what g does. To show the result of first applying f and then g, you would first stretch by a factor m_1 and then, without relaxing the tension, stretch by a factor m_2. As a result, you would have stretched the thread from its original length by a factor $m_2 \cdot m_1$. It follows that, since multiplication of real numbers is commutative, f and g are permutable.

If you are fortunate in your students, some may ask why you are sure that stretching about a really has the effect of a linear mapping which leaves a fixed. The answer to this amounts to showing that if $a = -b/(m - 1)$ then, for all real numbers x and y, $y = mx + b$ if and only if $y - a = m(x - a)$. Your students should, of course, be able to show that this is the case.

Answers for Part E

1. Since each member of f is an ordered pair, S is a set of ordered pairs. Since no two members of f have the same first component, neither do any two members of S. Hence, S is a function. In short, any subset of a function is a function.
2. (a) $\mathcal{D}g \subseteq \mathcal{D}f$

   (b) $\mathcal{D}g = \mathcal{D}f$ [Since $g \subseteq f$, if $g \neq f$ there must be an ordered pair in f which is not in g. Since no two members of f have the same first component it follows that there is a member of $\mathcal{D}f$ which does not belong to $\mathcal{D}g$. Hence, if $\mathcal{D}g = \mathcal{D}f$ then $g = f$.]

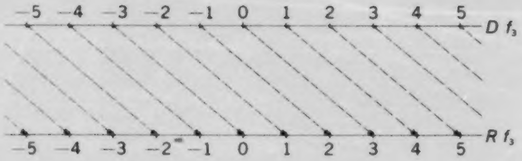


Fig. 1–2

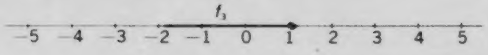


Fig. 1–3

for any real number a, $f_3(a) - a = 3$, we can also picture f_3 by drawing one picture of the number line and one arrow. [Of course, to understand this picture, we must know that f_3 is a linear mapping of slope 1 and, so, "treats all points alike."] It is convenient to use "geometric language" and say that the mapping f_3 moves each point of the number line a distance 3 in the positive sense. More briefly, we shall say that f_3 is a *translation of the number line*—the translation of magnitude 3 in the positive sense.

As another example, consider the linear mapping f such that, for each real number x, $f(x) = x - 2$. We may picture this mapping as in Fig. 1–4 and call it *the translation of the number line of magnitude 2 in the negative sense.*

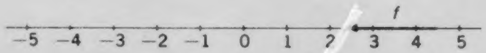


Fig. 1–4

[By now, you are probably tired of reading the words 'real number' and 'of the number line'. From now on we shall use '$\mathscr{R}$' as a name for the set of all real numbers.]

‖ A translation of $\mathscr{R}$ is a linear mapping with slope 1.

There is one linear mapping of slope 1 which you may have some question about calling a "translation". This is the mapping of $\mathscr{R}$ into itself which leaves each point of $\mathscr{R}$ fixed. Definitions, however, are adopted for convenience and, as you will see, it is convenient to include this mapping under the definition of 'translation'. [So convenient, indeed, that if we excluded it now, we would almost certainly decide later to change the definition.]

We use the phrase 'the number line' to refer to the set of all real numbers endowed with the structure which is defined by taking the distance between real numbers a and b to be $|a - b|$. In mathematical terminology, then, the number line is a certain metric space. [By definition, a metric space is a set for whose members one has adopted a definition of 'distance'.] More specifically, the number line is a 1-dimensional Euclidean metric space. Current use of the word 'space' by mathematicians is perhaps best described by saying that any set on which a structure has been specified — for example, by defining various properties of and relations among its members — is called a space if study of this structure is facilitated by the use of "geometric intuition". In the case of such structures terminology is often borrowed from geometry. For example, the members of the set in question are called points; and if a numerical-valued function whose arguments are pairs of points, and which has certain standard properties of ordinary length-measures, can be defined in terms of the given structure then this function is called a distance function. [The "standard properties" which characterize a distance function, d, are that, for any points a and b, $d(a, b) = 0$ if and only if $a = b$ and, for any points a, b, and c, $d(b, a) + d(b, c) \geq d(a, c)$. It follows that, for any points a and b, $d(b, a) = d(a, b) \geq 0$.]

In the definition at the beginning of the preceding paragraph we have endowed the set of real numbers with more structure than is appropriate for the study of Euclidean geometry. For, in Euclidean — as contrasted with Euclidean metric — geometry, there is no preferred distance function. In Euclidean geometry only ratios of distances — or, more properly, ratios of segments — are relevant.

In the main portions of this course we shall study Euclidean geometry, basing all our ideas on the notion of a translation of a Euclidean space. Although we shall have in mind a 3-dimensional Euclidean space [and the lines and planes which are its 1-dimensional and 2-dimensional subspaces], the concepts we introduce will not depend on dimension, and many of our theorems will be true for Euclidean spaces of any dimension. In particular, what in this section we call 'translations of the number line' are precisely those mappings which come under our general notion of translations as it applies to the case of this particular 1-dimensional space. So, the present section foreshadows, to a very limited extent, our later discussion of translations.

In speaking, above, about distance functions, we have said that they are "numerical valued". It is often helpful [we believe] to take account of the fact that the nonpolar ["unsigned"] "numbers of arithmetic" which, in a logical development of number concepts, are precursors of the polar ["signed"] real numbers, are also different entities from the nonnegative real numbers. If one does so then the values appropriate to a distance function are these nonpolar numbers [a distance being no more positive than it is negative], and the same applies to magnitudes of translations. In this course, however, we ignore the nonpolar numbers and, in place of them, use the nonnegative reals.

For precision, one should distinguish between the set of all real numbers and the number line, since the latter is the former together with a certain specified structure. Thus, if we use '$\mathcal{R}$' to denote this set of all real numbers, 'translation of $\mathcal{R}$' is not quite proper as a synonym for 'translation of the number line'. It is customary, however, to use the same symbol to denote both a given space and the set of its points.

The translation of $\mathcal{R}$ which leaves each real number fixed has previously been referred to in this commentary as 'the identity mapping of $\mathcal{R}$ into itself'. This is a special case of a terminology which is introduced in the text on page 25. If you wish, you may introduce it now [or even earlier] and denote it by '$i_{\mathcal{R}}$'.

Exercises

Part A

1. Show that, for any $a \in \mathcal{R}$ and any $b \in \mathcal{R}$, there is a unique translation of $\mathcal{R}$ which maps a on b.
2. Show that if f is a translation of $\mathcal{R}$ then, for any $a \in \mathcal{R}$ and any $b \in \mathcal{R}$,

$$f(a) - a = f(b) - b.$$

[*Hint*: If you wish, you may use the result of an earlier exercise.]

3. The statement you proved in Exercise 2 is a way of saying that any translation of $\mathcal{R}$ moves all points of $\mathcal{R}$ the same distance in the same sense. Show that any mapping, whose domain is $\mathcal{R}$, which does this is a translation. [*Hint*: Choose some real number b, and suppose that, for each $x \in \mathcal{R}, f(x) - x = f(b) - b$. Does it follow that f is a translation?]
4. When we use geometric language to discuss real numbers, the *distance* between a point a and a point b of the number line is, by definition, their "absolute difference", $|a - b|$. [Recall that, for each $x \in \mathcal{R}$, $|x| = x$ if x is nonnegative, and $|x| = -x$ if x is negative.]
   - **(a)** What does a linear mapping with slope 2 do to distances? [That is, in terms of the distance between two points, what can you say about the distance between their images under such a mapping?]
   - **(b)** What does a translation of $\mathcal{R}$ do to distances?
   - **(c)** What does a linear mapping with slope m do to distances?
5. In Exercise 5 of Part D, page 14, you found that a linear mapping leaves just one point fixed if and only if its slope differs from 1. It follows that translations of $\mathcal{R}$ are just those linear mappings which either leave no point fixed or leave more than one point fixed. Is there a translation which leaves more than one point fixed? Is there more than one such translation?

Part B

As you know, a function is a set of ordered pairs; but, not every set of ordered pairs is a function. For any a and b, the *converse* of the ordered pair (a,b) is the ordered pair (b,a). [What is the converse of (2,3)? Of (3,2)? Of (2,2)?] Given any set S of ordered pairs, the *converse* of S is the set which consists of the converses of the members of S.

1. For each of the following sets, describe its converse.
   - **(a)** $\{(1,3),(2,5),(3,-1),(\sqrt{4},5)\}$
   - **(b)** $\{(1,3),(2,5),(3,-1),(4,5)\}$
   - **(c)** $\emptyset$
   - **(d)** $\{(x,y): y = x\}$
   - **(e)** $\{(x,y): y = |x|\}$
   - **(f)** $\{(x,y): y = x + 3\}$
   - **(g)** $\{(x,y): y = x - 5\}$

Answers for Part A

1. f is a translation of $\mathcal{R}$ which maps a on b if and only if f is a linear mapping with slope 1 and $f(a) = b$. f is a linear mapping with slope 1 if and only if $\mathcal{D}_f = \mathcal{R}$ and, for some $c \in \mathcal{R}$, $f(x) = x + c$ for each $x \in \mathcal{R}$. For such a mapping f, $f(a) = b$ if and only if $b = a + c$ — that is, if and only if $c = b - a$. Hence, the unique translation which maps a on b is that one such that, for each $x \in \mathcal{R}$, $f(x) = x + (b - a)$.

2. From the definition of 'translation' and the result of Exercise 5(d) on page 14, it follows that if f is a translation then, for any real numbers a and b, $f(b) - f(a) = 1 \cdot (b - a)$. So, $f(b) - b = f(a) - a$.

3. Yes. [slope = 1, intercept = $f(b) - b$]

4. (a) A linear mapping of slope 2 doubles all distances.
   (b) A translation leaves distances unchanged.
   (c) A linear mapping with slope m multiplies all distances by $|m|$.
   [All three answers drop out of the result established in Exercise 5(d) on page 14. Make sure that students understand the somewhat abstruse language used in these answers. Nothing can change the distance between two given points of a space. Nevertheless, one says that a mapping doubles the distance between two points, meaning that the distance between their images is twice that between the given points.]

Some of your students may be interested in the fact that linear mappings may be characterized as the mappings of $\mathcal{R}$ into itself each of which multiplies all distances by a given number. In view of the answer for part (c), this can be established by showing that if f is a mapping of $\mathcal{R}$ into itself such that for each $x \in \mathcal{R}$ and each $y \in \mathcal{R}$, $|f(x) - f(y)| = m|x - y|$ [where $m \neq 0$] then f is a linear mapping whose slope is either m or $-m$. To do so, suppose that f is a mapping of $\mathcal{R}$ into $\mathcal{R}$ and that, for each x and y in $\mathcal{R}$, $|f(x) - f(y)| = m|x - y|$. Let a and b be two points of $\mathcal{R}$. By hypothesis, there are two cases, that in which $f(a) - f(b) = m(a - b)$ and that in which $f(a) - f(b) = m(b - a)$. It will be sufficient to show, in the first case, that, for any $c \in \mathcal{R}$, $f(c) - f(b) = m(c - b)$ and, in the second case, that, for any $c \in \mathcal{R}$, $f(c) - f(b) = m(b - c)$. Consider, then, the first case — that in which $f(a) - f(b) = m(a - b)$ — and suppose that c is any point of $\mathcal{R}$. As was the case with a and b, we know that if $f(c) - f(b) \neq m(c - b)$ then $f(c) - f(b) = m(b - c)$. But, supposing that $f(c) - f(b) = m(b - c)$, it follows, since $f(a) - f(b) = m(a - b)$, that $f(c) - f(a) = m(2b - a - c)$. Since $f(c) - f(a)$ is either $m(c - a)$ or $m(a - c)$ [and $m \neq 0$] it follows that $c = b$ or $a = b$. Since $a \neq b$ [a and b being two points of $\mathcal{R}$] it follows that $c = b$. So [even in this least favorable situation], $f(c) - f(b) = m(c - b)$. Hence, in the first of our two cases, $f(x) = mx + [f(b) - mb]$ for each $x \in \mathcal{R}$ and, so, f is a linear mapping with slope m. [The second case may be dealt with in a similar manner.]

5. There is such a translation and only one. It is the mapping of $\mathcal{R}$ into itself which leaves each point of $\mathcal{R}$ fixed.

Answers for Part B

[These exercises are exploratory for section 1.03. Note our uses of the word 'con'verse'. Although these have been standard for years, some authors of recently published texts use the word 'inverse' as a synonym for 'converse'. This arbitrary change in the long-accepted meaning of 'inverse' [see page 19] is deplorable.]

1. (a) $\{(3, 1), (5, 2), (-1, 3), (5, \sqrt{4})\}$ [Alternate answer: $\{(3, 1), (5, 2), (-1, 3)\}$]

   (b) $\{(3, 1), (5, 2), (-1, 3), (5, 4)\}$ (c) $\emptyset$

   (d) $\{(x, y)\colon\ y = x\}$ (e) $\{(x, y)\colon\ |y| = x\}$

   (f) $\{(x, y)\colon\ y = x - 3\}$ (g) $\{(x, y)\colon\ y = x + 5\}$

   [As an example of a formal procedure for justifying the answers for the parts (d) - (g) we take part (e). By definition, (a, b) belongs to the converse of $\{(x, y)\colon\ y = |x|\}$ if and only if $(b, a) \in \{(x, y)\colon\ y = |x|\}$. The latter is the case if and only if $a = |b|$ — that is, if and only if $|b| = a$. Hence, the converse of $\{(x, y)\colon\ y = |x|\}$ is $\{(x, y)\colon\ |y| = x\}$.

   Students will presumably guess that the answers can be obtained by interchanging the occurrences of 'x' and 'y' after the colon [or, by replacing '(x, y)' by '(y, x)' — but not by doing both]. They should, however, be able to give some explanation, analogous to the preceding, of why this mechanical procedure works.]

2. **(a)** Which of the sets in Exercise 1 are functions?
   (b) Which of the sets in Exercise 1 have converses which are functions?
3. Complete:
   (a) A function whose converse is a function is a set of ordered pairs such that no two have the same first component and no two have ______.
   (b) The converse of a mapping is a mapping if and only if no two objects have ______________ under the given mapping.
4. Is the converse of the squaring operation for real numbers a function?
5. **(a)** Must the converse of a linear mapping be a mapping?
   (b) Must the converse of a translation of $\mathscr{R}$ be a translation of $\mathscr{R}$?

1.03 Function Inversion

Suppose that f is a mapping of a set S onto a set T. Without further specifying f, S, or T we can picture a situation like that just described as is done in Fig. 1-5(a). The dots inside the loop labeled 'S' represent members of S; those inside the loop labeled 'T' represent members of T; the arrows indicate which members of T are images of the various members of S. Figure 1-5(b) suggests the converse of f. The first components of the members of the converse of f are located in T; the second members are in S; the arrows run from first components to corresponding second components. Assuming that different dots inside the left-hand loop represent different members of S, the converse of f is not a mapping.

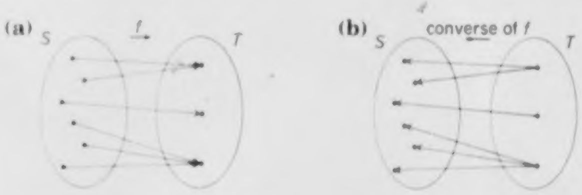


Fig. 1–5

Fig. 1-6 shows a different situation. Again, f is a mapping which maps S onto T. But, this time, the converse of f is also a mapping [and so, of course, maps T onto S]. More specifically, it makes sense to say that the mapping which is the converse of f "undoes" what the mapping f "does". [Run your eye from left to right along one of the arrows in (a), then, from right to left along the corresponding arrow in (b).] Notice, also, that f matches the members of S in a one-to-one manner with those of T—or, if you prefer, f matches the members of T

---

2. (a) All the sets described in Exercise 1 are functions.
   (b) The sets described in parts (a), (c), (d), (f), and (g) have converses which are functions. Those described in parts (b) and (e) do not.
3. (a) the same second component
   (b) the same image
4. No.; since, for example, both (2, 4) and (−2, 4) belong to the squaring operation, its converse contains the two ordered pairs (4, 2) and (4, −2), both of which have first component 4.
5. (a) Yes.; for $m \neq 0$, the converse of $\{(x, y): y = mx + b\}$ is $\{(x, y): y = x/m - b/m\}$ and $1/m \neq 0$.
   (b) Yes.

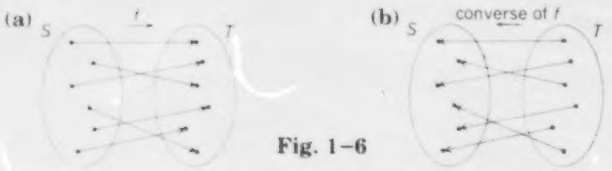


Fig. 1–6

in a one-to-one manner with those of S. Of course, the converse of f matches the members of S and those of T in the same manner.

When, as in the second situation, the converse of a mapping f is, itself, a mapping, this converse is called the *inverse* of f. Evidently, a mapping has an inverse if and only if it matches the members of its domain in a one-to-one way with those of its range. For this reason, a mapping which has an inverse is said to be a *one-to-one* mapping. Instead of 'the inverse of f' one often writes 'f^{-1}'. [Read 'f^{-1}' as 'the inverse of f' or, more easily, as 'f inverse'.]

Exercises

In order to show that a function f has an inverse, it is sufficient to show that, for any $a \in Df$ and any $b \in Df$, if $f(a) = f(b)$ then $a = b$. Another equivalent procedure is to show that, for any *two* arguments, a and b, of f, $f(a) \neq f(b)$.

To show that a function does not have an inverse, merely find two arguments of f at which f has the same value.

Use one of these methods to show that

1. squaring of real numbers does not have an inverse,
2. the function described in Exercise 1(a) of Part B on page 17 has an inverse,
3. the function described in Exercise 1(b) of Part B does not have an inverse,
4. any linear mapping has an inverse.

1.04 Function Composition

Suppose that f and g are mappings. For any $a \in Df$ which is such that $f(a) \in Dg$, there is a uniquely determined member, $g(f(a))$, of the range

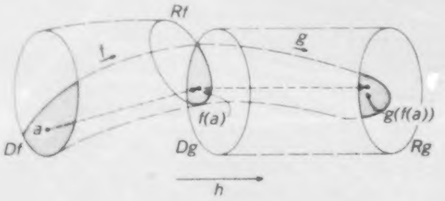


Fig. 1–7

Answers for Exercises

1. Since (2, 4) and (−2, 4) both belong to the squaring function, 2 and −2 are two arguments of this function at both of which it has the same value. Consequently, the squaring function does not have an inverse.
2. The arguments of the function in question are 1, 2, and 3 and the corresponding values are 3, 5, and −1, respectively. Since the function has different values at different arguments it follows that it has an inverse.
3. Since the function in question has the same value, 5, at both of its arguments 2 and 4 it follows that it does not have an inverse.
4. [This has already been shown, in the answer for Exercise 5(a) of Part B on page 18, by showing that the converse of any linear mapping is a linear mapping. This, however, does not satisfy the instructions for the present exercise.]

Suppose that a and b are arguments of the linear function f with slope m. Then, $m \neq 0$ and, by an earlier exercise, for any arguments a and b of f, $f(a) - f(b) = m(a - b)$. It follows that if $f(a) = f(b)$ then $a = b$. Consequently f — and, so, any linear mapping — has an inverse.

of g. So, f and g together determine a mapping h such that $Dh = \{x \in Df: f(x) \in Dg\}$ and such that, for each $x \in Dh$, $h(x) = g(f(x))$. The mapping h with these properties is called *the resultant of f followed by g*, or *the composition of g on f*. Evidently, composition of functions is an operation on functions in the same sense as, say, addition of real numbers is an operation on real numbers. Just as we use the operator '+' in referring to the result of adding real numbers, so we need an operator to use in referring to the result of composing functions. We introduce such an operator in the following definition.

> For any functions f and g,
> (i) $D[g \circ f] = \{x \in Df: f(x) \in Dg\}$, and
> (ii) for each $x \in D[g \circ f]$, $[g \circ f](x) = g(f(x))$.

[Read '$g \circ f$' as 'the resultant of f followed by g' or, for short, as 'g circle f'.]

You may recall that you computed the resultants of several pairs of linear functions in an earlier exercise.

Exercises

Part A

1. Suppose that $f = \{(1,-1),(-2,4),(5,\pi),(6,-3)\}$ and
$g = \{(-1,1),(-2,-2),(-3,5),(10,5)\}$.
Since $1 \in Df$ and $f(1) = -1 \in Dg$ it follows that $1 \in D[g \circ f]$ and that $[g \circ f](1) = g(-1) = 1$. So, $(1,1) \in g \circ f$. Since, although $-2 \in Df$, $f(-2) = 4 \notin Dg$ it follows that $-2 \notin D[g \circ f]$. So, there is no y such that $(-2,y) \in g \circ f$. Also, since $3 \notin Df$, $3 \notin D[g \circ f]$ and there is no y such that $(3,y) \in g \circ f$.
Complete:
(a) $g \circ f = \{(1,1),$ ____$\}$ **(b)** $f \circ g =$ ________
2. Your work in Exercise 1(a) brings out a point which, by part (i) of the definition, is obvious enough. For any functions h and k, $D[k \circ h] \subseteq Dh$; but, in general $D[k \circ h] \neq Dh$. In fact, as is illustrated in Exercise 1(b), for any functions h and k, $D[k \circ h] = Dh$ if and only if Rh ____ Dk. [Complete.]
3. Your work in Exercise 1 also shows that there exist functions k and h such that $k \circ h \neq h \circ k$. In other words, function composition is not a ________ operation.

Part B

1. Let f be the translation of $\mathcal{R}$ which maps each point of the number line a distance 1 in the positive sense. Let g be the linear mapping under which the image of each real number is its double. You could picture either of these functions by drawing two horizontal lines which represent its domain and range, respectively, and

Since the result of composing any function g on any function f is a function — in "most" cases, the function $\emptyset$ since, more often than not, $\mathcal{R}f \cap \mathcal{D}g = \emptyset$ — it follows that function composition is a binary operation on the set of all functions. As is shown in the exercises, it is an example of a binary operation which is associative but not commutative.

Function composition is a special case of a very basic operation on relations which is called relation multiplication or relative multiplication. A somewhat typical "relative multiplication fact" is that "niecehood" is the relative product of "daughterhood" by "siblinghood" — b is a niece of a if and only if b is a daughter of some person who is a sibling of a. [Although, of course, function notation cannot be used when referring to arbitrary relations, function composition can be defined without use of function notation. The appropriate definition — (☆☆) on page 23 — is readily transformed into a definition of 'relative multiplication'. If we then construe the relation "niecehood" to be the set N of all ordered pairs of persons the second of whom is a niece of the first, and construe "daughterhood" and "siblinghood" similarly as sets D and S of ordered pairs, it is clear that $N = D \circ S$.]

Although it is convenient to define function composition as we have, so that it is an operation on the set of all functions, in our uses of it we shall be dealing with mappings of a given set S — as a matter of fact, the set of all points of Euclidean space — into itself. If f and g are mappings of this special kind then, since $\mathcal{R}f \subseteq S = \mathcal{D}g$, so is $g \circ f$. The set of all one-to-one mappings of a given set S onto itself is also easily seen to be closed under function composition. This and the fact that the same set of functions is obviously closed with respect to function inversion are of fundamental importance for the "algebra" of such mappings.

As remarked earlier in the commentary, we shall later have much to do with the set $\mathcal{T}$ of all translations of Euclidean space $\mathcal{E}$. [The notations '$\mathcal{E}$' and '$\mathcal{T}$' are introduced in sections 1.05 and 1.07, respectively.] It will turn out that the members of $\mathcal{T}$ are one-to-one mappings of $\mathcal{E}$ onto itself and that $\mathcal{T}$ is closed with respect to both function inversion and function composition. It follows that each of these can be thought of as an operation on $\mathcal{T}$. [More precisely, the restriction of each of them to $\mathcal{T}$ is an operation on $\mathcal{T}$.] The second of these operations is not only associative — as always — but, as applied to members of $\mathcal{T}$, is commutative. As a result, composition and inversing of translations of $\mathcal{E}$ have algebraic properties very similar to those of addition and oppositing of real numbers. In fact, in the 1-dimensional case in which $\mathcal{E}$ is the real number line $\mathcal{R}$, the two pairs of operations have exactly the same properties.

The choice of the topics discussed in this and the preceding section is motivated largely by a desire to prepare students to recognize analogies between the set $\mathcal{T}$ of translations of $\mathcal{E}$ [illegible] the set of translations of $\mathcal{R}$.

When reminding students that they have already dealt with examples of function composition in Exercise 4 of Part D on page 14, you might also lead them to recognize that your elastic thread experiments were based on the recognition that any linear mapping is the resultant of a stretching [or shrinking] followed by a translation. By recalling the definitions of the mappings h and f, in the exercise just mentioned, you can lead them to see that the inverse of the resultant of a stretching s followed by a translation t is the resultant of the inverse of t followed by the inverse of s $[(t \circ s)^{-1} = s^{-1} \circ t^{-1}]$. The relation between composition and inversion suggested by this example does indeed hold. A resultant of mappings which have inverses does itself have an inverse which is related to the inverses of the "factors" as the formula suggests. The fact that, for real numbers a and b, $-(a + b) = -b + -a$ is a special case of this general result. The fact that $-(a + b) = -a + -b$, on the other hand, is not. It depends essentially on the commutativity of addition.

Answers for Part A

1. (a) (6, 5) (b) $\{(-1, -1), (-2, 4), (-3, \pi), (10, \pi)\}$
2. $\subseteq$
3. commutative

drawing a few arrows connecting some arguments with the corresponding values. Since $Rf = Dg$, you can picture both functions by drawing three horizontal lines – one for the Df, the next below it for both Rf and Dg, and the third, below these, for Rg. Do so, drawing dashed arrows, some to indicate the effect of f on its arguments and others to indicate the effect of g on the corresponding values of f. On the same picture, draw solid arrows to show the action of $g \circ f$.

2. In Exercise 1 you pictured two functions f and g:

$$f = \{(x,y)\colon y = x + 1\} \text{ and } g = \{(x,y)\colon y = 2x\}$$

You also pictured the resultant $g \circ f$. Now, give a brace-notation description [like those of f and g, above] of $g \circ f$. Also, give such a description of $f \circ g$.

3. In Exercise 4 of Part D on page 14 you computed resultants for several pairs of the functions h, f_1, f_2, f_3, and f_4. For example, in that exercise, $g_1 = h \circ f_1$ and $g_2 = f_1 \circ h$. Reacquaint yourself with this exercise and answer the following questions.
   - **(a)** Is $h \circ f_1$ the same function as $f_1 \circ h$? If so, what function? From what you have learned since doing this exercise, what can you say about h and f_1?
   - **(b)** Is $h \circ f_2$ the same function as $f_2 \circ h$?
   - **(c)** Is $h \circ f_3$ the same function as $f_3 \circ h$?
   - **(d)** Is $f_2 \circ f_3$ the same function as $f_3 \circ f_2$?

4. From your results in Exercise 3 you see that some pairs of linear functions have the same resultant as their converse pairs, and some do not. [In other words, some pairs of linear mappings are *permutable* and some are not.] Let's investigate permutability for linear mappings.

Suppose that f and g are linear mappings. By definition, $Df = Dg = \mathscr{R}$ and we may assume that, for each $x \in \mathscr{R}$,

$$f(x) = ax + b \text{ and } g(x) = cx + d \text{ [where } a \neq 0 \neq c].$$

By the definition of function composition it follows that $D[g \circ f] = \mathscr{R}$ and that, for each $x \in \mathscr{R}$,

$$[g \circ f](x) = c(ax + b) + d = (ca)x + (cb + d).$$

Similarly [complete], $D[f \circ g] =$ ________ and, for each $x \in \mathscr{R}$,

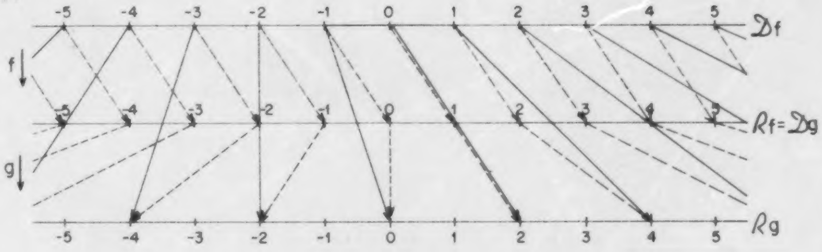
$$[f \circ g](x) = \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_.$$

It follows that both $g \circ f$ and $f \circ g$ are ____________ mappings and that they have the same ____________. Moreover,

$$g \circ f = f \circ g \text{ if and only if } (c - 1)b = \_\_\_\_\_\_\_\_\_\_\_\_.$$

Answers for Part B

1.



2. $g \circ f = \{(x,y)\colon y = 2x + 2\}$; $f \circ g = \{(x,y)\colon y = 2x + 1\}$

3. (a) Yes; f_4; They are inverses of one another.
   - (b) Yes.
   - (c) No.
   - (d) No.

*

Textbook writers sometimes confuse permutability of singulary operations with commutativity of a binary operation. [For example, the nonpermutability of the operations of putting on shoes and socks may (wrongly) be used as an illustration of noncommutativity because "putting on" is a singulary operation.]

*

4. $\mathfrak{R}$; $a(cx + d) + b$; $(ac)x + (ad + b)$; linear; slope; $(a - 1)d$

5. The results you obtained in Exercise 4 have several important consequences. Here are some of them. For each, explain how it follows from the results of Exercise 4.
   - (a) The set of all linear mappings is closed under function composition.
   - (b) The set of all translations is closed under function composition.
   - (c) Any two translations are permutable.
   - (d) *Two* linear mappings with the same slope are permutable if and only if they are translations.
   - (e) The only linear mapping which permutes with every such mapping is the translation which maps each real number on itself. [*Hint*: By part (d), any linear mapping which permutes with every linear mapping must be a translation. Is there a translation which permutes with the mapping g of Exercise 2?]
   - ◦ (f) Two linear mappings which are not translations are permutable if and only if they leave the same point fixed.

*

You have seen that, for any functions f and g, there is a function $f \circ g$ and there is, also, a function $g \circ f$. And, you have seen that, for some choices of f and g, $f \circ g = g \circ f$ while for other choices this is not the case. It follows from this that function composition is not commutative.

Similarly, for any functions f, g, and h, there is a function $f \circ [g \circ h]$ and there is, also, a function $[f \circ g] \circ h$. For example, suppose that f, g, and h are the linear mappings such that, for each $x \in \mathscr{R}$,

$$f(x) = 2x - 3,\ g(x) = -x + 5, \text{ and } h(x) = 3x + 1.$$

It follows, by the definition of function composition, that, for any $a \in \mathscr{R}$,

$$[g \circ h](a) = g(h(a)) = -(3a + 1) + 5, \text{ and}$$
$$[f \circ [g \circ h]](a) = f([g \circ h](a)) = 2(-(3a + 1) + 5) - 3.$$

Similarly [complete],

$$[f \circ g](a) = \underline{\qquad\quad} = \underline{\qquad\qquad}, \text{ and}$$
$$[[f \circ g] \circ h](a) = [f \circ g](h(a)) = \underline{\qquad\qquad\quad}.$$

Part C

1. Do you think that function composition is associative—that is, do you think that, for *any* functions f, g, and h, $f \circ [g \circ h] = [f \circ g] \circ h$?
2. Note that, by Exercise 5(a) of Part B, all functions which can be obtained from linear functions by composition are linear functions

Answers for Part B [cont.]

5. (a) It has been shown that, for any linear mappings f and g, the resultant of g followed by f is a linear mapping. This is what is meant when one says that the set of all linear mappings is closed under function composition.

   (b) By definition, a translation is a linear mapping with slope 1. Since, as has been shown, the resultant of a linear mapping g followed by a linear mapping f is a linear mapping whose slope is the product of the slopes of f and g, it follows that a resultant of translations is a translation.

   (c) This follows from the final result in Exercise 4, since if f and g are translations then $c - 1 = 0 = a - 1$.

   (d) From the final result, for $c = a \neq 1$, it follows that $g \circ f = f \circ g$ if and only if $b = d$.

   (e) Following the hint, for $a = 1$, $c = 2$, and $d = 0$, $g \circ f = f \circ g$ if and only if $b = 0$. So, the only translation which permutes with the mapping g of Exercise 2 is the identity mapping of $\mathscr{R}$ onto itself.

   ☆(f) [See answer for Exercise 5(h) of Part D on page 15.]

*

The fill-ins on the last two lines are as follows:

$f(g(a))$; $2(-a + 5) - 3$; $2(-(3a + 1) + 5) - 3$

Answers for Part C

1. Yes.

and so, in particular, have the same domain, $\mathcal{R}$. So, if f, g, and h are any linear functions, it will follow that $f \circ [g \circ h] = [f \circ g] \circ h$ if you can show that, for each $x \in \mathcal{R}$, $[f \circ [g \circ h]](x) = [[f \circ g] \circ h](x)$. Do so by using part (ii) of the definition of function composition.

* 3. From your work in Exercise 2 it follows that, for any mappings f, g, and h, $f \circ [g \circ h] = [f \circ g] \circ h$ *if* $f \circ [g \circ h]$ and $[f \circ g] \circ h$ have the same domain. One way of showing that function composition is associative is to show that this condition is satisfied by any functions f, g, and h. Do so. [A quite different proof of the associativity of function composition will be given presently.]

*

There is an alternative definition of function composition which it is useful to know. To discover this, suppose that $(a,b) \in g \circ f$. It follows that $a \in D[g \circ f]$ and that $b = [g \circ f](a)$. So, by the definition of function composition which we are using, $a \in Df$, $f(a) \in Dg$, and $b = g(f(a))$. Since $a \in Df$, $(a, f(a)) \in f$; and, since $f(a) \in Dg$, $(f(a), g(f(a))) \in g$. Since $g(f(a)) = b$ it follows that $(f(a), b) \in g$. Since $(a, f(a)) \in f$ and $(f(a), b) \in g$ it follows that

$$(*) \qquad \exists_z((a,z) \in f \text{ and } (z,b) \in g).$$

Hence, if $(a,b) \in g \circ f$, then $(*)$.

On the other hand, suppose that a and b are such that $(*)$ is satisfied. Let c be some object such that $(a,c) \in f$ and $(c,b) \in g$. It follows that $a \in Df$ and $c = f(a)$, and that $c \in Dg$ and $b = g(c)$. So, $a \in Df$, $f(a) \in Dg$, and $b = g(f(a))$. It follows by our definition of function composition that $a \in D[g \circ f]$ and $b = [g \circ f](a)$. In other words, it follows that $(a,b) \in g \circ f$. Hence, if $(*)$, then $(a,b) \in g \circ f$.

Combining our results we have:

$$(**) \qquad (a,b) \in g \circ f \Longleftrightarrow \exists_z((a,z) \in f \text{ and } (z,b) \in g)$$

[Read '$\Longleftrightarrow$' as 'if and only if'.] It is now easy to write a new definition of function composition which is equivalent to the one we are using:

$$g \circ f = \{(x,y) \colon \exists_z \underline{\qquad\qquad\qquad}\}.$$

Using $(**)$ it is easy to show that function composition is associative. For, suppose that f, g, and h are any functions. It follows from $(**)$ that

$$\begin{aligned}(a,b) \in f \circ [g \circ h] &\Longleftrightarrow \exists_u((a,u) \in g \circ h \text{ and } (u,b) \in f)\\ &\Longleftrightarrow \exists_u(\exists_v((a,v) \in h \text{ and } (v,u) \in g) \text{ and } (u,b) \in f)\\ &\Longleftrightarrow \exists_u\exists_v(((a,v) \in h \text{ and } (v,u) \in g) \text{ and } (u,b) \in f)\end{aligned}$$

[Each of the first two transformations – or, steps – is justified by $(**)$.

2. For each $x \in \mathcal{R}$, $[f \circ [g \circ h]](x) = f([g \circ h](x)) = f(g(h(x)))$,

and $[[f \circ g] \circ h](x) = [f \circ g](h(x)) = f(g(h(x)))$.

☆3. For any $a \in \mathcal{D}[f \circ [g \circ h]]$ it follows [by part (i) of the definition of 'function composition'] that $a \in \mathcal{D}[g \circ h]$ and that $[g \circ h](a) \in \mathcal{D}f$. Hence [by part (i)], $a \in \mathcal{D}h$, $h(a) \in \mathcal{D}g$, and [by part (ii)] $g(h(a)) \in \mathcal{D}f$. Since $h(a) \in \mathcal{D}g$ and $g(h(a)) \in \mathcal{D}f$ it follows [by part (i)] that $h(a) \in \mathcal{D}[f \circ g]$. Since $a \in \mathcal{D}h$ and $h(a) \in \mathcal{D}[f \circ g]$ it follows that $a \in \mathcal{D}[[f \circ g] \circ h]$. Consequently, $\mathcal{D}[f \circ [g \circ h]] \subseteq \mathcal{D}[[f \circ g] \circ h]$. [A similar argument establishes the reverse inclusion, and, so, completes the proof of identity.]

*

The standard reading of (☆) is:

There exists a z such that $(a, z) \in f$ and $(z, b) \in g$.

It might, however, be helpful to read it — once — as:

There is something such that the ordered pair $(a, \text{it}) \in f$
and the ordered pair $(\text{it}, b) \in g$.

Following the preceding suggestions may help to develop in students the [correct] feeling that the indices 'x', etc. on quantifiers serve the same reference — or "linking" — purposes as do the words 'something' and 'it' in the suggested reading. [For a rather extended discussion of quantifiers, see High School Mathematics, Course I, pp. 116-142. What is said there applies to our use of quantifiers in the present text. We shall here, however, use open sentences as well as universal generalization sentences to express universal generalities. You will find more about this in the commentary for section 1.07.]

The fill-in three lines below (☆☆) is: $((x, z) \in f \text{ and } (z, y) \in g)$

We have supposed students to be sufficiently aware of the meaning conveyed by 'if and only if'. [This, and some other points of logic are reviewed in Chapter 2.] The introduction of '$\Longleftrightarrow$' as an abbreviation for this phrase should cause no extra difficulty.

The "matters of grammar" referred to in the bracketed analysis of the first part of the proof of associativity is, more properly, a rule of logic. In case your students are already acquainted with logic as developed in High School Mathematics, Course I, you may wish to go into the rules governing the use of existential generalizations and use them in justifying the step in question. As is the case for universal generalizations, there are two basic rules for dealing with existential generalizations. One is to the effect that an existential generalization is a consequence of any of its instances [for example, the sentence '$1 \cdot 2 = 2$' implies the sentence '$\exists_x\ 1 \cdot x = 2$'] and the other is a test-pattern rule. To arrive at the latter we shall first exemplify its use in a typical argument having an existential generalization as a premiss. For the purposes of this argument we shall assume that we know that addition of real numbers is associative; that, for any real number a, $a + 0 = a$; and that, for any real number a, there exists a real number

x such that $a + x = 0$. On the basis of this information we wish to conclude that, for any real numbers a, b, and c, if $b + a = c + a$ then $b = c$. Part of the argument might be formulated as follows:

> Suppose that a, b, and c are any real numbers such that $b + a = c + a$. Let d be a number such that $a + d = 0$. Since $b + a = c + a$ it follows that $(b + a) + d = (c + a) + d$ and, so, that $b + (a + d) = c + (a + d)$. Since $a + d = 0$ it follows that $b + 0 = c + 0$ and, so, that $b = c$. Hence, if $b + a = c + a$ then $b = c$.

This much of the argument shows that the conclusion 'if $b + a = c + a$ then $b = c$' is a consequence of the assumption '$a + d = 0$', the associative principle for addition, and the principle for adding 0. It is especially to be noted that 'd' occurs only in the first of these three premisses and that, while it occurs elsewhere in the argument, it does not occur in the conclusion. We may then think of the displayed argument as a pattern, with 'd' as pattern variable, which shows that the conclusion is a consequence of any instance whatever of '$\exists_x\ a + x = 0$', the associative principle for addition, and the principle for adding 0. Since the existential generalization asserts that there is some number which has the property which we have assumed d to have, and, since the steps in the argument are valid, and the conclusion remains the same, no matter what number d may be, it is reasonable to grant that the argument shows that the conclusion is a consequence of the existential generalization '$\exists_x\ a + x = 0$' [itself], the associative principle for addition, and the principle for adding 0.

Generalizing on the preceding example, we may say that a pattern which shows that a given conclusion follows from any instance whatever of a certain existential generalization [together with other premisses] shows, also, that the conclusion follows from the existential generalization itself [and the other premisses]. As pointed out, it is essential that the conclusion and the "other premisses" be the same no matter what instance of the existential generalization may be in question. This means, in brief, that the "pattern variable" must not occur in the conclusion or in the "other premisses".

On the basis of the two rules for existential generalizations we can now justify the third step in the first half of the proof of associativity of function composition. What needs justification is the shifting of '$\exists_v$' from one side to the other of the left-most parenthesis. In the following, suppose 'Fc' to be replaced by a sentence containing the variable 'c', '$\exists_v$ Fv' replaced by the existential generalization of this sentence with respect to 'c', and 'p' by a sentence in which 'c' does not occur. With this in mind, the first tree-diagram below shows that '$\exists_v$ (Fv & p)' is a consequence of '$\exists_v$ Fv & p', while the second shows that the first of these two sentences implies the second. Together they show that '($\exists_v$ Fv & p) $\Longleftrightarrow$ $\exists_v$ (Fv & p)' is logically valid.

$$\dfrac{\dfrac{\exists_v Fv \,\&\, p}{\exists_v Fv} \qquad \dfrac{\dfrac{\overset{*}{Fc} \qquad \dfrac{\exists_v Fv \,\&\, p}{p}}{Fc \,\&\, p}}{\exists_v (Fv \,\&\, p)}}{\exists_v (Fv \,\&\, p)}\ *$$

[The double bar signalizes an application of the test-pattern rule for existential generalizations, and the '*'s indicate that through use of this rule the premiss 'Fc' has been "discharged" (in favor of '$\exists_v$ Fv').]

$$\dfrac{\exists_v (Fv \,\&\, p) \qquad \dfrac{\dfrac{\dfrac{Fc \overset{*}{\&} p}{Fc}}{\exists_v Fv} \qquad \dfrac{Fc \overset{*}{\&} p}{p}}{\exists_v Fv \,\&\, p}}{\exists_v Fv \,\&\, p}\ *$$

There is a good deal more which needs to be said about the test-pattern rule for existential generalizations, but the foregoing is enough for the present.

The third is just a matter of grammar.] Similarly,

$$\begin{aligned}(a,b) \in [f \circ g] \circ h &\Longleftrightarrow \exists_v((a,v) \in h \text{ and } (v,b) \in f \circ g)\\ &\Longleftrightarrow \exists_v((a,v) \in h \text{ and } \exists_u((v,u) \in g \text{ and } (u,b) \in f))\\ &\Longleftrightarrow \exists_v \exists_u((a,v) \in h \text{ and } (v,u) \in g \text{ and } (u,b) \in f))\end{aligned}$$

Comparing our two results [and noting that to say 'for some u, for some v,' amounts to the same thing as saying 'for some v, for some u,'] we see that, for any a and b,

$$(a,b) \in f \circ [g \circ h] \Longleftrightarrow (a,b) \in [f \circ g] \circ h.$$

Consequently, $f \circ [g \circ h] = [f \circ g] \circ h$.

‖ Function composition is associative.

Exercises

Part A

Suppose that f is a function which has an inverse—that is, suppose that the converse of f is a function. Recall the notation 'f^{-1}' for the inverse of f; it follows that, for any a and b,

$$(a,b) \in f \Longleftrightarrow (b,a) \in f^{-1}.$$

1. In terms of 'Df' and 'Rf', what is $D[f^{-1}]$? What is $R[f^{-1}]$? $D[f^{-1} \circ f]$? $R[f^{-1} \circ f]$?
2. Show what the mapping $f^{-1} \circ f$ does. [*Hint*: If $a \in D[f^{-1} \circ f]$ then, by Exercise 1, $a \in Df$ and, so, $(a,f(a)) \in f$. It follows that $(?,?) \in f^{-1}$ and, so, that $f^{-1}(f(?)) = ?$ Hence, for any $a \in D[f^{-1} \circ f]$, $[f^{-1} \circ f](a) = ?$]

*

It follows from Exercises 1 and 2 that, for any function f which has an inverse,

$$f^{-1} \circ f = \{(x,y): x \in Df \text{ and } y = x\}.$$

[Another way to establish this is to use (**) on page 23. According to this, $(a,b) \in f^{-1} \circ f$ if and only if $\exists_z((a,z) \in f$ and $(z,b) \in f^{-1})$. But, the latter is the case if and only if $\exists_z[(z,a) \in f^{-1}$ and $(z,b) \in f^{-1}]$. By the definitions of 'range' and 'function', this last is the case if and only if $(a \in R[f^{-1}]$ and $b = a)$—that is, if and only if $(a \in Df$ and $b = a)$.]

We have already come across $\{(x,y): x \in Df \text{ and } y = x\}$ in the case in which $Df = R$. In this case it is the translation which maps each point

Answers for Part A

1. $\mathfrak{D}[f^{-1}] = \mathfrak{R}f$, $\mathfrak{R}[f^{-1}] = \mathfrak{D}f$, $\mathfrak{D}[f^{-1} \circ f] = \mathfrak{D}f = \mathfrak{R}[f^{-1} \circ f]$
2. [The hint becomes a satisfactory answer if the five '?'s are replaced by 'f(a)', 'a', 'a', 'a', and 'a', respectively.]

of the number line on itself. For any set S, the mapping of S into itself under which each member of S is its own image is called *the identity mapping of S onto itself* and is often denoted by 'i_S'. Using this notation, we see that

$$i_S = \{(x,y): x \in S \text{ and } y = x\}.$$

So, the result displayed above may be written:

$$f^{-1} \circ f = i_{Df}$$

It follows [explain why] that

> For any function f which has an inverse,
> $f^{-1} \circ f = i_{Df}$ and $f \circ f^{-1} = i_{Rf}$.

Part B

1. Show that, for any function f and any set S,
   (a) if $Df \subseteq S$ then $f \circ i_S = f$,
   (b) if $Rf \subseteq S$ then $i_S \circ f = f$, and
   (c) if $Rf \subseteq Df$ then $f \circ i_{Df} = f = i_{Df} \circ f$.
2. It follows from a result you established in Part A that
   if f has an inverse then there exists a function g
   such that $g \circ f = i_{Df}$.

   [One such function g is f^{-1}.] Suppose, now, that f and g are any functions such that $g \circ f = i_{Df}$. Does it follow that f has an inverse? [*Hint*: Suppose that $g \circ f = i_{Df}$. It follows that $D[g \circ f] = Df$. [Why?] So, for any $a \in Df$, $g(f(a)) = a$. Recall that in order to show that f has an inverse it is sufficient to show that, for any $a_1 \in Df$ and any $a_2 \in Df$, if $f(a_1) = f(a_2)$ then $a_1 = a_2$. Try to show that this is the case.]

⁂

It follows from Parts A and B that

> For any function f,
> f has an inverse if and only if there is a function g such that
> $g \circ f = i_{Df}$.

You established the "only if" part of this theorem in Part A and the "if" part in Part B. We can learn more about inverses by giving a different proof of the "if" part. To do so, suppose that $g \circ f = i_{Df}$. It follows [as in the hint for Exercise 2] that, for any $a \in Df$, $g(f(a)) = a$. In other words, for any a and b,

$$\text{if } (a,b) \in f \text{ then } (b,a) \in g.$$

Answers for Part B

1. (a) Suppose that $\mathcal{D}f \subseteq S$. It follows that, for $a \in \mathcal{D}f$, $a \in S$ and, by the definition of 'i_S', $a \in \mathcal{D}[i_S]$ and $i_S(a) = a$. So, by the definition of 'function composition', $\mathcal{D}f \subseteq \mathcal{D}[f \circ i_S]$ and, for $a \in \mathcal{D}f$, $[f \circ i_S](a) = f(a)$. On the other hand, if $a \notin \mathcal{D}f$ then either $a \notin \mathcal{D}[i_S]$ or, if $a \in \mathcal{D}[i_S]$, $i_S(a) \notin \mathcal{D}f$. In either case, $a \notin \mathcal{D}[f \circ i_S]$. Consequently, $\mathcal{D}f = \mathcal{D}[f \circ i_S]$ and $f \circ i_S = f$.

   (b) Suppose that $\mathcal{R}f \subseteq S$. It follows that $\mathcal{R}f \subseteq \mathcal{D}[i_S]$ and, so, that $\mathcal{D}[i_S \circ f] = \mathcal{D}f$ and, for $a \in \mathcal{D}f$, $[i_S \circ f](a) = i_S(f(a)) = f(a)$. Hence, $i_S \circ f = f$.

   (c) Suppose that $\mathcal{R}f \subseteq \mathcal{D}f$ and let $S = \mathcal{D}f$. Since $\mathcal{D}f \subseteq S$ it follows by (a) that $f \circ i_S = f$. Since $\mathcal{R}f \subseteq S$ it follows by (b) that $i_S \circ f = f$. So since $S = \mathcal{D}f$, $f \circ i_{\mathcal{D}f} = f = i_{\mathcal{D}f} \circ f$.

2. Suppose that $g \circ f = i_{\mathcal{D}f}$. Since $g \circ f$ and $i_{\mathcal{D}f}$ are the same function, they have the same domain. So, $\mathcal{D}[g \circ f] = \mathcal{D}f$. So, for any $a \in \mathcal{D}f$, $g(f(a)) = i_{\mathcal{D}f}(a) = a$. Suppose that $a_1 \in \mathcal{D}f$ and $a_2 \in \mathcal{D}f$ and that $f(a_1) = f(a_2)$. It follows that $g(f(a_1)) = a_1$, $g(f(a_2)) = a_2$, and [since $f(a_1) = f(a_2)$ and g is a function] that $g(f(a_1)) = g(f(a_2))$. So, $a_1 = a_2$. Hence, for any arguments a_1 and a_2 of f, if $f(a_1) = f(a_2)$ then $a_1 = a_2$. Consequently, f has an inverse.

Now, by the definition of 'converse', $(a,b) \in f$ if and only if $(b,a) \in$ [the converse of f]. So,

$$\text{if } (b,a) \in [\text{the converse of } f] \text{ then } (b,a) \in g.$$

From this it follows that the converse of f is a subset of g. So, since g is a function, and since [by Part E on page 15] any subset of a function is a function, it follows that the converse of f is a function—that is, that f has an inverse.

The preceding argument shows, as it was intended to, that if there exists a function g such that $g \circ f = i_{Df}$, then f has an inverse. In addition to this, the argument shows that, for any functions f and g,

$$\text{if } g \circ f = i_{Df} \text{ then the inverse of } f \text{ is a subset of } g.$$

Part C

1. Suppose that you know, for some function f and some function g, that $g \circ f = i_{Df}$. Would it be safe to conclude from this information alone that g is the inverse of f? If your answer is 'Yes.', justify it. If your answer is 'No.', what additional information would enable you to arrive at the conclusion in question?
2. You know that if f has an inverse, then $f^{-1} \circ f = i_{Df}$ and, also, $f \circ f^{-1} = i_{Df^{-1}}$. Suppose that you know, for some function f and some function g, that $g \circ f = i_{Df}$ and $f \circ g = i_{Dg}$. Would you be justified in concluding that g is the inverse of f? Explain.

*

The next two theorems about inverses of functions summarize the results in the exercises in the preceding exercises.

> For any functions f and g,
> g is the inverse of f if and only if (i) $g \circ f = i_{Df}$ and (ii) $Dg = Rf$.

> For any functions f and g,
> g is the inverse of f if and only if (i) $g \circ f = i_{Df}$ and (ii) $f \circ g = i_{Dg}$.

1.05 Some Geometry

In this course you will be studying the geometry of space.

If you don't understand the preceding sentence, you're normal. To understand it you would need to know the meaning of the phrase 'the geometry of space'. Part of what this phrase refers to you will learn in this course. Nobody knows all of it; you already know a little of it.

Answers for Part C

1. No. Although we may conclude that f has an inverse and that this inverse is a subset of g, g may contain ordered pairs which do not belong to the inverse of f. In fact, for any function h such that $\mathfrak{D}h \cap \mathfrak{R}f = \emptyset$ the union of h and f^{-1} is a function g such that $g \circ f = f^{-1} \circ f = i_{\mathfrak{D}f}$.

   In addition to knowing that $g \circ f = i_{\mathfrak{D}f}$ [which assures us that $f^{-1} \subseteq g$] it is sufficient to know that $\mathfrak{D}[f^{-1}] = \mathfrak{D}g$ in order to be sure that $g = f^{-1}$. [Since $\mathfrak{D}[f^{-1}] = \mathfrak{R}f$, it is sufficient to know that $\mathfrak{D}g = \mathfrak{R}f$.]
2. Yes. For, suppose that $g \circ f = i_{\mathfrak{D}f}$ and $f \circ g = i_{\mathfrak{D}g}$. From the first assumption if follows that f has an inverse and that $f^{-1} \subseteq g$. So, in particular, $\mathfrak{R}f = \mathfrak{D}[f^{-1}] \subseteq \mathfrak{D}g$. From the second assumption it follows that $\mathfrak{R}[f \circ g] = \mathfrak{R}[i_{\mathfrak{D}g}] = \mathfrak{D}g$ and, so, since $\mathfrak{R}[f \circ g] \subseteq \mathfrak{R}f$, that $\mathfrak{D}g \subseteq \mathfrak{R}f$. Hence, from the two assumptions it follows that $f^{-1} \subseteq g$ and that $\mathfrak{D}[f^{-1}] = \mathfrak{R}f = \mathfrak{D}g$. Consequently, if $g \circ f = i_{\mathfrak{D}f}$ and $f \circ g = i_{\mathfrak{D}g}$ then $g = f^{-1}$. [Alternatively, it follows from the two assumptions that $f^{-1} \subseteq g$ and $g^{-1} \subseteq f$. From the definition of 'converse' and 'inverse' it is clear that if $g^{-1} \subseteq f$ then $g \subseteq f^{-1}$. So, $g = f^{-1}$.]

Using the second of the boxed theorems it is easy to establish a result mentioned earlier in this commentary:

> For any functions f and g which have inverses,
> $g \circ f$ has an inverse and
> $[g \circ f]^{-1} = f^{-1} \circ g^{-1}$.

What needs to be shown is that $[f^{-1} \circ g^{-1}] \circ [g \circ f] = i_S$ where $S = \mathfrak{D}[g \circ f]$, and that $[g \circ f] \circ [f^{-1} \circ g^{-1}] = i_T$ where $T = \mathfrak{D}[f^{-1} \circ g^{-1}]$. Once the first is established [for any f and g which have inverses] the second can be obtained from it by substituting 'g^{-1}' for 'f' and 'f^{-1}' for 'g'. We proceed to establish the first. Since

$[f^{-1} \circ g^{-1}] \circ [g \circ f] = f^{-1} \circ [g^{-1} \circ [g \circ f]] = f^{-1} \circ [[g^{-1} \circ g] \circ f] = f^{-1} \circ [i_{\mathfrak{D}g} \circ f]$, what we need to show is that $f^{-1} \circ [i_{\mathfrak{D}g} \circ f] = i_S$ when $S = \mathfrak{D}[g \circ f]$. Now, clearly, $f^{-1} \circ [i_{\mathfrak{D}g} \circ f]$ is the identity mapping of its domain onto itself. So, what needs to be shown is merely that this domain is $\mathfrak{D}[g \circ f]$. Since $\mathfrak{R}[i_{\mathfrak{D}g} \circ f] = \mathfrak{D}g \cap \mathfrak{R}f \subseteq \mathfrak{R}f = \mathfrak{D}[f^{-1}]$ it follows that the domain of $f^{-1} \circ [i_{\mathfrak{D}g} \circ f]$ is the domain of $i_{\mathfrak{D}g} \circ f$. Since $i_{\mathfrak{D}g}$ and g have the same domain, the domain of $i_{\mathfrak{D}g} \circ f$ is $\mathfrak{D}[g \circ f]$. Q.E.D.

Sample Quiz

Consider the functions f, g, and h, where, for each x, $f(x) = x^2 + 1$ and $g(x) = 2x$, and $h = \{(x, y): y = x + 2\}$. Answer these questions.

1. What is $[f \circ g](3)$? What is $[g \circ h](-2)$? What is $[h \circ g](-2)$?
2. For what values of 'a' is it the case that $f(a) = 2$?
3. Which of the functions f, g, and h do not have inverses? Explain.
4. For what values of 'a' is it the case that $f(a) = g(a)$?
5. Is $f \circ g = g \circ f$? Explain.
6. What is the range of f?

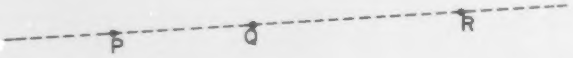
Answers for Sample Quiz

1. 37; 0; −2
2. 1 and −1
3. f does not have an inverse, because f is not one-to-one [e.g. $f(-1) = 2 = f(1)$].
4. 1
5. No. $[f \circ g](a) = 4a^2 + 1$ and $[g \circ f](a) = 2a^2 + 2$. Since $4a^2 + 1 \neq 2a^2 + 2$, $f \circ g \neq g \circ f$.
6. $\{x: x \geq 1\}$

We begin with some general comments on this section and on the course as a whole. More detailed comments on the content of this section begin on page TC 26, 27 (3)

The purpose of section 1.05 is to lead students to think about lines and planes and about some of their simpler subsets — rays, half-lines, intervals, and segments; half-planes and closed half-planes. Students need a clear intuitive understanding of these notions as a preparation for the formal development which begins in Chapter 2. The emphasis now is on intuition and experiment. In the formal development we shall give very different definitions of 'line', etc. than those which are implicit in the present section. But, students need to have developed a feeling for lines, etc. to enable them to appreciate these later definitions.

The approach which we adopt here to lines and planes through the notion of a ray is unconventional but brings out best some ideas which will be helpful in the sequel. The notion of a ray is introduced in terms of the notion of a first point, P, being beyond a second point, Q, from a third point, R. [Intuitively, this is equivalent to R being beyond Q from P and to Q being between P and R (or between R and P).] So, a formalization of geometry completely different from the one developed in this text might be based on postulates concerning this ternary relation of "beyondness". Because such a different formalization is of interest in itself, we go into somewhat more detail, in parts of this commentary, into how such a formalization might be developed. These parts are preceded by '* * *' and followed by '*'. They are largely irrelevant to your students' proper interests at this time, and dwelling on them in class would be very likely to lead to some confusion. Your study of them may, however, give you a better basis for answering questions which may arise. For a much more adequate treatment of such an alternative formulation of geometry, see Basic Concepts of Geometry, Prenowitz and Jordan, Blaisdell Publishing Company, New York, 1965.



The exercises at the end of the section are meant to serve two purposes. In part they furnish much needed practice in drawing — particularly, in using a parallel ruler. They have, however, been chosen to prepare students for the work with translations in the following section. It is to be hoped that the discoveries students make in that section will appear familiar to them because of their work on these exercises. In the commentary on the individual exercises, we shall point out what we hope students will learn from them.

Sections 1.05 and 1.06 are examples of a kind of section which will occur at several points in the text. As remarked earlier, students need to develop concepts before formulating their intuitive beliefs in a formal manner. Section 1.05 is a concept-development section dealing with space and some basic geometric figures, and section 1.06 is a concept-development section dealing with a kind of mapping of space onto itself which is of extremely basic importance to us in this course. Other such concept-development sections will occur where they are

A good way to begin is to think of space as the set of all locations around [and in] you, in whatever direction, and however far away, they may be. As you learn more you may adopt other ways of thinking of space; but, at any rate, space is a set – we shall call it '$\mathscr{S}$' – and its members [which we have suggested you might think of as locations] are customarily called *points*. To carry these ideas just one step further, when you think of points as locations you should think of them as absolutely precise locations. You may have been thinking that some locations "overlap"; two points never do.

So much for 'space' – space is the set $\mathscr{S}$ of all points. How about 'geometry'? Etymologically, this word means earth-measurement and you might suspect that, originally, geometry had to do with ways of computing distances between points of the surface of the earth and, perhaps, of computing area-measures of regions. Today, 'geometry' has a somewhat similar but much broader meaning. To tell a little of what this meaning is it is convenient to introduce another word – 'figure'. A [geometric] *figure* is a set of points – usually a fairly simple set, such as a triangle or a straight line. In other words, a figure is a subset of $\mathscr{S}$. Briefly [and inadequately] geometry is the study of figures and of relations among figures. This includes – although it is not immediately obvious that it does – the study of mappings of one figure onto another. You may see why if you ask yourself what it means to say that two figures have the same shape.

Before we can begin a serious study of geometry we need to take note of some important kinds of figures. To begin with, we shall attempt to clarify our intuitive notions of lines and planes. Think of some point P – some precise location in space – and of all the points which are, from your location, directly behind *P*. [For short, we shall say 'beyond *P*'.] The set which consists of *P* and all these points is a *ray* with *vertex* *P*. Clearly, given any point *P*, there are lots of rays which have *P* as vertex. In fact, given any point $Q \neq P$ there is a unique ray with vertex *P* which contains *Q*. To describe such a ray it is sufficient to find a viewpoint *E* such that *Q* is beyond *P* from *E*.

Fig. 1–8

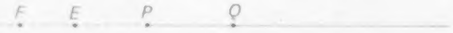


Fig. 1–9

needed. Sections of this kind are, like the Introduction, of an entirely different nature than those others which contain our formal organization of geometry. In a sense, the text contains two courses — a course in intuitive geometry and a much more extensive one in formal geometry. The formal course might, by itself, be appropriate for students who had had the intuitive course in earlier grades. For such [largely nonexistent] students the concept-development sections might be omitted — with considerable gain in continuity. For real students, however, these sections are a necessity. But, it is essential to realize that the concept-development sections set the stage for the formal treatment and one must be able to distinguish between the stage-hands and the actors. To help students make the distinction, concept-development sections sometimes bristle with the word 'intuition' and its derivatives. To help you, we refer to them in the commentary as concept-development sections.

We believe that students can learn through reading [in addition to doing exercises and participating in classroom discussions] and should be taught to do so. So, the textual material in this book is meant for student consumption and rumination [as well as for classroom discussion]. The extent of the reading matter in section 1.05 is, however, atypical. We strongly suggest the following treatment for the purpose of beginning to train students to read accurately and to learn by reading. Ask a student to begin reading this section aloud in class, the others listening and reading silently. By interrupting and asking questions when you think necessary, establish the meaning of words [ordinary, as well as technical ones] which you think may be unfamiliar or misunderstood. Interrupt the reading when you think class discussion may be profitable. Perhaps, without interrupting, you may think it helpful to make sketches on the chalkboard and, by pointing, "act out" what is being read. It is probably better to stop too frequently for discussion than not to stop frequently enough. You will, naturally, take care not to tire out any one reader. Give each student a chance.

Your aim is, of course, to give each student the idea that when he reads by himself he should stop frequently and think about what he has read, and that understanding is aided and tested by making his own sketches and examples as he reads. Reading mathematics is likely to be relatively unproductive unless one has a paper beside him, a pencil in one hand, and makes frequent use of them.

This procedure we suggest is time consuming, but worthwhile. If you need to spend two class periods in reading section 1.05 and discussing it, this can be time well spent. Be sure that each point which seems to need it gets some discussion when it comes up, but it may be well to postpone the completion of some discussions until the second day. Tell students, truthfully, that a short discussion which clarifies the nature of a problem is often more profitably continued later after the problem has "sunk in". Hopefully, you will then be able to reach the experiment on the existence of parallel lines which is suggested on page 32 at the end of the first of two sessions, and leave the experiment as homework to be discussed during the second session. Completing the reading, tieing up loose ends, and a preliminary discussion of the exercises — perhaps with some done as seat-work – will take up the second session.

We come now to the more detailed discussion of the content of section 1.05.

Our starting point is that the study of geometry arises out of an analysis of the results of experiments in physical space. [In this section, most of the experiments in question involve merely sighting — sighting beyond a ''point'', or sighting across a ''line''. Toward the end there is an experiment which involves drawing.] Such an analysis leads one to envision certain ''abstractions'' such as our ''absolutely precise locations'' and to endow them with certain properties and relations. This is a dubious process, but is justified by its undoubted pragmatic value and by the fact that, by another kind of ''abstraction'' one arrives at mathematical — as opposed to physical — geometry. In mathematical geometry one deals with sentences which contain undefined — or, better, uninterpreted — terms, and one studies the logical relations among these sentences. Subject to the dubiety of the first-mentioned abstraction process, the abstractions which it is supposed to yield furnish interpretations for the undefined terms of the mathematical theory and, so, constitute a model of it.

Since an appreciation of mathematical geometry comes late, we adopt the somewhat mendacious practice of speaking of such abstractions as ''absolutely precise locations''. Doing so enables us to avoid the — for us — irrelevant difficulties which arise out of the fact that real locations do overlap, and line up with one another only approximately. Fortunately, the sentences we use to express our fantasies about our abstractions are the same ones we would consider in studying mathematical geometry, and the logical relations which we wish students to explore are independent of whether the terms which, in mathematical geometry we would treat as undefined, are so treated or are interpreted in any way whatsoever.

So much in explanation and apology. Presumably, students will swallow the fiction of absolutely precise locations [and it would be cruel, at this point, to attempt to disabuse them]. No harm will result if they continue thinking in these terms throughout the course. It is not our purpose to explicitly call to their attention the beauty of mathematical geometry. It is enough to give them the opportunity of experiencing it unknowingly while they are thinking of something else.

The richness of geometry is only partly due to the existence of Euclidean and noneuclidean geometries. These geometries are mentioned briefly at the end of the section and, as is pointed out there, the differences among them result from different assumptions as to what space is like. We shall, of course, study Euclidean geometry and, so, assume that space is euclidean. But, this latter assumption by itself would still leave us with a wide choice of ''geometries''. In choosing Euclidean geometry from among these, we limit ourselves to considering properties of figures which are possessed by all similar figures. [Congruence is, of course, a special case of similarity.] As a sample of the alternatives, we might choose to study the topology of Euclidean space, in which case we would concern ourselves with many fewer properties — those properties of figures which ''are preserved'' under all continuous mappings which have continuous inverses. For example, the property of being a spherical surface is a Euclidean property — anything similar to a sphere is a sphere — it is not a topological property. A spherical surface does, however, have various

topological properties. For one, any spherical surface does separate the rest of space into two disjoint regions, one consisting of the points inside the sphere and the other of the points outside the sphere. This separation property is topological. [Roughly and inadequately put, however badly you dent or stretch a spherical surface, the surface which results will still separate space.]

The question, in the text, as to what it means to say that two figures have the same shape, is one which is well worthy of discussion sometime during the period you spend on Chapter 1. But, an extended discussion is not appropriate at this time. Merely establish consensus as to the meaning of 'same shape' by drawing some similar and dissimilar triangles and some circles. [Concentric circles might give a useful hint toward the answer.] You may wish to suggest that answering a similar question as to same size and shape may be simpler. The answer to the question in the text is that two figures have the same shape if and only if one of them can be mapped on the other in such a way that the distance between any two points of the first is the same nonzero multiple of the distance between their images. [The nonzero multiplier is of course, the ratio of similitude (or, depending on one's point of view, its reciprocal).] Two figures have the same size and shape if and only if one can be mapped onto the other in such a way that the distance between any two points is the same as that between their images. [If you wish, introduce the words 'similar' and 'congruent'.] If, ultimately, you obtain satisfactory answers from your students [don't be concerned if you don't — just drop the subject] bring out the fact that, for example, any two lines are congruent and that any two segments are similar. Also, bring out the fact that the conditions imposed on the mappings referred to in the two definitions ensure that these mappings are one-to-one. If you have an interested class and, on some future day, have plenty of time, bring out the fact that congruence is a symmetric and transitive relation, and so is similarity.

Clearly, any other viewpoint F which is beyond P from Q will do [explain], and, from either of two such viewpoints, the points beyond P are the same.

Two rays with the same vertex "point different ways" or, as we shall say, have different *senses*. Rays which "point opposite ways" are said to have *opposite senses*. If they have the same vertex then each is called the *opposite* of the other. For example, the ray $\overrightarrow{PQ}$ [from P through Q] which is pictured above and the ray $\overrightarrow{QP}$ have opposite senses but, since they have different vertices, they are not opposites. In fact, the opposite of $\overrightarrow{PQ}$ is a proper subset of $\overrightarrow{QP}$. What is it? What is the opposite of $\overrightarrow{PE}$? What is the opposite of $\overrightarrow{EP}$? Clearly, the opposite of a ray consists of its vertex together with all points which are beyond the vertex from any other point of the ray.

The union of two opposite rays is a *line*. For example, the line l pictured here is the union of two opposite rays with vertex P. If Q is any point of l other than P then, since the opposite of $\overrightarrow{PQ}$ is a subset of $\overrightarrow{QP}$ and $\overrightarrow{QP} \subseteq l$, $l = \overrightarrow{PQ} \cup \overrightarrow{QP}$. Finally, the line l which, by definition, is the union of $\overrightarrow{PQ}$ and its opposite is, also, the union of $\overrightarrow{QP}$ and its opposite. So, given *any* point of a line, the line is the union of two opposite rays which have this point as vertex. It follows that there is only one line containing two given points.

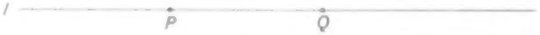


Fig. 1–10

Given any line l and any point $P \in l$, there are just two rays which have P as vertex and are subsets of l. Each point of l belongs to one or the other of these rays, and P is the only point which belongs to both rays. It is natural to speak of a point of l other than P as being on one side or the other of P according as it belongs to one or the other of these two rays. This suggests saying that the points other than P which belong to one of these rays constitute a *side* of P, with respect to l. The line l is, then, the union of three disjoint sets – the two sides of P with respect to l and the set $\{P\}$. The set which consists of the points of a ray $\overrightarrow{PQ}$ other than P is often called a *half-line* – the half-line $\overrightarrow{PQ}$. [It has P as its *vertex*, even though P does not belong to it.] So, each side of a point, with respect to a line, is a half-line.

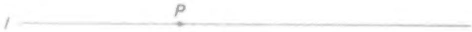


Fig. 1–11

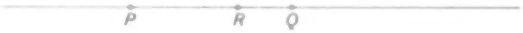
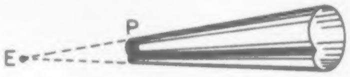


Fig. 1–12

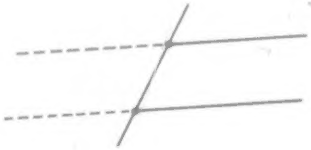
As remarked in the general comments, the various statements made in this section are meant to be accepted — after appropriate amounts of thought, discussion, and just plain quibbling — as being intuitively obvious. For example, if P, in our initial description of a ray, is not a precise location then the points directly behind P will constitute a frustum of a narrow cone. This will also be the case if the pupil of the pictured eye is not a point. But, P is a point and we agreed to think of points as precise locations; and we shall impose the same limitation on pupils. So, if $Q \neq P$ there is a unique ray which contains Q and has P as vertex and, moreover, if R is a point not on this ray, the ray with vertex P which contains R has only P in common with the former ray. [There are exactly as many rays with vertex P as there are points on a spherical surface with P as center; and no two of these rays have any point other than P in common.] The explanation asked for amounts to pointing out that if E is beyond P from Q then Q is beyond P from E — being "beyond P from" is symmetric.



The notation '$\bullet\!\!\rightarrow$' which we introduce, in passing, to refer to rays is sufficiently useful and self-explanatory to out-weigh the fact that it conflicts with a notation frequently used in referring to vectors. This conflict will certainly not trouble students at this time. Later, after they have come to possess clear concepts of rays and vectors, they will be able to cope with the various notations used in various texts.

The notion of sense is one which will come up frequently later. For the present, the sense of a ray is "the way it points" and whatever this means to a student is probably enough meaning for 'sense' to have now. Actually, we deal here only with collinear rays, and it is clear that two such rays either point the same way or point opposite ways.

It may occur to you that 'direction' would be a more natural word to use than 'sense'. Perhaps it would; but, unfortunately, we need a word to fill the blank in 'Parallel lines have the same _______', and 'direction' seems to be it. So, in our use of 'sense' and 'direction', a line will have a direction and each ray which is a subset of that line will have one of two opposite senses. Two rays will have the same sense if and only if they are collinear and one is a subset of the other or they are subsets of two parallel lines and are contained in a closed half-plane whose edge is the line which contains their vertices. Obviously, none of the preceding is appropriate for class-discussion



at present. Whatever intuitive notion of sense students get from thinking of the sense of a ray as "the way the ray points" is sufficient. When, in a later chapter, 'ray' is defined formally in terms of 'translation', its sense will be defined in similar terms.

A third word which might be used in place of 'sense' or 'direction' is 'orientation'. This word is also needed for another purpose. We shall, for example, orient a line by associating with it a chosen one of the two senses which are associated with its direction. Thus, a line may be given one of two orientations. Of itself, a line has a direction; and, of itself, a ray has a sense.

From the intuitive view-point of this section, the phrase 'the union of two opposite rays' should be considered as an adequate description of what a line is. In some formalizations of geometry it might serve as the formal defining phrase for 'line'. Since the formalization for which we are setting the stage is not of this kind, it would be confusing to students to refer to 'A line is the union of two opposite rays.' as a definition. Use 'description' instead. In the formal part of the course 'line' will be defined in terms of 'translation'.

According to that intuitive notion of a line for which we are propagandizing here, to recognize that a set is a line one must pick out one of its points and convince himself that the set is the union of two opposite rays which have this point as their common vertex. One might, for example, pick two points P and Q, which belong to the set and convince himself that the set is $\overrightarrow{PQ} \cup -\overrightarrow{PQ}$. [We have not introduced the customary oppositing sign, '−', in the text. Do so in class if you wish. Allowing students to suggest notation gives them a valuable sense of power over notation. This is an occasion where a minimal amount of prodding on your part will give them a chance to feel that they are contributing. (Some of your students may have previously used this or a similar operation in referring to complements of sets. If so, you may need to point out that oppositing of rays is certainly not complementing, with respect to any universe, but that there is no harm in adopting a new meaning for a symbol as long as you don't use it with both meanings in the same context.)] That one can also recognize a line by noting that, for two of its points, P and Q, it is the union $\overrightarrow{PQ} \cup \overrightarrow{QP}$ is a slightly different intuitive notion of a line — even though it is as "obvious" as the former one. Finally, noting that if a set is a line then, given any of its points, it is the union of two opposite rays with this point as vertex is still another intuition. If Q belongs to the union of two opposite rays with vertex P, it is at least sensible to ask whether this union is also the union of two opposite rays with vertex Q. On the intuitive level, any doubt one may have as to the correct answer should be momentary — but the question deserves recognition.

Although the three notions concerning lines which have just been discussed are all equally obvious, the arguments sketched in the text for acceptance of the last two of them are of value because they offer an opportunity to review some concepts from the algebra of sets. As suggested in the text, one can prove that $\overrightarrow{PQ} \cup -\overrightarrow{PQ} = \overrightarrow{PQ} \cup \overrightarrow{QP}$ if one accepts that $-\overrightarrow{PQ} \subseteq \overrightarrow{QP} \subseteq \overrightarrow{PQ} \cup -\overrightarrow{PQ}$. [From the first inclusion it follows that $\overrightarrow{PQ} \cup -\overrightarrow{PQ} \subseteq \overrightarrow{PQ} \cup \overrightarrow{QP}$; from the second, that $\overrightarrow{PQ} \cup \overrightarrow{QP} \subseteq \overrightarrow{PQ} \cup -\overrightarrow{PQ}$.] From this result it follows that, given any point of a line, the line is the union of two opposite rays which have this point as vertex. For, consider a line ℓ which is the union of two opposite rays with vertex P, and let Q be any other point of ℓ. As just shown, $\ell = \overrightarrow{PQ} \cup \overrightarrow{QP}$. But, also, $-\overrightarrow{QP} \subseteq \overrightarrow{PQ} \subseteq \overrightarrow{QP} \cup -\overrightarrow{QP}$ and, by the same argument, $\overrightarrow{QP} \cup -\overrightarrow{QP} = \overrightarrow{QP} \cup \overrightarrow{PQ}$. Since $\overrightarrow{QP} \cup \overrightarrow{PQ} = \overrightarrow{PQ} \cup \overrightarrow{QP} = \ell$ it follows that ℓ is the union of the two opposite rays, $\overrightarrow{QP}$ and $-\overrightarrow{QP}$, with vertex Q.

That there is at most one line containing two given points Q and R follows from the result just proved. For, if ℓ is such a line then, since $Q \in \ell$, ℓ is the union of two opposite rays with vertex Q. Since $R \in \ell$, R belongs to one of these rays and, since $R \neq Q$, the only ray with vertex Q which contains R is $\overrightarrow{QR}$. Hence, the opposite rays with vertex Q whose union is ℓ must be $\overrightarrow{QR}$ and its opposite. Since these are uniquely determined by Q and R, so is ℓ.

We have seen that, given a line ℓ and a point $P \in \ell$, ℓ is the union of two rays with vertex P. Each of these is, of course, a subset of ℓ. Should students suggest that there might, somehow, be another ray with vertex P which is a subset of ℓ, recall that there is a unique ray with vertex P and containing a given point. Any ray with vertex P which contains another point of ℓ must, then, be that one of the two rays first mentioned which contains this other point.

The notion of 'side' is an important one. A point has two sides with respect to any line to which it belongs, a line has two sides with respect to any plane which contains it, and a plane has two sides with respect to space. It is convenient to identify these sides as sets of points. The essence of this notion is that, given a line and a point in it, you can't get from one side of the point to the other without passing through the point [or going off the line]; given a plane and a line in it, you can't get from one side of the line to the other without crossing the line [or leaving the plane]; given a plane in space, you can't get from one side of the plane to the other without passing through the plane [or going out of space]. The separation of planes by lines and of space by planes is taken up later in this section.

The notation '$\longrightarrow$' we choose to use in referring to half-lines is, of course, derived from that for rays. In the case of rays the dot at the left end of the arrow indicates that the vertex belongs to the ray; in the case of half-lines the absence of the dot indicates that the vertex does not belong to the half-line. If students have difficulty with the distinction between 'has' and 'belongs to', recall that we are using the latter with the meaning of 'is a member of', and point out that it is customary to say that a circle has a center, despite the fact that the center is not on the circle.

Be sure that students understand the meaning of 'disjoint'. Two or more sets are disjoint if and only if no two of them have a common member — that is, if and only if each two of them have the empty set, ∅, as their intersection. It may also be worthwhile to point out again that, in this book 'two' means, literally, two. Two things are, for us, always different things.

If Q is beyond R from P [or, equivalently, if P is beyond R from Q] it is natural to say that R is *between* P and Q. Intuitively, the points between P and Q are just those which belong to both of the two half-lines $\overrightarrow{PQ}$ and $\overrightarrow{QP}$. The set of all points between P and Q is called the *interval*, $\overline{PQ}$, with *endpoints* P and Q. As just remarked, $\overline{PQ} = \overrightarrow{PQ} \cap \overrightarrow{QP}$. From our description of opposite rays, two points belong to opposite half-lines if and only if the common vertex of the half-lines is between the points. Notice, also, that if two points belong to a given half-line then so do all points between them. Finally, the *segment* $\overset{\bullet\!-\!\bullet}{PQ}$ with endpoints P and Q consists of the points between P and Q together with P and Q themselves—$\overset{\bullet\!-\!\bullet}{PQ} = \overline{PQ} \cup \{P,Q\}$. Evidently, $\overset{\bullet\!-\!\bullet}{PQ} = \overset{\bullet\!\rightarrow}{PQ} \cap \overset{\bullet\!\rightarrow}{QP}$.

Intuitively, a plane is a flat surface without holes or edges. The surface of a flat sheet of paper makes a pretty good model of part of a plane; but, since the paper has edges—and might have holes in it—it is not a model of an entire plane. Let's try, now, to relate this rather vague notion of what a plane is to the notions we have developed of ray, line, etc. One way to begin is by thinking of a test you might use to determine how nearly flat some surface—say, the surface of your desk—is. Suppose that you have a ruler whose edge is straight—no dents! You could test your desk top for flatness by holding your ruler so that two points of its edge are in contact with the desk top. If the desk top is flat then—unless the ruler extends over an edge of the desk—all points of the edge of the ruler should be in contact with the top of the desk. On the other hand, if the desk top is not flat, you can find a way of holding your ruler so that [at least] two points of its edge are in contact with the desk top, but some points of the edge are not. This suggests that we might define a flat surface without holes or edges—that is, a plane—to be a surface such that, given any two points which belong to it, each point of the line which contains these points also belongs to the surface.

We might, then, think of a plane as a surface which contains each line through any two of its points. But, this still leaves us with the question as to what a surface is. This is a rather difficult question to answer. To say that a surface is thin is not of much help, but it may suggest that a surface is easily cut. To cut a surface you don't need a saw—scissors will do the job. This might remind you that we noticed earlier that it is even easier to cut a line. For this, one doesn't even need scissors. All that is necessary is to punch out a point. Just as a point of a line separates the line into two half-lines, so a line in a plane separates the plane into two half-planes. This suggests that, in investigating planes, we might begin by considering half-planes. Just as we found that a line consists of a point P together with the points of two opposite half-lines which have P as vertex, we might expect that a

The fact that all points between two points of a half-line belong to that half-line is expressed in more technical terms by saying that a half-line is a convex set. Rays, lines, intervals, and segments are also, in the same sense, convex. ['Convex' is sometimes used with other, closely related meanings. For example, a convex polygon is a plane polygon which is such that the points of its plane which are "inside" it constitute a convex set.] Other examples of convex sets are the half-planes, closed half-planes, and planes [which are discussed next in the text], space itself, and the set consisting of the points of space which are inside a given spherical surface. [A spherical surface itself is not convex, but the ball which is the union of such a surface and its interior is convex.] Convexity is a very important geometric property, but we shall not have many occasions to refer to it.

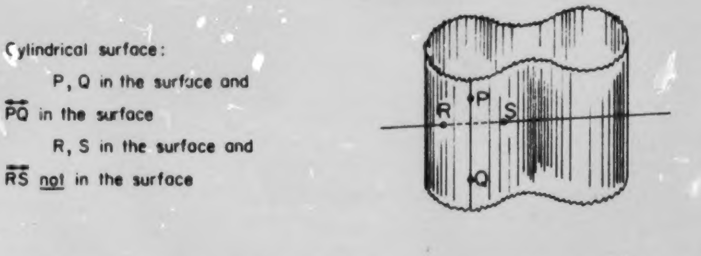
Before going on to the discussion in the text of planes you might ask students to mark two points, label them 'A' and 'B' and draw the ray $\overset{\bullet\!\rightarrow}{AB}$. Then, on other parts of their papers, draw pictures of the ray $\overset{\bullet\!\rightarrow}{BA}$, the segment $\overset{\bullet\!-\!\bullet}{AB}$, and the interval $\overline{BA}$. While they are so engaged, write on the board:

(a) $\overset{\bullet\!\rightarrow}{AB} \subseteq \overset{\bullet\!-\!\bullet}{AB}$ (b) $\overset{\bullet\!-\!\bullet}{AB} \subseteq \overset{\bullet\!\rightarrow}{AB}$ (c) $\overline{AB} \subseteq \overset{\bullet\!-\!\bullet}{AB}$
(d) $\overset{\bullet\!-\!\bullet}{AB} \subseteq \overset{\bullet\!\rightarrow}{BA}$ (e) $\overset{\bullet\!-\!\bullet}{AB} = \overset{\bullet\!-\!\bullet}{BA}$ (f) $\overline{AB} = \overset{\bullet\!-\!\bullet}{AB}$

Then, ask for true-false answers.

In refining one's intuitive notion of what a plane is, the notion that it is a flat surface without holes or edges seems a good one to start from. The phrase 'without holes or edges' is somewhat more adequate than, say, 'which goes on forever'. To arrive at an "adequate description" of plane: we begin by noting that being flat and without holes or edges suggests that any line which contains two points of such a set should be completely contained in that set. [Technically, a set which has this property is said to be linear, but the use of 'linear' in the present context would scarcely be helpful.] Although not only planes, but lines and space itself have this property, it should be intuitively obvious that the only "surfaces" with this property are planes — "flat surfaces without holes or edges".

Some surfaces have the property that it is possible to find two points of the surface such that the line containing these points is in the surface, and to find, also, two points of the surface such that the


Cylindrical surface:
P, Q in the surface and
$\overleftrightarrow{PQ}$ in the surface
R, S in the surface and
$\overleftrightarrow{RS}$ not in the surface
R
P
S
Q

line containing these points is not in the surface. The cylindrical surface pictured above is an example.

plane consists of the points of a line l together with the points of two opposite half-planes each of which has l as its edge.

A half-line with vertex P consists of all the points which are beyond P from some point E. Analogously, a *half-plane* with *edge* l consists of all points which are across the line l from some point E. [The phrase 'across l from E' is short for 'beyond one or another point of l from E'.] A half-plane is analogous to a half-line, and the edge of a half-plane is analogous to the vertex of a half-line. So, the analogue of a ray is the union of a half-plane and its edge. Such a set is called a *closed half-plane*. It should seem intuitively obvious that, given a line l and a point $Q \notin l$, there is a unique closed half-plane which contains Q and has edge l. The set of all points which are across l from Q is another half-plane, and its union with l is a closed half-plane—the *opposite* of the one with edge l which contains Q. [The two half-planes—as well as the *closed* half-planes—are opposites of one another.] Evidently, if Q and R belong to opposite half-planes then there is a point of the common edge of these half-planes between Q and R. Also, any half-plane which contains two given points also contains all points between them.

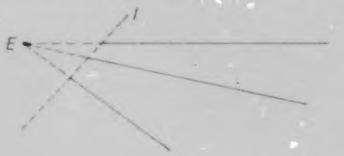


Fig. 1–13

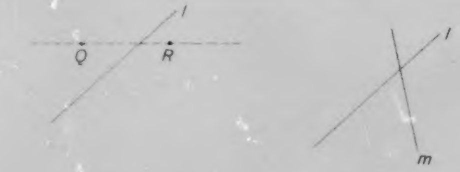


Fig. 1–14

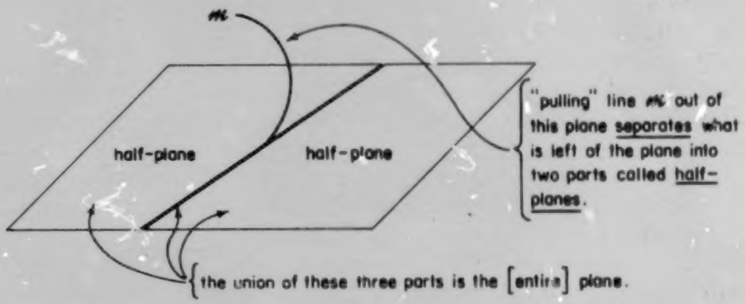
A little thought should convince you that a given union of two opposite closed half-planes seems to satisfy our two requirements that a set be a plane. These were:

> If Q and R are any two points of a plane then the line $\overleftrightarrow{QR}$ which contains Q and R is a subset of that plane. ["Planes are flat."]
>
> If m is any line which is a subset of a plane then this plane is the union of m and two half-planes which have m as edge. ["Planes are thin."]

TC 29 (2)

It remains to analyze what distinguishes a surface from other kinds of sets. Intuitively, this is their "thinness". In the case of "linear" sets [see the preceding bracketed sentences] a thin set is one which is separated by any line contained in it. In summary, one adequate description of planes is that a plane is a set such that

> each line which contains two of its points is contained in it, and each line which is contained in it separates it



"pulling" line m out of this plane separates what is left of the plane into two parts called half-planes.

the union of these three parts is the [entire] plane.

[The second property is closely related to the modern theory of dimension. A two-dimensional set is one out of which one can separate arbitrarily small neighborhoods of any point by removing appropriate one-dimensional sets; a one-dimensional set is one out of which one can separate arbitrarily small neighborhoods of any point by removing appropriate zero-dimensional sets; a zero-dimensional set is one out of which one can separate arbitrarily small neighborhoods of any point without removing anything.]

TC 30

Our description of lines in terms of rays — or, more conveniently now, in terms of half-lines — suggests that there may be an analogous description of planes in terms of "half-planes". This together with the separation property in the description we have just arrived at suggests the appropriate description of half-planes. A half-plane is somewhat like a half-line, but differs by having an edge rather than a vertex. Since there seems to be no appropriate word for the analogue of a ray, we call the union of a half-plane and its edge a closed half-plane. [Incidentally, the meanings of 'ray', 'half-line', 'interval', 'segment' and 'half-plane' are not standardized. In reading other authors you may find any of these words prefixed by either 'open' or 'closed'. If by 'open', the set referred to does not contain its vertex, end points, or edge, as the case may be. If by 'closed', it does.]

As a matter of fact, a union of two opposite closed half-planes does have these properties; and that it does can be shown to follow from what we already know about rays, lines, and half-planes. We shall not take time to show this here because there are other properties of planes which we need to investigate.

First, let's notice that the "thinness property" does tell us that removing a line "cuts" a plane into two disjoint pieces. This is because *two* half-planes with the same edge cannot both contain any given point. [No half-plane contains a point of its edge and, as we know, given m and a point $Q \notin m$, there is a unique half-plane which contains Q and has m as its edge.] Actually, as you probably suspect, we know more:

> If m is any line which is a subset of a plane then this plane is the union of m and two *opposite* half-planes which have m as an edge.

This is usually called 'the line-plane separation property' and follows rather easily from the flatness property, the thinness property, and the uniqueness property for half-planes. [What is the point-line separation property? Can you think of a third separation property?]

Next, let's investigate some uniqueness properties for planes. As you know, there is a unique line containing any two given points. What would be an analogous uniqueness property for planes? Recalling the earlier analogies, you might think of this one:

> Given any line l and any point P not on l, there is a unique plane which contains l and $\{P\}$.

One plane containing l and $\{P\}$ is, of course, the union of the closed half-plane containing P whose edge is l and the opposite of this closed half-plane. Using the line-plane separation property, it is easy to see that this is the only plane containing l and $\{P\}$.

Another uniqueness property for planes which you may already have thought of is:

> Given three points not on the same line, there is a unique plane which contains all of them.

This follows from the preceding uniqueness property and the flatness property. [Let l be the line containing two of the given points, and let P be the third of the given points.]

For our final property, consider a line l and a point $P \notin l$. As you know, there is a unique plane—say, π—which contains P and each point of l. It is the union of l and two half-planes—the half-plane which consists of the points which are across l from P and the half-plane which is the opposite of this one. From the meaning of 'across' it follows that each point of the first of these two half-planes belongs to a line which contains P and some point of l. [The same is, of course, the case for each point of l.] From the flatness property we know that

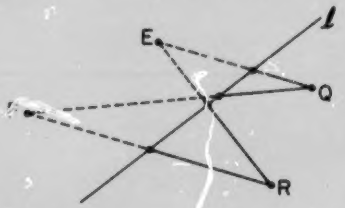
We take it as intuitively obvious

> that given a line ℓ and a point $Q \notin \ell$, there is a unique closed half-plane which contains Q and has edge ℓ,
>
> that the opposite of a closed half-plane, as described in the text, is uniquely determined, and
>
> that a union of opposite closed half-planes does have the flatness and thinness properties which [as we have decided] characterize planes.

Presumably, students will agree to this. If there are doubts then — although you will not have time to discuss proofs — knowing how the first, and least dubious, of these three assertions can be used in establishing the other two may give you ideas as to how to proceed. So, in the next six paragraphs we shall do what we say in the text "We shall not take time" to do there. [As the asterisks indicate, you probably need not take time to read them.]

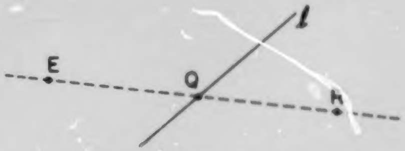
* * *

Suppose then, that it is granted that, given a line ℓ and a point $Q \notin \ell$, there is a unique closed half-plane which contains Q and has edge ℓ. [Notice that it makes no real difference as to what we are granting if we delete the word 'closed'. Given two half-planes with a common edge, their unions with this edge are two closed half-planes; given two closed half-planes with a common edge, removing this edge from each of them leaves two half-planes.] What is it, exactly, that we are granting? According to our notion of half-planes, the only way to obtain a half-plane with edge ℓ which contains Q is to choose a point E such that Q is across ℓ from E, and take, as our half-plane, the set of all points which are across ℓ from E. What we are granting is that, given two points, E and F, such that Q is across ℓ from each of them, the points which are across ℓ from either are the same as those which are across ℓ from the other. This, now, implies at once that a half-plane has a unique opposite. To see this, consider a half-plane with edge ℓ. There is a point Q such that the points of the given half-plane are just those which are across ℓ from Q. So, given any two points E and F of the given half-plane, the same points are across ℓ from either. Hence, in describing the opposite of the given half-plane we are at liberty to choose any of its points as our "view-point". The result is independent of this choice.

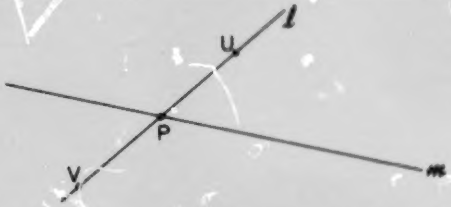


In showing that a union of two opposite closed half-planes is both flat and thin, it is most economical to begin by establishing "part" of its flatness, use this to establish thinness, and then use thinness to complete the proof of flatness. [Actually, rather than thinness, we shall establish the stronger line-plane separation property which, in the text, is shown to be implied by the properties of flatness and thinness.]

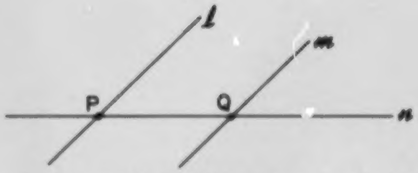
Suppose, then, that π is the union of two opposite closed half-planes with edge ℓ. For flatness, we wish to show that if Q and R belong to π then the line $\overleftrightarrow{QR}$ which contains them is a subset of π. To begin with, we consider the case in which $Q \in \ell$. In case $R \in \ell$, $\overleftrightarrow{QR} = \ell$ and, so, is a subset of π. Suppose then that $R \notin \ell$. It follows [since $R \in \pi$] that R belongs to one of the two opposite closed half-planes whose union is π. Since $R \notin \ell$, R belongs to one of the two corresponding half-planes. From our uniqueness assumption, the points of this half-plane can be described as those which are across ℓ from any point E we may choose as long as R is beyond some point of ℓ from E. So, for E, we may choose a point which is beyond Q from R. It follows that all points of $\overrightarrow{QR}$ belong to the half-plane with edge ℓ which contains R. All points of $\overrightarrow{QE}$ belong to the opposite of this half-plane. Since $Q \in \ell$ and $\overleftrightarrow{QR}$ is the union of the opposite rays $\overrightarrow{QR}$ and $\overrightarrow{QE}$, it follows that $\overleftrightarrow{QR} \subseteq \pi$.



Before completing the proof of flatness by considering the case in which neither Q nor R belong to ℓ, we use what we have already proved to show that π has the line-plane separation property. What we wish to show is that if m is any line contained in π then π is the union of m and two opposite half-planes with edge m. The case in which $m = \ell$ is trivial. We consider first the case in which m intersects ℓ at a single point P. This point separates ℓ into two half-lines, $\overrightarrow{PU}$ and $\overrightarrow{PV}$. The half-plane with edge m which contains U consists of all points which are across m from V. It contains each point of $\overrightarrow{PU}$. The opposite of this half-plane consists of all points across m from U, and contains each point of $\overrightarrow{PV}$. The union of m and these two half-planes is a plane — let's call it 'π_m' — which, as we have seen, contains ℓ. It should turn out to be π. Let's, first, show that $\pi_m \subseteq \pi$. By assumption, each point of π_m is either a point of m — and, so, belongs to π — or is across m from U or across m from V. Now, each point which is across m from U belongs to a line which contains U and a point of m — that is, it belongs to a line which contains two points of π, one of them being a point of ℓ. Since we have shown that each such line is a subset of π, each point which is across m from U belongs to π. Similarly, each point which is across m from V belongs to π. So [since $m \subseteq \pi$] it follows that each point of π_m belongs to π. Hence, $\pi_m \subseteq \pi$. Since we have shown that $\ell \subseteq \pi_m$, we can almost repeat the preceding argument — beginning with 'By assumption' — to show that $\pi \subseteq \pi_m$. To begin with, let Q and R be points of m which are on opposite sides of P. Since $m \subseteq \pi$, both Q and R belong to π. So, since $P \in \ell$, the two opposite closed half-planes with edge ℓ whose union is π must be the unique closed half-planes with edge ℓ which contain Q and R. So, each point of π is either a point of ℓ — and, so, belongs to π_m — or is across ℓ from Q or across ℓ from R. Now, each point which is across ℓ from Q belongs to a line which contains Q and a point of ℓ — that is, it belongs to a line which contains two points of π_m. Since we have shown that each such line is a subset of π_m, each point which is across ℓ from Q belongs to π_m. Similarly, each point which is across ℓ from R belongs to π_m. So [since $\ell \subseteq \pi_m$] it follows that each point of π belongs to π_m. Hence, $\pi \subseteq \pi_m$. Consequently [since we showed earlier that $\pi_m \subseteq \pi$], $\pi = \pi_m$.

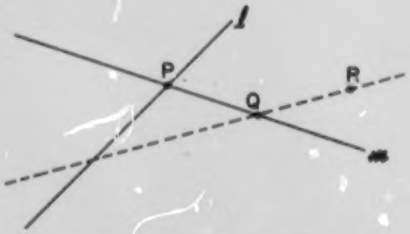


To complete our proof of the line-plane separation property we must free ourselves from the assumption that the line m intersects ℓ. This is now easy. Let P and Q be two points such that $P \in \ell$ and $Q \in m$.



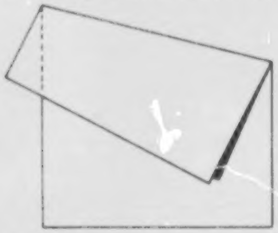
Since $P \in \ell$ and $Q \in m \subseteq \pi$ we know [from the case of flatness already established] that the line n which contains P and Q is a subset of π. From what we have established as to line-plane separation, we know that $\pi = \pi_n$, where π_n is the union of n and two opposite half-planes with edge n. Since, by assumption, $m \subseteq \pi$ it follows that $m \subseteq \pi_n$. Again from the special case of line-plane separation already established, $\pi_n = \pi_m$, where π_m is the union of m and two opposite half-planes with edge m. Since $\pi = \pi_n = \pi_m$, we are finished.

We return now to flatness. Suppose that Q and R are any two points of π, neither of which belongs to ℓ. [As before, π is, by assumption the union of ℓ and two opposite half-planes with edge ℓ.] Let P be a point of ℓ and let m be the line through P and Q. By the case of flatness already dealt with, $m \subseteq \pi$. By the first case of line-plane separation, $\pi = \pi_m$, where π_m is, as usual, the union of m and two



opposite half-planes with edge m. Since $Q \in m$ and $R \in \pi = \pi_m$ it follows from the already established case of flatness that $\overleftrightarrow{QR} \subseteq \pi_m$. So, as we wished to show, $\overleftrightarrow{QR} \subseteq \pi$.

Returning now to discussion of the text itself, some words may be in order as to the difference between what we have called 'thinness' and the line-plane separation property. An analogy may help. Make a fold in a sheet of paper. The result is a model of a surface on which one can draw many straight lines — or, at least, segments running from edge to edge. These are the lines which one could draw on the unfolded paper without crossing the fold. Cutting along any of these lines separates the paper into two pieces. Only cutting it along the fold separates it into two "half-planes". No cut separates it into "opposite half-planes". If the paper were flat, each cut would separate it into "opposite half-planes".

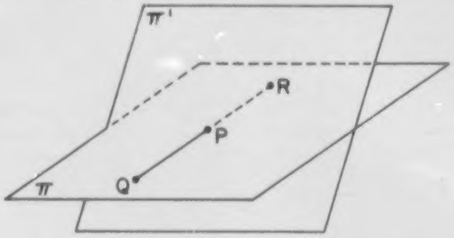


The point-line separation property is that if Q is any point which belongs to a line then this line is the union of $\{Q\}$ and two opposite half-lines which have Q as vertex.

The plane-space separation property is that if π is any plane in $\mathcal{E}$ then $\mathcal{E}$ is the union of π and two opposite half-spaces each of which has π as its face. Your students can probably guess this third separation property, and suggest an adequate description of half-spaces with face π, and opposite half-spaces.

Accepting the plane-space separation property amounts — as you might point out to your students — to agreeing that space is three-dimensional. [For analogy, had we agreed that if ℓ is any line in $\mathcal{E}$ then $\mathcal{E}$ is the union of ℓ and two opposite half-planes each of which has ℓ as edge, we would have limited "space" to be a plane.] On the other hand, if we wished to study four-dimensional space we would, following the now-familiar pattern, have described a hyperplane as the union of a plane and two opposite half-hyperplanes having this plane as face. [The descriptions would be those students should have suggested for half-spaces.] We would then have, in place of the plane-space separation property, a plane-hyperplane separation property obtained from the former by replacing the first '$\mathcal{E}$' by 'a hyperplane' and the second by 'this hyperplane'. Finally, we would have a hyperplane-space separation property.

Although we do not discuss it in the text at this point, this is a convenient place in the commentary to point out that adoption of the plane-space separation property has the consequence that two planes which intersect do so in at least two points. [Using the flatness property and the three-point uniqueness property of planes one sees, then, that the intersection of two planes is either empty or a line. (In four-dimensional space, on the contrary, two planes can intersect in a single point.)] To show this, assume the plane-space separation property, and suppose that π and π' are two planes which have a point P in common. Since π' has [only] two sides in [3-dimensional] space, π must contain a point Q on one of these sides. Since $P \in \pi'$, a point R beyond P from Q belongs to the other side of π'. Since P and Q belong to π, so does R. Now, choose a point $S \in \pi$ which is not on $\overleftrightarrow{QR}$. If $S \in \pi'$, we have found a second point of intersection of π and π'. If not, S belongs to one of the two sides of π'. We may suppose it to belong to the same side as Q. It follows that S and R belong to opposite sides of π' and, so, that there is a point of π' between them. Since each point between S and R belongs to π it follows that this is a second point of intersection of π and π'.



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The "final property" discussed in the text concerns the existence and uniqueness of parallel lines. In the formal part of the course we shall define the direction of a line, and parallel lines will be defined to be lines which have the same direction. [In consequence, any line will be parallel to itself.] In this section we deal with the more usual notion of parallelism — the question being, first, whether, given a line ℓ and a point $P \notin \ell$, there are points of the plane of ℓ and P which are not on any line through P which intersects ℓ. If — as we finally urge students to agree — there are such points then, by the flatness property, there is at least one line in the plane of ℓ and P which contains P and does not intersect ℓ. Such a line we call a parallel to ℓ through P. Whether we go on to assume that there is at most one or more than one such parallel depends on whether we decide to study Euclidean or Lobachewskian geometry. Although devices such as your students parallel rulers may predispose them to believe in the existence of parallel lines in physical space, we know of no device which would tip the scales in favor of Euclidean as opposed to Lobachewskian geometry. If driven to the wall, your arguments in favor of the former are that it's simpler and that it is what everybody else has studied first, and, finally, that, like it or not, it's what we are going to study in this course.

each of these lines is a subset of π, and we also know that "half" of each line is made up of points of the second half-plane. Question: Is each point of the second half-plane on a line which contains P and a point of l?

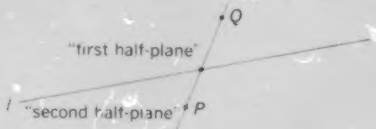


Fig. 1–15

Before trying to answer the preceding question you should experiment by doing some careful drawing. [For one thing, use a *sharp* pencil.] A flat sheet of paper is a pretty good model of part of the plane π, and by using a ruler and a sharp pencil you can draw pretty good pictures of parts of lines. Working as carefully as you can, draw a picture of part of l and of parts of some lines through P which intersect l.

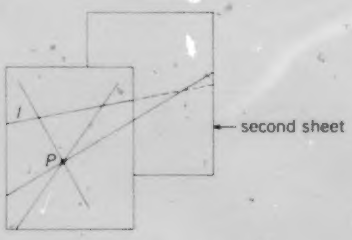


Fig. 1–16

The figure suggests how you can use a second sheet of paper to draw lines which intersect l at points which don't show on your finished drawing. Continue with your drawing until you think you know whether there are points of π which do not belong to any line through P and a point of l. If you decide that there are such points, try to locate some of them as accurately as you can, on your paper. [You can spoil the fun by reading on, and you may learn more if you complete your drawing first.]

If you have done the job suggested in the preceding paragraph, you probably have some ideas on the question: Is the union of all lines

There are several references in the text and exercises to a drawing instrument called a parallel ruler. This instrument, which is essentially a ruler which is mounted on rollers, is a handy device for drawing lines which are parallel to a given line. There are quite a few models available, and any well-stocked drawing supply store will have a supply of them. If you cannot manage to have a classroom supply of such instruments as departmental property for students to borrow and use in solving the exercises in questions, you can get by with one such instrument that can be used in conjunction with the overhead projector or with a large model designed for use on the chalkboard.

Although the point need not arise, it may be worth examining the evidence for parallel lines which is furnished by parallel rulers. Using a parallel ruler one can, on rolling it from ℓ until P is on its edge, draw a line which, by symmetry, has the property that if either of its rays with vertex P intersects ℓ then so does the other. So, if this line intersects ℓ at all, it would seem to intersect it in two points. Since there is only one line through two points, and since $P \notin \ell$, it would follow that the line in question cannot intersect ℓ at all. This ointment has its own fly. We have tacitly assumed that a ray is not its own opposite — that, sighting from E through P, one will never see the back of one's own head — or, less spectacularly put, that E is not beyond P from E. If, for physical space, this assumption is incorrect then our argument based on the behavior of parallel rulers breaks down. For, in this case, the line through P which we draw "parallel" to ℓ may intersect ℓ in a single point which we can sight toward by looking "either way" from P. If this should turn out to be so then lines "close up on themselves" — topologically, they are indistinguishable from circles. And, there are no parallel lines. The assumption that this is the case commits one to a third kind of geometry — called, as one wishes, Riemannian, or elliptic.

through P which intersect l a plane? In particular, does each point you could mark on your paper belong to such a line? In doing your drawing, you may have thought of a way of using your parallel ruler to draw [part of] a line through P which seems not to intersect l. [If you didn't, try to think of one now.] If so, you will be ready to agree with us that the answer to the question we have been considering is 'No.' And you will agree that there is a line through P which is a subset of the plane π and which does not intersect l. A line which is a subset of a plane containing l and which does not intersect l is said to be *parallel* to l. Such a line can also be described as one which has the same *direction* as l. Later, this second meaning of 'parallel' [same direction] will be more convenient than the other. Then we shall also agree that any line is parallel to itself [it certainly has its own direction!]. So, for now, let's say that l and m are parallel if and only if $l = m$ or l and m are subsets of a single plane and have no point in common.

There is, of course, more to be said about our question. Although experimenting with your parallel ruler should give you fairly convincing evidence that, given a line l and a point $P \notin l$, there is a line through P and in the plane of l and P which does not intersect l, the evidence may not seem as convincing as that which led you to accept the other statements we have made about rays, half-planes, planes, etc. Making drawings on a rather small sheet of paper may give one a cozy feeling; but physical space is, to say the least, large. When we agree to assume that through a point not on a line there is a parallel line, we are agreeing that the geometry we shall study is of a special kind. For the usual *Euclidean* geometry which we are going to study in this book, we shall assume even more:

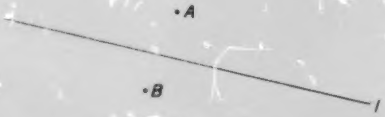
> Given a line l and a point $P \notin l$ there is a *unique* line through P which is a subset of the plane of l and P and does not intersect l.

In terms of the question we asked earlier this means that not all points of the plane of l and P belong to lines through P and points of l, but those which are not all belong to one line through P. If, instead, we assumed that there is more than one line through P parallel to l then we should be studying a non-Euclidean geometry which is called Lobachevskian, or hyperbolic, geometry.

Since parallel lines are going to play a considerable role in our study, it will be of help for you to know quite a bit about them before we begin our real work of developing geometry. The following exercises give you a chance to notice some of the things you should know. They also give you a chance to develop some skill in drawing which will be useful to you.

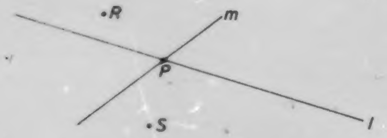
Exercises

1. On your paper make a picture, similar to the one given, of line l and points A and B not on l.



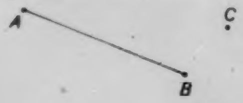
(a) Draw a line m through A which is parallel to l.
(b) Draw a line n through B which is parallel to l.
(c) What relation [or relations] exists between lines m and n?

2. Make a picture, similar to the one given, of lines l and m which intersect in the point P and of points R and S which are not on l or on m.



(a) Draw a line n_1 through R which is parallel to line l.
(b) Draw a line n_2 through S which is parallel to line m.
(c) Are the lines n_1 and n_2 parallel? Do these lines intersect?
(d) Try to describe the location of a point T such that a line n_3 through T parallel to l does not intersect the line n_2. What is a relation between the lines n_3 and n_1?

3. On your paper, make a picture similar to the one given of a segment $\overleftrightarrow{AB}$ which is two inches long and of a point C which is not on the line $\overleftrightarrow{AB}$.



(a) Draw a segment $\overleftrightarrow{CD}$ such that $\overleftrightarrow{CD}$ is parallel to $\overleftrightarrow{AB}$ and $\overleftrightarrow{BD}$ is parallel to $\overleftrightarrow{AC}$. How long is $\overleftrightarrow{CD}$? Compare the lengths of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$.
(b) Draw a segment $\overleftrightarrow{CE}$ such that $\overleftrightarrow{CE}$ is parallel to $\overleftrightarrow{AB}$ and $\overleftrightarrow{AE}$ is

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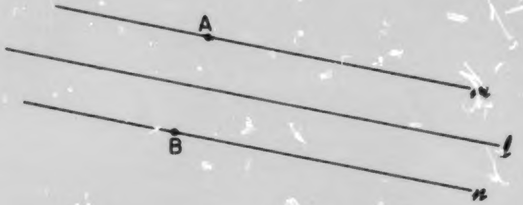
Answers for Exercises

1. After doing parts (a) and (b), the student's diagram should look something like this:



(c) Some answers you might receive here are:

The lines m and n are parallel.
The lines m and n won't intersect.
The lines ℓ, m, and n are parallel.

[The conclusion that, for any lines ℓ, m, and n, if m and n are both parallel to ℓ then m is parallel to n amounts, in view of the obvious symmetry of parallelism, to the conclusion that parallelism is transitive [if $m \| \ell$ and $\ell \| n$ then $m \| n$]. It is in order that parallelism have this very desirable property that we insist on each line being parallel to itself.

Using the notion that m and n are parallel if and only if $m = n$ or m and n are coplanar and $m \cap n = \emptyset$ it is easy to see that our parallel postulate implies that coplanar lines which are parallel to a third line are parallel to each other. For, suppose that m and n are coplanar and that both are parallel to ℓ. In case $m = n$ it follows by definition that m and n are parallel. Suppose, then, that $m \neq n$. It follows that if there exists a point common to m and n then there exists a point through which there are two lines parallel to ℓ. Since, by the parallel postulate, this is not the case it follows that $m \cap n = \emptyset$. Since, by hypothesis, m and n are coplanar it follows, by definition, that m and n are parallel.

Since the lines m and n of Exercise 1 are coplanar this argument shows that, having accepted the parallel postulate, students are committed to agreeing that these lines are parallel.]

* * *

The more general proposition that [any] lines which are parallel to a given line are parallel requires an additional argument. In view of the preceding, it is sufficient to show that lines which are parallel to a given line are coplanar. We first dispose of trivial cases by noting that if m and n are parallel to ℓ then, in case $m = n$ or $m = \ell$ or $n = \ell$, m and n are parallel, either by definition or by assumption. So, what remains to be shown in that if two of three lines are parallel to a third then these two are parallel. Suppose, then, that ℓ, m, and n are three lines such that $m \| \ell$ and $n \| \ell$. Since $m \neq \ell$ and $m \| \ell$ it follows, by definition, that m and ℓ are coplanar in some plane π_m. By the point-line uniqueness property of planes it follows, since $m \cap \ell = \emptyset$, that π_m is the unique plane which contains ℓ and intersects m. Similarly, n and ℓ are coplanar in some plane π_n, and and π_n is the unique plane which contains n and intersects ℓ. Since $m \neq n$, there is a point — say, A — of m which does not belong to n. Let π be the unique plane which contains n and A. Since π_m and π are planes [in 3-dimensional space] which both contain A it follows that either $\pi_m = \pi$ or $\pi_m \cap \pi$ is a line. In case $\pi_m = \pi$ it follows, since $m \subseteq \pi_m$ and $n \subseteq \pi$, that m and n are coplanar. Suppose, then, that $\pi_m \cap \pi$ is a line — say p. Suppose that p intersects ℓ. It follows, since $p \subseteq \pi$, that π intersects ℓ. Since π contains n it follows that $\pi = \pi_n$. Since $A \in \pi$, $A \in \pi_n$. Since $A \in m$ and $\ell \subseteq \pi_n$ it follows that $\pi_n = \pi_m$. Since $\pi = \pi_n$ and $\pi_n = \pi_m$ it follows that $\pi = \pi_m$. But, since $\pi \cap \pi_m = p$, $\pi \neq \pi_m$. Hence, $p \cap \ell = \emptyset$. Since p and ℓ are coplanar in π_m it follows that $p \| \ell$. Since $A \in p \cap m$ and $m \| \ell$ it follows by the parallel postulate that $m = p$. Since $p \subseteq \pi$ it follows that $m \subseteq \pi$ and, so, that m and n are coplanar in π.

Hence, if ℓ, m, and n are three lines such that $m \| \ell$ and $n \| \ell$ then m and n are coplanar.

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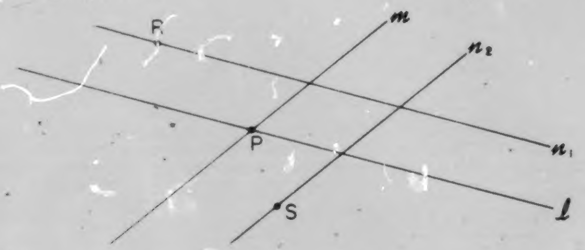
2. After doing parts (a) and (b), the student's diagram should look something like this:



(c) The lines n_1 and n_2 are not parallel.

[An argument that might be given to support this is the following: From the conditions stated, we know that $n_1 \| \ell$ and $n_2 \| m$. Assuming that $n_1 \| n_2$ it follows, since parallelism is symmetric and transitive, that $\ell \| m$; for $\ell \| n_1 \| n_2 \| m$. But, this contradicts the notion that two lines which are parallel cannot intersect, for we were given that $\ell \cap m = \{P\}$. Hence, n_1 is not parallel to n_2 [or, as the symmetry of parallelism allows us to put it, n_1 and n_2 are not parallel].]

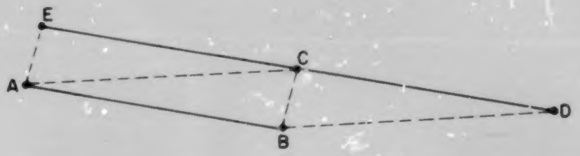
The lines n_1 and n_2 intersect.

[Call to students attention the fact that the argument given above to justify the conclusion that $n_1 \| n_2$ does not suffice for the conclusion that $n_1 \cap n_2 \neq \emptyset$. For this conclusion we need to know, also, that n_1 and n_2 are coplanar. To establish this, make use of the assumption — implicit in the exercise — that m, ℓ, R, and S are coplanar.]

(d) Choosing T "out of the plane of ℓ and m" will do the job here. What we intend to do here is to make the student begin to think about the points "above" and "below" any plane in which he happens to be working or, drawing].

Answers for Exercises [cont.]

3. The student's completed diagram should look something like this:



(a) $\overleftrightarrow{CD}$ is 2 inches long. $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ have the same length.

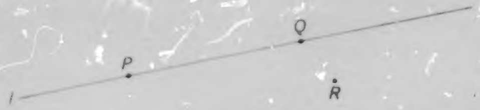
(b) $\overleftrightarrow{CE}$ is 2 inches long. $\overleftrightarrow{AE}$ and $\overleftrightarrow{BC}$ have the same length.

parallel to $\overleftrightarrow{BC}$. How long is $\overline{CE}$? Compare the lengths of $\overline{AE}$ and $\overline{BC}$.

(c) Compare the senses of ray $\overrightarrow{CD}$ and ray $\overrightarrow{CE}$.

(d) One of the rays $\overrightarrow{CD}$ and $\overrightarrow{CE}$ has the same sense as $\overrightarrow{AB}$; the other has the same sense as $\overrightarrow{BA}$. Which is which?

4. Make a picture, similar to the one given, of points P and Q on line l and of point R not on l.



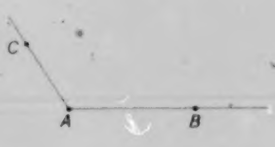
(a) Draw $\overleftrightarrow{PR}$, the line containing points P and R. Draw a line q through Q parallel to $\overleftrightarrow{PR}$. Draw a line r through R parallel to l.

(b) Let T be the point of intersection of lines q and r. What can you say about the segments $\overline{PQ}$ and $\overline{RT}$? About the rays $\overrightarrow{PQ}$ and $\overrightarrow{RT}$?

(c) Let S be a point on $\overleftrightarrow{PR}$ such that P is between R and S. Draw a line s through S parallel to l. Let U be the point of intersection of q and s. What can you say about $\overline{PQ}$ and $\overline{SU}$? About $\overline{RT}$ and $\overline{SU}$? About $\overrightarrow{SU}$ and $\overrightarrow{PQ}$?

(d) Choose a point V on $\overleftrightarrow{PQ}$ such that Q is between P and V. Locate a point W on $\overleftrightarrow{PQ}$ such that $\overleftrightarrow{TW}$ is parallel to $\overleftrightarrow{RV}$. Is there more than one such point? Compare $\overline{VW}$ and $\overline{PQ}$. Compare $\overrightarrow{VW}$ and $\overrightarrow{PQ}$.

5. Make a picture, similar to the one given, of angle BAC [for short: $\angle BAC$].



(a) Mark a point D about 2 inches from A. Draw a ray $\overrightarrow{DE}$ through D which has the same sense as the ray $\overrightarrow{AB}$. [Start by drawing $\overleftrightarrow{DE}$ parallel to $\overleftrightarrow{AB}$.]

(b) Draw a ray $\overrightarrow{DF}$ which is parallel to the ray $\overrightarrow{AC}$ and such that $\angle EDF$ is congruent to $\angle BAC$.

(c) Draw a ray $\overrightarrow{DG}$ which is parallel to $\overrightarrow{AC}$ and such that $\angle EDG$ is *not* congruent to $\angle BAC$.

(d) Compare the rays $\overrightarrow{DF}$ and $\overrightarrow{DG}$.

(c) $\overrightarrow{CD}$ and $\overrightarrow{CE}$ have opposite senses.

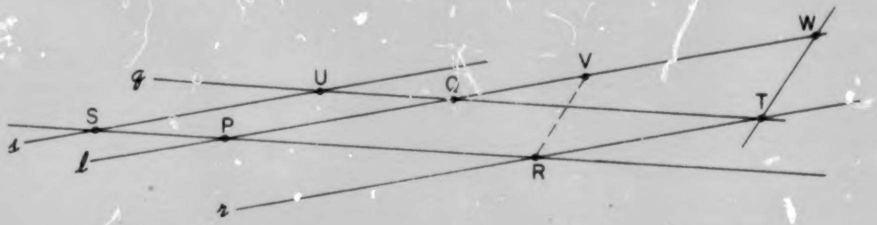
(d) $\overrightarrow{CD}$ has the same sense as $\overrightarrow{AB}$; $\overrightarrow{CE}$ has the same sense as $\overrightarrow{BA}$.

*

The main purpose of Exercises 3, 4, and 5 is to enable students to discover how, given three points P, Q, and R, to find the point T such that $\overline{RT}$ has the same length as $\overline{PQ}$ and $\overrightarrow{RT}$ has the same sense as $\overrightarrow{PQ}$.

*

4. The student's completed diagram should look something like this:



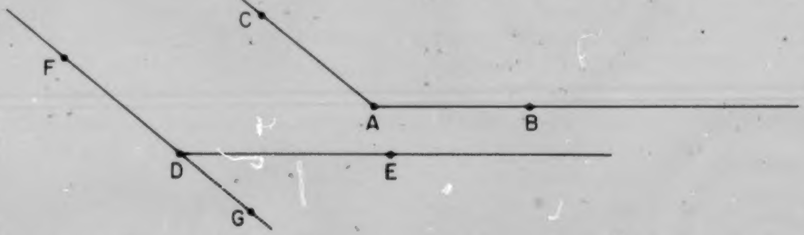
(a) [See the diagram.]

(b) $\overline{PQ}$ is parallel to $\overline{RT}$ and has the same length as $\overline{RT}$. $\overrightarrow{PQ}$ has the same sense as $\overrightarrow{RT}$.

(c) $\overline{PQ}$ and $\overline{SU}$ are parallel and have the same length. $\overline{RT}$ and $\overline{SU}$ are parallel and have the same length. $\overrightarrow{SU}$ and $\overrightarrow{PQ}$ have the same sense.

(d) There is just one point W on $\overleftrightarrow{PQ}$ such that $\overleftrightarrow{TW}$ is parallel to $\overleftrightarrow{RV}$. The segments $\overline{VW}$ and $\overline{PQ}$ have the same length. The rays $\overrightarrow{VW}$ and $\overrightarrow{PQ}$ have the same sense.

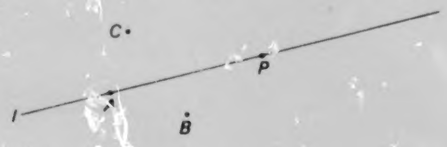
5. The student's completed diagram should look something like this:



(a), (b), (c) [In the diagram.]

(d) The rays $\overrightarrow{DF}$ and $\overrightarrow{DG}$ are opposite rays.

6. Make a picture, similar to the one given, of points A and P on line l and of points B and C such that A, B, and C are not collinear.



(a) Draw the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Draw lines m through B and r through C which are parallel to l.
(b) Draw a line s through P parallel to $\overleftrightarrow{AC}$. Let D be the point of intersection of lines r and s. What can you say about the segments $\overline{PD}$ and $\overline{AC}$?
(c) Draw a line t through P parallel to $\overleftrightarrow{AB}$. Let E be the point of intersection of lines m and t. Draw lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{ED}$. What can you say about these lines?
(d) Compare the segments $\overline{CD}$, $\overline{AP}$, and $\overline{BE}$.

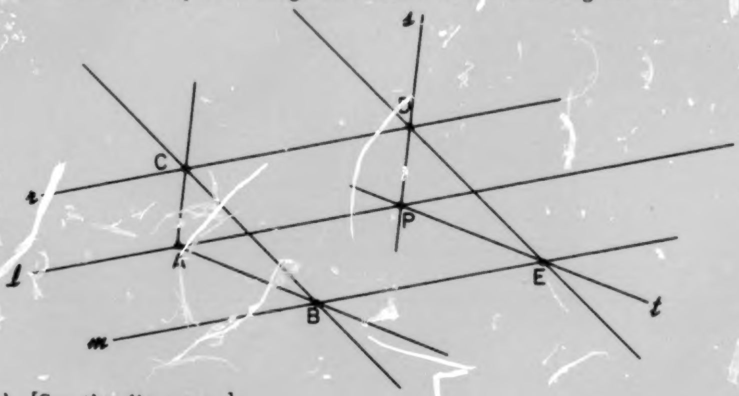
7. Make a picture, similar to the one given, of the segment $\overline{AB}$.

(a) Mark a point A' about 2 inches from A. Draw the line $\overleftrightarrow{AA'}$.
(b) Draw the line through A' which is parallel to the line $\overleftrightarrow{AB}$. Draw the line through B which is parallel to $\overleftrightarrow{AA'}$. Let B' be the point of intersection of these lines.
(c) Compare the segments $\overline{A'B'}$ and $\overline{AB}$. Compare the rays $\overrightarrow{A'B'}$ and $\overrightarrow{AB}$.
(d) Mark a point A''. Repeat parts (a), (b), and (c).
(e) Compare the segments $\overline{A'A''}$ and $\overline{B'B''}$.

8. Mark three noncollinear points P, C_1, and C_2. Draw the rays $\overrightarrow{PC_1}$ and $\overrightarrow{PC_2}$.
(a) On half-line $\overrightarrow{PC_1}$, mark two points A_1 and B_1 which are different from C_1. Draw $\overline{A_1C_2}$ and $\overline{C_2B_1}$.
(b) Draw a line l through C_1 parallel to $\overleftrightarrow{A_1C_2}$. Let A_2 be the point of intersection of l and $\overrightarrow{PC_2}$. Draw $\overline{A_2B_1}$.
(c) Draw a line m through C_1 parallel to $\overleftrightarrow{B_1C_2}$. Let B_2 be the point of intersection of m and $\overrightarrow{PC_2}$. Draw $\overline{A_1B_2}$.
(d) Make a guess at a relation between $\overleftrightarrow{A_2B_1}$ and $\overleftrightarrow{A_1B_2}$. Check your guess with your parallel ruler.

9. Let l_1 and l_2 be two lines which intersect in the point P. Mark three points A_1, B_1, and C_1 on l_1 which are different from P and which are not all on the same side of P. Mark a point C_2 on l_2 which is different from P. Locate points A_2 and B_2 on l_2, as in Exercise 8, and compare the lines $\overleftrightarrow{A_2B_1}$ and $\overleftrightarrow{A_1B_2}$ as you did in that exercise.

6. The student's completed diagram should look something like this:



(a) [See the diagram.]
(b) Segments $\overline{PD}$ and $\overline{AC}$ are parallel and have the same length.
(c) The lines $\overleftrightarrow{DE}$ and $\overleftrightarrow{BC}$ are parallel.
(d) $\overline{CD}$, $\overline{AP}$, and $\overline{BE}$ are parallel and have the same length.

* * *

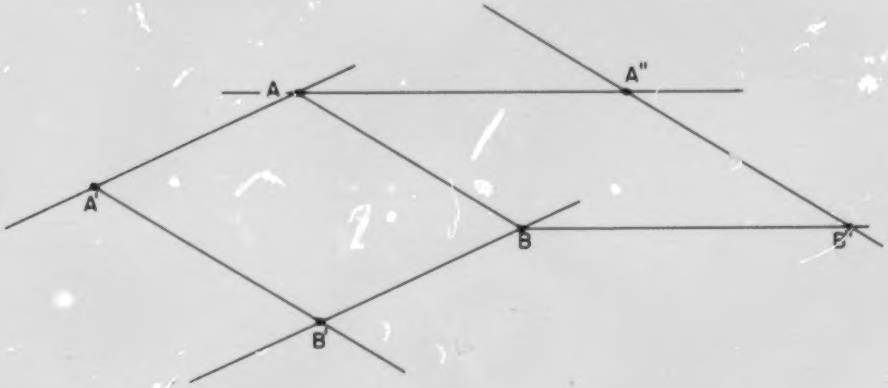
Part (c) of Exercise 6 illustrates a case of Desargues' theorem. This is one of two very fundamental theorems [the other — see Exercises 8 and 9 — is Pappus' theorem] which may suggest to you other exercises. Desargues' theorem is more easily described than stated formally. We begin by describing the kind of situation to which Desargues' theorem applies. Suppose given two triangles and a matching of the vertices of one with those of the other such that corresponding vertices are distinct and the lines containing corresponding sides are distinct. Suppose, further, that the lines which contain corresponding vertices are either parallel or concurrent. It follows [of course] that the lines which contain a pair of corresponding sides are coplanar and, so, either are parallel or intersect at a unique point.

In the case of parallelism the lines are distinct since, otherwise the same line would contain the vertices of two pairs of corresponding vertices. There are, then, two cases: for each of two pairs of corresponding sides, the lines containing those sides are two parallel lines, and for each of two pairs of corresponding sides, the lines containing these sides are two intersecting lines. In the second case, the points of intersection are distinct and, so, determine a line ℓ — otherwise, the same point would be each of a pair of corresponding vertices. Desargues' theorem, now, relates to the lines containing the remaining corresponding sides. It asserts that in the first case these lines are parallel and that in the second case they either are parallel to ℓ or intersect at a point of ℓ.

*

Answers for Exercises [cont.]

7. The student's completed diagram should look something like this:



(a)
(b) } [See the diagram.]

(c) $\overleftrightarrow{A'B'}$ and $\overleftrightarrow{AB}$ are parallel and have the same length. The rays $\overleftrightarrow{A'B'}$ and $\overleftrightarrow{AB}$ have the same sense.

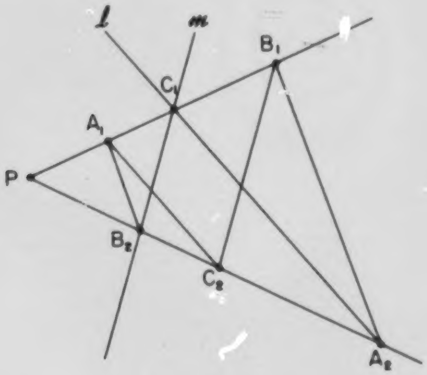
(d) [Same answers as in (a), (b), (c).]

(e) $\overleftrightarrow{A'A''}$ and $\overleftrightarrow{B'B''}$ are parallel and have the same length.

Parts (a) - (d) show, again, how to find points "at a given distance in a given sense" from given points — the points A′ and A″. Part (e) suggests that the points so found — B′ and B″ — are the same distance apart as the given points, and that the line containing them is parallel to the line containing the given points. In the terminology of the next section: A translation of $\mathcal{E}$ preserves distance and maps each line into a parallel line.

Answers for Exercises [cont.]

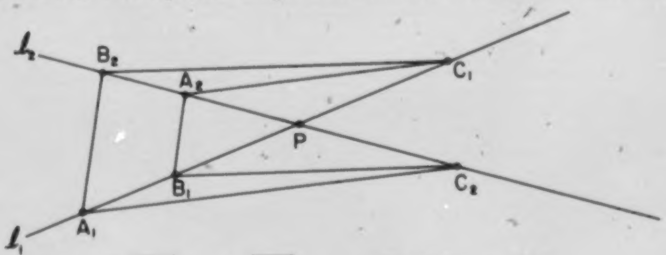
8. The student's completed diagram should look something like this:



(a)
(b) } [See the diagram.]
(d)

(d) $\overleftrightarrow{A_2B_1}$ and $\overleftrightarrow{A_1B_2}$ are parallel.

9. The student's completed diagram should look something like this:



The segments $\overleftrightarrow{A_2B_1}$ and $\overleftrightarrow{A_1B_2}$ are parallel.

* * *

Exercises 8 and 9 illustrate Pappus' theorem. Like Desargues', this is a theorem which may suggest to you many exercises. The kind of situation to which Pappus' theorem applies can be described in the following manner. Suppose one is given coplanar lines ℓ and ℓ', three points A, B, and C each of which is on ℓ but not on ℓ', and three points A′, B′, and C′ each of which is on ℓ' but not on ℓ. It follows [of course] that $A \neq C' \neq B \neq A' \neq C \neq B' \neq A$. Furthermore, the lines of the pair $\{\overleftrightarrow{A'C}, \overleftrightarrow{AC'}\}$ are coplanar, as are those of $\{\overleftrightarrow{B'C}, \overleftrightarrow{BC'}\}$ and those of $\{\overleftrightarrow{A'B}, \overleftrightarrow{AB'}\}$. So, as in the case of Desargues' theorem, there are two cases: for each of two pairs the lines of that pair are parallel, and: for each of two pairs, the lines of that pair intersect. In the second case the points of intersection are distinct and, so, determine a line m. Pappus' theorem asserts that in the first case the lines of the third pair are parallel and that in the second case these lines either are parallel to m or intersect at a point of m.

1.06 Translations of $\mathscr{E}$

In the preceding sections you have been improving your acquaintance with some simple geometric figures – rays, segments, half-planes, etc. – and reviewing some of what you know about functions [or: mappings]. Most of the time we used as examples functions whose domains and ranges consisted of real numbers – that is, functions which mapped real numbers on real numbers. In particular you studied some especially simple mappings – *translations of* $\mathscr{R}$ – of the set $\mathscr{R}$ of all real numbers onto itself. [Another name for these mappings is 'linear functions with slope 1'.]

Here are statements of some of the things you learned about translations of $\mathscr{R}$:

(a) For any real numbers a and b, there is just one translation of $\mathscr{R}$ which maps a on b. [In fact, this is the translation $\{(x,y)\colon y = x + (\_\_\_)\}$.]

(b) A resultant of translations of $\mathscr{R}$ is a translation of $\mathscr{R}$.

(c) Composition of translations of $\mathscr{R}$ is commutative.

(d) The converse of a translation of $\mathscr{R}$ is a translation of $\mathscr{R}$.

You also discovered that one way of describing translations of $\mathscr{R}$ is to say:

> A translation of $\mathscr{R}$ is a mapping f of $\mathscr{R}$ into itself such that, for any real numbers a and b, $f(a) - a = f(b) - b$ [and any such mapping is a translation of $\mathscr{R}$].

In short, the identity mapping of $\mathscr{R}$ onto itself is a translation of $\mathscr{R}$, and the other translations of $\mathscr{R}$ are those mappings each of which moves all points of the number line a given distance in a given sense.

The reason for having spent so much time on mappings is that in our study of geometry we shall make considerable use of functions which map points on points – that is, of mappings whose domains and ranges consist of points of $\mathscr{E}$. In particular, although points are very different from real numbers, we shall find that there are mappings of space onto itself which have properties similar to those which you have found for translations of $\mathscr{R}$. We shall call these mappings *translations of* $\mathscr{E}$, or, for short, merely 'translations'. The remainder of this chapter will be spent in becoming acquainted with these mappings.

One way of getting ideas about geometry is by drawing and studying pictures. In the preceding section you practiced picturing geometrical figures in a given plane by making drawings on a sheet of paper. We can begin to get acquainted with translations of $\mathscr{E}$ in a similar way. To begin with, our pictures will show us what a translation does to the points of a given plane. When we understand this we will be able to see how a translation acts on all of $\mathscr{E}$.

The purpose of this section is to lead students to become aware of certain mappings of Euclidean space $\mathscr{E}$ onto itself and of some of the properties of these mappings. The mappings in question are the identity mapping of $\mathscr{E}$ onto itself and those mappings each of which moves all points of $\mathscr{E}$ a given distance in a given sense. They are, then, analogous to translations of $\mathscr{R}$ and will be called *translations of* $\mathscr{E}$. The properties of these mappings which students will discover in this section are summarized on page 47. Of these, (1) - (3) will be the basis of the first three postulates which we shall adopt when, in Chapter 2, we begin our formal development of geometry; and (4) will suggest the most important part of our fourth postulate. [It should be apparent that section 1.06 is, like section 1.05, a concept-development section whose role is to help prepare students for the formal development.]

* * *

Student's curiosity as to the meaning of the word 'vector' should remain unsatisfied — at least as far as you are concerned — until this word is introduced later in the course. It is, however, appropriate now to relate, for you, the notion of a translation with the notion you may already have obtained of a vector from other sources. As you may recall, vectors are often introduced, in elementary texts on vector algebra, as directed segments which are represented pictorially by arrows. Next, two such vectors are said to be equivalent if and only if they have the same length and the same direction — in our terminology, the same sense. Finally, it may be mentioned — perhaps in a footnote — that a vector is not really a directed segment — rather, it is a class of equivalent directed segments. A better procedure which is sometimes followed is to begin by saying that a vector is *represented* by a directed segment, that directed segments with the same length and sense represent the same vector, and, finally, that a vector *is* the set of all directed segments which represent it. This latter procedure is logically unexceptionable, but may be a bit sticky, pedagogically.

The procedure used in this text to introduce translations is essentially the reverse of the method for introducing vectors outlined above. Note, first, with regard to the latter, that nothing but perhaps the pictorial representation would be changed essentially if one spoke of ordered pairs of points rather than of directed segments. Matching each directed segment with the ordered pair whose first and second components are the initial and terminal points of the directed segment, respectively, gives us a one-to-one correspondence between ordered pairs of points [with distinct components] and directed segments. So, in the context of the preceding paragraph, a vector might as well be defined as a set of "equivalent" ordered pairs of points. Since the equivalence relation in question is such that, given a point and a set of equivalent ordered pairs, the latter contains just one ordered pair which has the former as its first component it follows that a vector as so defined is a mapping of the set $\mathscr{E}$ of all points into itself. In fact, it is precisely a translation of $\mathscr{E}$. Also, each translation of $\mathscr{E}$ [with the exception of the identity mapping] is, in this sense of the word, a vector. [The identity mapping corresponds, of course, with the zero vector. In the usual developments the latter is somewhat mysterious since it is obvious that it is not, like other vectors, a set of directed segments. The former, on the other hand, while it may seem of a somewhat different nature than other translations, is just another mapping.] Since [as your students will discover] a translation is determined when one is given any point and the image of that point under the translation in question, a translation is determined by any one of the ordered pairs of points which are its members. Consequently, a translation may be represented pictorially by any arrow whose initial point represents the first component of some such ordered pair and whose terminal point represents the second component of this ordered pair.

It should now be evident that in our study of translations we shall end up with essentially the same results as are obtained through the usual elementary approach to vectors. The difference between our approach and the usual is that we begin with the notion of a translation and, on the basis of this, investigate the members of a translation, instead of beginning with the potential membership of vectors and obtaining the latter as certain subsets of the former. The principle advantages of our approach are pedagogical. One advantage is due to the fact that translations are very simple mappings — easily understood and forming an initially reasonable subject for investigation — while the ''construction'' of vectors by classifying directed segments is a somewhat abstract procedure and there seems to be no very convincing motivation, initially, for choosing the appropriate equivalence relation.

The similarity between the notion of translation and the elementary notion of vector might be taken as sufficient ground for referring to translations as vectors. The word 'vector' has, however, many meanings and our reasons for using the word in this connection are quite different. Although 'vector' is introduced much later in this course, it may be helpful to go into the matter now [but, of course, *not now in class*]. As you will recall, elementary vectors are subject to a binary operation called *addition* and, for each real number, to a singulary operation called *multiplication by* the real number in question. These operations have a number of familiar properties — for example, addition is associative and commutative, each of the singulary operations just mentioned is distributive over addition, and any two of these singulary operations are permutable. There is also a singulary operation of oppositing and a zero vector whose properties with relation to addition are those which oppositing of real numbers and 0 have with relation to addition of real numbers. Also, the operation oppositing coincides, as in the real number case, with multiplication by -1, multiplication by 0 always yields the zero vector, and the result of multiplying the zero vector by any real number is the zero vector. Because of all this there is an algebra of elementary vectors — of which the algebra of real numbers is a special case. The usefulness of elementary vectors is due to the existence of this algebra.

It turns out that, in addition to the set of all elementary vectors, there are many sets on which one can in natural ways define algebras of exactly the same kind as the algebra of elementary vectors. As is obvious from our earlier remarks, the set of all translations of $\mathcal{E}$ is one such example. Here the natural addition operation is function composition, oppositing is function inversion, and the zero element is the identity mapping. [Multiplication of translations by real numbers will be introduced in a later chapter after an appropriate concept-development section.] As a source of other examples, consider the set of all real-valued functions with a given domain. For the members of such a set, addition can be defined by saying that the sum of two real-valued functions with the given domain is the function whose value at any argument in that domain is the sum of the values of the given functions at that argument. Oppositing and multiplication by a real number can be defined in an analogous manner, and the zero element is the function whose value at each argument is 0. The resulting algebra turns out to be of the same kind as that of the elementary vectors. If, instead of considering all real-valued functions on a given domain, one considers only those of a subset of these which contains the zero function and is closed under the operations in question, one obtains another example of the same kind of algebra. For example, not only does the set of all real-valued functions whose domain is the set of all real numbers have an algebra of this kind, but, also, the set of all such functions as are represented by polynomials has such an algebra. As a final example — one to which we shall wish to refer shortly — the set of all ordered pairs of real numbers may be subjected to the same kind of algebra. To do so, define the sum of two such ordered pairs to be the ordered pair whose components are the sums of the components of the given pairs, define the opposite of an ordered pair to be the ordered pair whose components are the opposites of those of the given pair, etc.

In view of the multiplicity of interesting and useful examples it is desirable to have a name to use in referring to a set on which one has imposed the kind of algebraic structure which is exemplified by the usual algebra of elementary vectors. Since the latter is the prototype, and since geometric intuition is helpful in developing the theory of such objects, the natural term to use is 'vector space'. So, by definition, a vector space is a set on which one has imposed an algebraic structure like the usual algebra of elementary vectors. [Of course, the phrase modifying 'structure' in the preceding definition must be replaced by something more precise. This will be done as the course develops.]

Knowing, more or less, the meaning of 'vector space', it is natural to ask: What is a vector? The best answer we can give is that, in a given context, a vector is a member of a set which is, in that context, being dealt with as a vector space. So, for example, when we shall have succeeded in defining a vector space-structure on the set of translations of $\mathcal{E}$ we shall refer to translations as vectors. To exemplify the point still further, let's return to the set of all ordered pairs of real numbers. As we have seen, this set can ''be made into'' a vector space by giving appropriate definitions of the relevant operations. In a context in which these operations are of interest, we should refer to an ordered pair of real numbers as a vector. As just pointed out, in the context of this text we shall refer to a translation of $\mathcal{E}$ as a vector. Does this mean that a translation is an ordered pair of real numbers? Of course not. What it means is that the word 'vector' has different meanings in different contexts.

Recall that we have defined the number line to be the set of all real numbers with the structure imposed by defining the distance between real numbers to be their absolute difference. Similarly, we shall take the number plane to be the set of all ordered pairs of real numbers with the structure imposed by using the Pythagorean formula to define the distance between ordered pairs. [In terms of this distance we can define such geometric terms as 'straight line', 'circle', 'angle measure', etc.] In this context it is natural to refer to an ordered pair of real

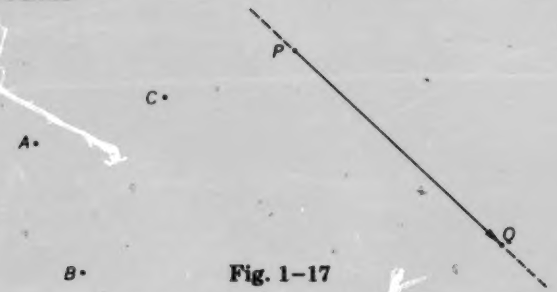


Fig. 1–17

One way of picturing the effect of a translation is to make use of a tracing sheet. As a first example, consider Fig. 1 – 17 as a picture of three points, *A*, *B*, and *C* and a segment $\overrightarrow{PQ}$, all in some plane π. [The arrowhead drawn at *Q* is part of the instructions of the exercise.] Copy the figure and trace your drawing on a tracing sheet. As you trace the letters, draw a star [*] after each tracing. [Read '*A**' as '*A* star'.] You will now have two copies of Fig. 1 – 17. Your work should look like this:

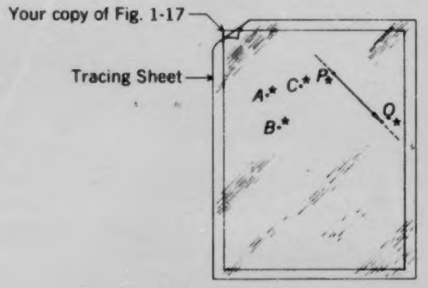


Fig. 1–18

Now slide the tracing sheet along the line through *P* and *Q* until the point *P** lies over the point *Q*. Clip the tracing sheet and paper together. Your work should look like this:

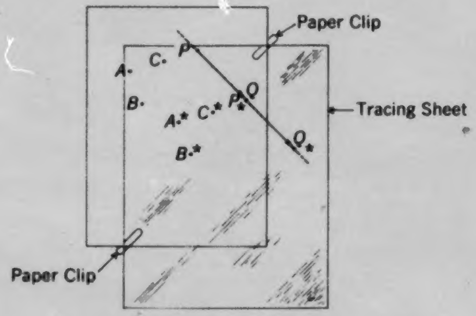


Fig. 1–19

numbers as a point. But, in the preceding paragraph we have said that an ordered pair of real numbers may be referred to as a vector. Does this mean that points and vectors are the same? Of course not. In different contexts we use different words in referring to ordered pairs. In one context we may use 'point', in another 'vector', and in a third 'complex number'. Which word we use will be determined not by the objects we are dealing with, but by the structure we have imposed on the set of these objects.

TC 38 (1)

In regard to the tracing sheets referred to in the text, various kinds of materials are readily available for this purpose. Among them are ordinary tracing paper, onionskin paper, velum, plain notebook paper, and clear acetate sheets. With the latter, the students will have to use grease pencils to make the tracings. [In the early trials of this text, acetate sheets were used. Later, we found that any appropriate tracing materials served just as well as the acetate sheets.]

The use of a tracing sheet, as described in the text and illustrated in Figures 1-17 and 1-18, seems to give students adequate intuitions as to how a translation acts on points of $\mathcal{E}$. It also leads rapidly to the discovery of the properties of translations which are of immediate importance. For example, the exercises of Part A on page 39 single out translations as those mappings of $\mathcal{E}$ into itself which move each point the same distance in the same sense. And, they show that any translation can be described pictorially by an arrow whose length shows the common distance through which points are moved and whose sense is the sense of the motion. The exercises of Part B call attention to the fact that translations map segments into segments of the same length and map each line into a parallel line.

For our purposes the only inadequacies of a tracing sheet are due to the fact that the portion of space dealt with in making use of the tracing sheet is subset of a single plane and is, in fact, a ''very small'' subset of a plane. Students' work in section 1.05 should make up for the second of these inadequacies — they have learned by now how a plane differs from the model which a sheet of paper may be thought to be. The fact that the domain of a translation is all of 3-dimensional space is brought out in the paragraph immediately preceding the exercises and, also, in Exercise 5 of Part A. One way of stressing this in class is to make a fold, parallel to the line of the arrow in both the tracing sheet and the underlying paper, in order to show how a translation acts on a pair of planes. [The need for making the fold parallel to the direction of the translation may bring out the fact that a translation maps certain planes into themselves and maps other planes into planes parallel to them. It is not essential that students become aware of this now; but, if they do, they will have a clearer notion of what a translation is.] Another way of indicating that we are talking of 3-dimensional space is to use a pencil held in one hand as an arrow describing a translation, and the tip of a finger of the other hand to indicate, first, a given point and, then, after moving the fingers appropriately, the image of this point under the given translation. You can check for understanding by asking students to do this.

Other devices will no doubt occur to you, including various uses for an overhead projector. Many teachers have found it helpful to illustrate on an overhead projector the "sliding" procedure described in the text. An initial demonstration of sliding a tracing helps to avoid procedural errors later on.

One final warning. Note that although it is helpful to speak of a translation as "moving" points onto their images, it is literal nonsense to speak of points of space actually changing position. A point of space is where it is. The motion of the tracing sheet when it is used to show the result of a translation *may* prove misleading. It is for the purpose of lessening the likelihood of this that we ask students to use '☆'s in labeling marks they make on the tracing sheet and '′'s in labeling the marks on the paper which show the images of points. Note that the paper does not move. The movement of the tracing sheet is, really, only a device for locating images.

Many of the following exercises can profitably be done, individually, by students in the classroom. Hopefully, the class will be small enough so that you can check work, answer questions, and offer advice on an individual basis. This should certainly, if at all possible, be done for the preliminary drawing of Fig. 1-17 which is described in the text preceding the exercises.

By pushing a pin through the tracing sheet, make holes in the paper under the points $A*$, $B*$, and $C*$. Remove the tracing, mark the pin-holes in the paper with your pencil, and label them 'A'', 'B'', and 'C'', respectively. Your picture now shows three points, A, B, and C, and their images, A', B', and C' under a mapping which moves each point the same distance – the distance between P and Q, in the same sense – the sense of the ray $\overrightarrow{PQ}$. The mapping which moves each point of space in this same way is called a *translation*.

Given any point R in space, there is a plane which contains P, Q, and R. So, you could picture P, Q, and R on a sheet of paper and, repeating what you did with the tracing sheet, find the image of R under the same translation. Evidently, to find the image of any given point under this translation, all you need know are the locations of P and Q and that the sense of the translation – as indicated by the arrowhead – is that of the ray $\overrightarrow{PQ}$. So, we can describe the translation we have been discussing as *the translation from P to Q*.

Exercises

Part A

1. Mark a point D on your drawing and use your tracing sheet to find image, D', of D under the translation from P to Q.
2. (a) Compare the lengths of the segments $\overrightarrow{PQ}$, $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, $\overrightarrow{CC'}$, and $\overrightarrow{DD'}$. What may you say about the distance between any point and its image under the translation from P to Q?

   (b) Is there any point which is its own image under the translation from P to Q?
3. (a) Use your parallel ruler to draw the line through B parallel to $\overrightarrow{PQ}$. What do you guess about the line through a given point and its image under the translation from P to Q?

   (b) Check your guess in the cases where the given point is A, C, D, P, or Q.
4. Recall that two rays which "point the same way" are said to have the same sense. Would you say that the rays $\overrightarrow{AA'}$ and $\overrightarrow{PQ}$ have the same sense? How about $\overrightarrow{BB'}$ and $\overrightarrow{PQ}$? $\overrightarrow{AA'}$ and $\overrightarrow{BB'}$?
5. In the preceding exercises we have been dealing with points all of which lie in a single plane. We pointed out, just preceding the exercises, that the domain of the translation from P to Q is $\mathscr{E}$. Suppose, now, that R is any point not in the plane we have been considering and that its image under the translation from P to Q is R'.

   (a) How may you describe the distance between R and R'?

   (b) How may you describe the sense of the ray $\overrightarrow{RR'}$?

Answers for Part A

1. [While following the directions earlier in the text, students will have marked points A', B', and C' on the work-sheets in such a way that if arrows were drawn from A to A', from B to B', and from C to C', these would be of the same length and point the same way as the arrow from P to Q. On completing the present exercise they will have marked points of some other such pair (D, D').]
2. (a) The distance between any point and its image is the same as that between P and Q. [The pictured segments should turn out to have substantially the same lengths when students measure them with a ruler.]

   (b) No. [The distance between any such point and its image would be 0 and, so, unequal to the distance between P and Q.]
3. (a) The line through any point and its image is parallel to $\overleftrightarrow{PQ}$.

   (b) On completing this a student's worksheet will contain drawings of at least four parallel lines, $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$, and $\overleftrightarrow{PQ}$. There will also be a fifth such line unless $\overleftrightarrow{DD'}$ coincides with one of these four.]
4. Yes. [three times] The ray from any point through the image of that point has the same sense as $\overrightarrow{PQ}$ [and any two such rays have the same sense.]
5. (a) The distance between R and R' is that between P and Q.

   (b) The sense of $\overrightarrow{RR'}$ is that of $\overrightarrow{PQ}$.

TC 40 (1)

Answers for Part B

1. (a) [One procedure a student might use is to draw the segment $\overline{AC}$ on his worksheet and use a tracing sheet to find the image of the entire segment. Another student might mark one or two points between A and C and make the intended discovery by finding only their images.]

   (b) [The picture should be that of the segment $\overline{A'C'}$.]

   (c) The image of $\overline{AC}$ is the segment whose end points are the images of A' and C'. [The discovery summarized in the answer to part (c) actually amounts to two intuitions. The first is that the image of any point of $\overline{AC}$ is a point of $\overline{A'C'}$. The second is that each point of $\overline{A'C'}$ is the image of some point of $\overline{AC}$. It should not be surprising to find a student who, at this point, is ready to accept the first but is not quite ready to accept the second. (Such a student should be cherished!) If this occurs, you have the choice between leaving the question open for the student to think about and giving him a hint

Part B

1. (a) Use your tracing sheet to help you see the images of points between A and C under the translation from P to Q.
   (b) Picture [on your drawing] the set of images of the points of the segment $\overline{AC}$.
   (c) Describe this set of images.
2. (a) The set of all images of points of $\overline{AC}$ under the translation from P to Q is called *the image of* $\overline{AC}$. In Exercise 1 you have seen that the image of $\overline{AC}$ is $\overline{A'C'}$. What is the image of $\overline{AB}$?
   (b) What is the image of the ray $\overrightarrow{AC}$ under the translation from P to Q? [Use your tracing sheet to check your answer.]
   (c) What is the image of the line $\overleftrightarrow{AC}$ under the translation from P to Q?
3. (a) In Exercise 2 you have seen that the translation from P to Q maps the line $\overleftrightarrow{AC}$ onto the line $\overleftrightarrow{A'C'}$. From your drawing, you should be able to make a guess about these two lines. What guess?
   (b) Use your parallel ruler to draw the line through B' which is parallel to $\overleftrightarrow{BC}$. The result should give you evidence in favor of the guess you made in part (a). What is this evidence?

Part C

1. (a) On your drawing, use your parallel ruler to draw the line through Q parallel to $\overleftrightarrow{BP}$.
   (b) In Exercise 3 of Part B you probably guessed that the translation from P to Q maps each line onto a line parallel to it. Does your work in part (a) give you more evidence for this conjecture? [Of what point is Q the image under this translation from P to Q?]
   (c) Check your conjecture, again, by drawing the line through C' parallel to $\overleftrightarrow{CP}$.
   (d) To check a different case of your conjecture, consider the line $\overleftrightarrow{PQ}$. What line is its image under the translation from P to Q? Is this line parallel to $\overleftrightarrow{PQ}$?
   (e) Suppose that R is a point on $\overleftrightarrow{BB'}$ which is different from B. What line is the image of $\overleftrightarrow{BR}$ under the translation from P to Q? Is this line parallel to $\overleftrightarrow{BB'}$?

* 2. By now you should be convinced that the translation from P to Q maps each line onto a parallel line. You can use this knowledge about translations to confirm a guess which you made in Part A. This guess was that the line through a given point—for example, the point A—and the image of this point under the translation from P to Q is parallel to $\overleftrightarrow{PQ}$. To confirm this guess, suppose that the translation does map each line onto a parallel line. It follows that the translation maps $\overleftrightarrow{AA'}$ onto a parallel line through A'. So,

such as 'What happens to $\overline{A'C'}$ under the translation from Q to P?'. If you adopt the former course, you can make sure that the question is settled when you discuss Part H. Since, as is pointed out there, the translation from Q to P is the inverse of the translation from P to Q, and since we are agreed that a translation maps a segment onto [at least] a subset of a segment, we can see that the first mentioned translation maps any given point of $\overline{A'C'}$ on some point of $\overline{AC}$ and, so, be convinced that the translation from P to Q maps this latter point on the given point of $\overline{A'C'}$. Hence, each point of $\overline{A'C'}$ is the image, under the translation from P to Q, of some point of $\overline{AC}$.]

Answers for Part B [cont.]

2. (a) $\overline{A'B'}$ [The point of this exercise is not so much the answer as it is to introduce the notion of the image of a set, as opposed to that of the image of a point.]
   (b) $\overrightarrow{A'C'}$
   (c) $\overleftrightarrow{A'C'}$ [Here, and in part (b), one might raise the same question as we did in the discussion of Exercise 1(c) and answer it in the same way.]
3. (a) that $\overleftrightarrow{A'C'}$ and $\overleftrightarrow{AC}$ are parallel.
   (b) The line drawn should go through the mark representing the point C', thus suggesting that $\overleftrightarrow{B'C'}$ is parallel to $\overleftrightarrow{BC}$.

Answers for Part C

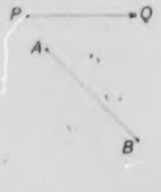
1. (a) [The students' drawings should indicate that B' belongs to the line through Q parallel to $\overleftrightarrow{BP}$. Students should begin to realize the connection of these exercises with some of those at the end of the preceding section. In the latter exercises they learned how to describe the point at a given distance, in a given sense, from a given point as the intersection of two lines which were parallel to lines determined by the given data. In short, students should realize that they might have known that B' is the point of intersection of the line through Q parallel to $\overleftrightarrow{PB}$ and the line through B parallel to $\overleftrightarrow{PQ}$.
   (b) Since, under the translation from P to Q, B' is the image of B and Q is the image of P it is to be expected [from Exercise 3 of Part B] that $\overleftrightarrow{B'Q}$ is the line through Q which is parallel to $\overleftrightarrow{BP}$. Since the work in part (a) seems to show that this is the case, this work does give additional evidence for the conjecture.
   (c) [The drawing should show that Q is a point of the line through C' parallel to $\overleftrightarrow{CP}$.]
   (d) $\overleftrightarrow{PQ}$ is its own image and is parallel to itself.
   (e) $\overleftrightarrow{BR}$ is $\overleftrightarrow{BB'}$, is its own image, and is parallel to itself.

the translation maps $\overleftrightarrow{AA'}$ onto itself. [Explain.] For the same reason, the translation maps $\overleftrightarrow{PQ}$ onto itself. It follows that any point in $\overleftrightarrow{AA'} \cap \overleftrightarrow{PQ}$ would have for its image a point in $\overleftrightarrow{AA'} \cap \overleftrightarrow{PQ}$. [Explain.] Since no point is its own image it follows that if $\overleftrightarrow{AA'} \cap \overleftrightarrow{PQ}$ is not empty then it contains at least two points. If $\overleftrightarrow{AA'} \cap \overleftrightarrow{PQ}$ is empty, then, since $\overleftrightarrow{AA'}$ and $\overleftrightarrow{PQ}$ are coplanar, $\overleftrightarrow{AA'} \parallel \overleftrightarrow{PQ}$. If $\overleftrightarrow{AA'} \cap \overleftrightarrow{PQ}$ contains two points then $\overleftrightarrow{AA'} = \overleftrightarrow{PQ}$ and, so, $\overleftrightarrow{AA'} \parallel \overleftrightarrow{PQ}$. So, in any case, $\overleftrightarrow{AA'} \parallel \overleftrightarrow{PQ}$.

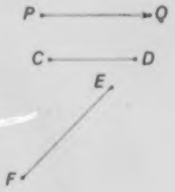
Part D

1. Below, you are given some figures together with arrows which describe the translation from P to Q. Your job is to sketch the images of the given figures under this translation.

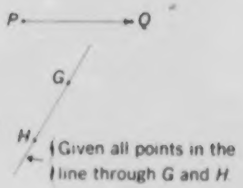
(a)



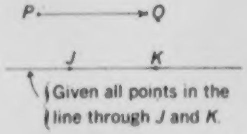
(b)



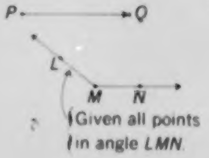
(c)



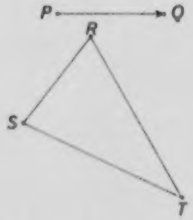
(d)



(e)



(f)



Answers for Part C [cont.]

☆2. [The two explanations asked for are, first, that since, by definition, a line is parallel to itself and two parallel lines are disjoint, the unique line through A′ which is parallel to $\overleftrightarrow{AA'}$ is $\overleftrightarrow{AA'}$; and, second, that since each of $\overleftrightarrow{AA'}$ and $\overleftrightarrow{PQ}$ is its own image, any point which belongs to both lines must have its image in both lines.]

Students may wonder what mappings other than translations map each line into a line parallel to it. Aside from the identity mapping [which is, in fact, eventually included among translations] and the constant mappings [each of which maps all points of ℰ on some single point] the only such mappings are the "uniform stretchings" about one or another fixed point. For such a mapping, the lines which are mapped into themselves are just the lines through the fixed point — rather than, as in the case of a translation, a family of parallel lines.

* * *

The real point of the optional Exercise 2 is that it shows that any mapping of ℰ into itself which maps each line into a parallel line and leaves no point fixed has the property that all lines each of which contains the image of one of its points are parallel. It follows that if a mapping is known to leave no point fixed and to map each line into a parallel line then, given the image Q of any given point P the image of any point A [not on $\overleftrightarrow{PQ}$] is the intersection of the line through Q parallel to $\overleftrightarrow{AP}$ and the line through A parallel to $\overleftrightarrow{PQ}$. The intuitions developed in working the preceding exercises [including those in section 1.05] strongly suggest that the mapping in question is none other than the translation from P to Q. This is, in fact, the case — the translations of ℰ [other than the identity mapping] are precisely those mappings of ℰ into itself which leave no point fixed and which map each line into a line parallel to it.

The preceding suggests an alternative definition of 'translation' which is simpler than the description ["moves all points the same distance in the same sense"] which we are using. This simpler definition has, however, the pedagogical disadvantage — always to be expected of simpler definitions — that one has to work with it longer before coming to appreciate its significance. So, for the text, we have adopted a description in terms of 'distance' and 'sense'. In the appendix referred to in the second paragraph of the commentary for section 1.05, we adopt the simpler definition and show that it [and the postulates adopted in the appendix] imply the desired theorems concerning translations.

*

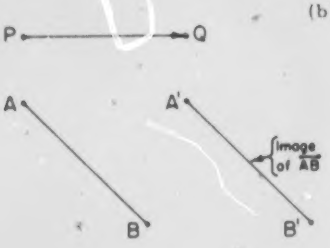
Answers for Part D

Each exercise in Part D can be checked with a tracing. A tracing also provides a very efficient way to resolve arguments.

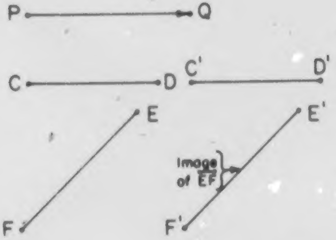
1. [If you have your students make careful drawings of the image sets in question, their answers should compare favorably with the diagrams given below. In any case the students should note that segments go into parallel segments of the same length, lines go into parallel lines, rays go into similarly sensed rays, noncollinear points go into noncollinear points, etc.]

TC 41 (2)

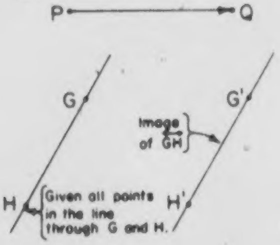
(a)


P
Q
A
A'
Image of $\overline{AB}$
B
B'

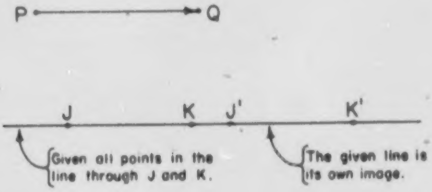
(b)


P
Q
C
D
C'
D'
E
E'
Image of $\overline{EF}$
F
F'

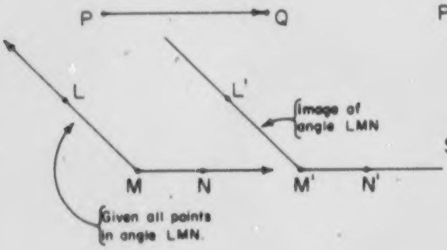
(c)


P
Q
G
G'
Image of $\overleftrightarrow{GH}$
H
H'
Given all points in the line through G and H.

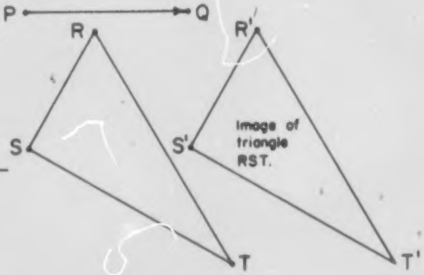
(d)


P
Q
J
K
J'
K'
Given all points in the line through J and K.
The given line is its own image.

(e)

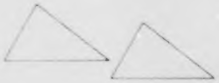

P
Q
L
L'
Image of angle LMN
M
N
M'
N'
Given all points in angle LMN.

(f)

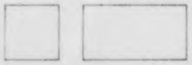

P
Q
R
R'
S
S'
Image of triangle RST.
T
T'

2. In each of the following, you are given two figures. You are to decide whether one of the figures is the image of the other *under some translation*. If you think that this is the case, draw an arrow from some point to its image under that translation. If you think that it is not the case, give a reason.

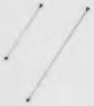
(a)



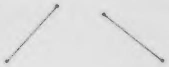
(b)



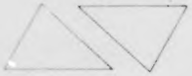
(c)



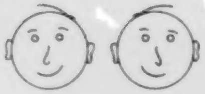
(d)



(e)



(f)



Part E

1. On a sheet of paper mark three noncollinear points P, Q, and R, two or three inches apart. Use your parallel ruler to draw the line through R parallel to $\overleftrightarrow{PQ}$ and the line through Q parallel to $\overleftrightarrow{PR}$. Mark the intersection.
2. On your picture, fill in the segment $\overleftrightarrow{PQ}$ and draw an arrowhead at Q. Now, with the aid of your tracing sheet, find the image of R under the translation from P to Q.
3. Compare the results you obtained in Exercises 1 and 2. Relate your finding to your previous discoveries.
4. (a) In Exercise 2 you have seen that, given two points P and Q, the image under the translation from P to Q of any point not on $\overleftrightarrow{PQ}$ is the point of intersection of two lines which you can draw with the help of your parallel ruler. Does this method work if you wish to find the image of a point on $\overleftrightarrow{PQ}$? Explain.
   (*)(b) Figure out how to use your parallel ruler to find the image, under the translation from P to Q, of a point on $\overleftrightarrow{PQ}$.

2. (a) Yes. The student should draw an arrow from a point of one of the triangles to the corresponding point in the other. The easiest points to choose in this regard are corresponding vertices. [Some students will have arrows drawn from the triangles on the left to the one on the right while others will have arrows drawn from the triangle on the right to the one on the left. Of course, the latter describe the inverse of the former, and conversely.]

   (b) No. [Translations preserve distance.]

   (c) No. [Translations preserve distance.]

   (d) No. [The image of a segment under a translation is parallel to the segment.]

2. (e) No. [Notice that corresponding sides are parallel and of the same length. But, a translation maps a ray onto a ray with the same sense.]

   (f) No. [Translations preserve distance. The given figures are mirror images of one another.]

Answers for Part E

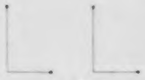
1. [This exercise amounts to using the parallel-line construction to locate the image of R under the translation from P to Q.]
2. [This exercise is to check, using a tracing sheet, that the point located in Exercise 1 is, in fact, the image of R under the translation from P to Q.]
3. [The purpose of this exercise is to give you a check on whether students do realize that the parallel-line construction does work. In case they don't as yet realize this, we tell them in the following exercise.]
4. (a) If $R \in \overleftrightarrow{PQ}$ then the image, R', of R under the translation from P to Q is, as before, on the line through Q parallel to $\overleftrightarrow{PR}$ and also on the line through R parallel to $\overleftrightarrow{PQ}$. But, in this case, these are the same line. So, in this case the method does not serve to determine R'.

   (b) [This optional exercise appears again as Exercise 7 of Part F. By that time most students should be able to figure out the simple trick: Choose any point S not on $\overleftrightarrow{PQ}$ and find its image S' under the translation from P to Q. The translation from S to S' is the same as the translation from P to Q. So, to find the image under the latter of a point R on $\overleftrightarrow{PQ}$ it is sufficient to find the image of R under the translation from S to S'. So, two parallel-line constructions suffice — one to find S', the other to find R'.]

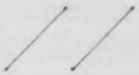
Part F

1. Mark four points *A*, *B*, *C*, and *D* on a sheet of paper and use a tracing sheet to find the images *A′* and *B′* of *A* and *B* under the translation from *C* to *D*.
2. Use your tracing sheet to find the image of *B* under the translation from *A* to *A′*. What point is this image?
3. Can you find a point whose image under the translation from *A* to *A′* is different from its image under the translation from *C* to *D*?
4. As you know, you can describe on your paper the translation from *C* to *D* by drawing an arrow from the mark for *C* to the mark for *D*. Of course, you can describe the translation from *A* to *A′* by drawing another arrow. Draw these two arrows. You saw in Exercise 3 that the translation from *A* to *A′* is the same mapping as the translation from *C* to *D*. So, your two arrows describe the same translation. Draw three more arrows which describe this translation. [The quickest way is to use your parallel ruler.]
5. In each of the following, you are given two figures and the information that one of the figures is the image of the other under a certain translation. In each case, draw three arrows from points to their images which describe the required translation.

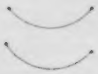
(a)



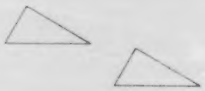
(b)



(c)



(d)



6. (a) Describe five translations by drawing five arrows.
   (b) Describe the same five translations by drawing five other arrows.
7. If you did Exercise 4(b) of Part E you were probably already aware of the answer to Exercise 3, above. Whether or not you were able to do Exercise 4(b), review it now and make sure you can do it.

Part G

1. Return to your work for Exercise 1 of Part F. Measure the distance between *A* and *B* and the distance between *A′* and *B′*. What do you guess to be true concerning the distance between two points and the distance between their images under a given translation?

Answers for Part F

1, 2. The image of B under the translation from A to A′ is its image, B′, under the translation from C to D.

3. No. The translation from A to A′ is the same mapping as the translation from C to D.

4. [Any three arrows which have the same length and the same sense as the arrow from C to D will do.]

5. [Any three arrows from points in one of the figures to the corresponding points in the other will do. These arrows will have the same length and same sense. The point of this exercise (and others like it) is to reinforce the notion that a translation may be described by any arrow from a point to its image under that translation.]

6. (a) [Any five arrows will do as long as no two have both the same length and the same sense.]

   (b) [Any five arrows which are "similar", respectively, to those drawn in answer to part (a).]

7. [See the earlier discussion of Exercise 4 of Part E.]

Answers for Part G

1. The distance between two points is the same as the distance between their images.

2. Mark another point, E, on your paper and find its image E' under the translation from C to D. Compare the lengths of $\overline{AE}$ and $\overline{A'E'}$ and those of $\overline{BE}$ and $\overline{B'E'}$. Do your results confirm the guess you you made in Exercise 1?
3. There are three other pairs of segments, whose end points are marked on your paper, which you can use to check your guess. So use them.
4. Draw a line l on another sheet of paper. Mark four points of l, A and B, two inches apart, and C and D, three inches apart. Use your parallel ruler – but, don't use the scale markings on it – to mark a point on l which is five inches from C. In a similar way, mark a point of l which is one inch from C.

*

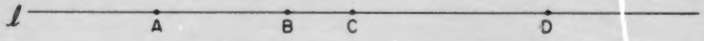
In Part A you have seen that, for any two points P and Q, there is a mapping of $\mathcal{E}$ into itself – the translation from P to Q – which maps P on Q and which moves all points the same distance [Exercise 2] in the same sense [Exercise 4]. You have seen in Parts B and C that such a translation maps each line into a parallel line. In Exercise 2 of Part C you have seen that this property of translations, together with the fact that no point is its own image, implies that the lines which join points and their images are parallel. In Part E you have seen how these two parallelism properties enable you to use your parallel ruler to mark images of given points under a given translation. In Part F you have seen that a translation can be described by giving any point and its image under the translation. In particular, there is just one translation which maps one given point on another. In Part G you have seen that a translation preserves distance. One consequence of this is that each translation is a one-to-one mapping.

You learned earlier that functions which are one-to-one have inverses – that is, have converses which are functions. Since each translation is a one-to-one mapping, it follows that each translation has an inverse.

Part H

1. You may have realized before this that a translation has an inverse. At any rate, with the help of your tracing sheet you can show even more than this. Mark two points – A and B – on a sheet of paper and draw the line $\overleftrightarrow{AB}$. Mark a third point – say, C. Use your tracing sheet to find the image C' of C under the translation from A to B. Now, use your tracing sheet to find the image of C' under the translation from B to A. What point is this?
2. Repeat Exercise 1, but this time apply the translation from B to A first, and then the translation from A to B. Where do you end up?

2. [They should.]
3. [The results of this exercise should confirm the answer given for Exercise 1.]
4. [Analysis: To find a point 5 inches from C. Assuming that the points are so chosen that the rays $\overline{AB}$ and $\overline{CD}$ have the same sense,

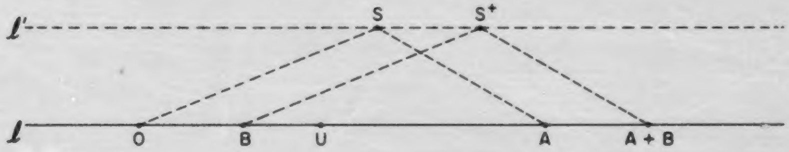


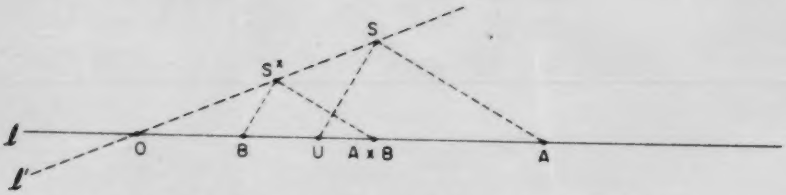
what is wanted is the image of D under the translation which maps A on B. (If the rays have opposite senses, we want the image of D under the translation which maps B on A.) Choose a point S not on l and find its image S' under the translation from A to B. Then find the image of D under the translation from S to S'.

To find a point 1 inch from C. Assuming that the points are so chosen that the rays $\overline{AB}$ and $\overline{CD}$ have the same sense, what is wanted is the image of D under the translation which maps B on A.

* * *

Exercise 4 shows how, with the aid of translations, one may "add" [or "subtract"] collinear segments. Using uniform stretching [see the discussion of Exercise 2 of Part C] one may also "multiply" collinear segments — supposing that one has previously chosen a unit segment. As an illustration and application of these procedures we show, below, how, having chosen an origin O and a unit-point U on a line l one can define addition and multiplication operations for points of l in such a way that if l is the number line, O is 0, and U is 1, these operations are the usual operations for real numbers. As in Exercise 4, S is an arbitrary point not on l. In the first of the following figures, S^+ is the image of S under the translation which maps O on B; while in the second, $S^\times$ is the image of S under the uniform stretching which maps U on B. Using these points, we find [as in Exercise 4] the point A + B which is the image of A under the translation which maps O on B and [in a similar manner] the point A × B which is the image of A under the uniform stretching which maps U on B. In the first case, l' is the line through S parallel to l; in the second l' is the line





through S which contains O. In the first case $\overleftrightarrow{BS^+} \parallel \overleftrightarrow{OS}$; in the second case $\overleftrightarrow{BS^\times} \parallel \overleftrightarrow{US}$. In the first case A + B is on the line through S^+ parallel to $\overleftrightarrow{SA}$; in the second case A × B is on the line through $S^\times$ parallel to $\overleftrightarrow{SA}$.

You will find it interesting to add to the above figures the constructions for B + A and B × A. It should turn out, of course, that B + A = A + B and B × A = A × B. If you do as suggested, you will find that the commutativity of addition and multiplication, when these operations are defined as above, is a consequence of Pappus' theorem. The fact that the operations depend only on O and U, and not on the choice of the point S, is a consequence of Desargues' theorem. This is also easy to see if you carry out the appropriate construction. Recall that Desargues' and Pappus' theorems are discussed in the commentary for Exercises 6, 8, and 9 of section 1.05.]

Warning: In the "algebra of points and translations" which we shall use in developing geometry [see section 1.07] we shall not use these addition and multiplication operations for points. If you show them to students now, you may later have to cope with students who wish to include them in this algebra.

*

Answers for Part H

1. The image of C′ under the translation from B to A is C.
2. At C.

3. In Exercise 1 you found that the resultant of two mappings – the translation from A to B followed by the translation from B to A – maps C on itself. Is there any point which is not mapped on itself by this resultant?
4. In Exercise 2 you found that the resultant of the translation from B to A followed by the translation from A to B maps C on itself. Is there any point which is not mapped on itself by *this* resultant?
5. In an earlier section you saw that if f and g are mappings such that $f \circ g$ maps each element of the domain of g on itself, and $g \circ f$ maps each element of the domain of f on itself, then each of f and g is the inverse of the other. What has this to do with the results of Exercises 3 and 4?
6. As you have known all along, the domain of any translation is $\mathcal{E}$. You have also known that the range of any translation is a subset of $\mathcal{E}$. What additional information do you now have about the range of a translation?

Part I

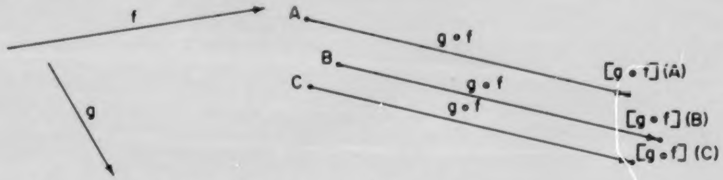
1. [Read this exercise all the way through *before* starting to work it.] Draw two arrows – one about 2 inches long and one about 1 inch long – so that the lines containing them are not parallel. These arrows describe two translations. Let's call the first translation 'f' and the second translation 'g'. You are going to study the mapping $g \circ f$. Since you will use your tracing sheet, it will be well to draw, lightly, the lines which contain your arrows. Mark three points – A, B, and C – on your paper. Cover your paper with the tracing sheet and mark on it the dots $A*$, $B*$, and $C*$ which lie above your marks for A, B, and C, and trace the arrow which you drew to describe f. Now, slide the tracing sheet so that the points $A*$, $B*$, and $C*$ are above the images $f(A)$, $f(B)$, and $f(C)$ of A, B, and C under the translation f.

   Next, being careful not to move the tracing sheet, copy the arrow you drew to represent g. Now, slide the sheet so that $A*$, $B*$, and $C*$ lie over the images $g(f(A))$, $g(f(B))$, and $g(f(C))$ of $f(A)$, $f(B)$, and $f(C)$ under the translation g. Use your pin to pinpoint these images on your paper. Remove the tracing sheet, mark the images, and label them '$[g \circ f](A)$', '$[g \circ f](B)$', and '$[g \circ f](C)$'.
2. There is, of course, a translation from A to $[g \circ f](A)$. Draw an arrow which describes this translation. Repeat, with 'B' for 'A' and, again, with 'C' for 'A'.
3. You now have three different arrows [in addition to the two you drew to describe f and g]. Do these three arrows describe different translations? [You can check by using either your parallel ruler or your tracing sheet.]
4. Is there a point D such that the translation from D to $[g \circ f](D)$ is different from the translation from A to $[g \circ f](A)$?

3. No.
4. No.
5. The translation from A to B and the translation from B to A both have $\mathcal{E}$ as domain. From Exercises 3 and 4 we know that the resultant of these translations, in either order, is the identity mapping of $\mathcal{E}$ onto itself. Hence, by the result quoted in this exercise. each of these translations is the inverse of the other.
6. The range of any translation is $\mathcal{E}$. [This question may surprise some students who have been taking the answer for granted. On the other hand, see an earlier comment on Exercise 1 of Part B.]

Answers for Part I

1, 2. A students' drawing might look like this:



3. No, they describe the same translation.
4. No. [Students may wish to do more drawing to be sure of this.]

5. Yes. [In terms of the operations with the tracing sheet, this should be obvious.]
6. If the identity mapping $i_{\mathcal{E}}$ is counted as a translation then, by Exercises 4 and 5 and the definition of 'inverse', the resultant of any two of the mappings we have previously called translations will be a translation. Also, if f is any of these mappings then $f \circ i_{\mathcal{E}} = f$ and $i_{\mathcal{E}} \circ f = f$. Since, finally, $i_{\mathcal{E}} \circ i_{\mathcal{E}} = i_{\mathcal{E}}$ it follows that if we include $i_{\mathcal{E}}$ among the translations then a resultant of any two translations [or, of any translation and itself] is a translation.
7. [This exercise should promote some classroom discussion. To find an arrow which represents $f_2 \circ f_1$ the only way to start is to mark a point P, find the location of $f_2(f_1(P))$ somehow, and draw the arrow from P to this latter point. One way to do this is to use a tracing sheets as in Exercise 1, but shifting the tracing sheet twice was probably not very easy. Could we do the job with just one shift of the tracing sheet? The first shifting of the sheet in Exercise 1 was for the purpose of finding images of points under the first applied translation [the translation f_1]. So, we could avoid this if we could choose for P a point whose image under f_1

5. One effect of our instructions to draw the arrows describing f and g so that the lines containing them are not parallel is to rule out the case in which f and g are inverses of one another. Suppose f and g are translations which move points in the same sense, or which move points in opposite senses but are not inverses. In this case, would $g \circ f$ be a translation?
6. You have probably concluded in Exercises 4 and 5 that if f and g are translations which are not inverses of one another then $g \circ f$ is also a translation. If we include one other mapping of $\mathscr{E}$ onto itself among the translations of $\mathscr{E}$ [which mapping?], it will follow that the resultant of *any* two translations of $\mathscr{E}$ is a translation of $\mathscr{E}$. Explain.
7. A resultant of translations which are not inverses of each other is a translation which is not the identity mapping and, so, can be represented by an arrow. Suppose you are given arrows which describe translations f_1 and f_2 which are not inverses of each other. What is the easiest way you can think of to draw an arrow which describes the translation $f_2 \circ f_1$?

*

In Part I you have seen that if, as we shall, we include the identity mapping of $\mathscr{E}$ onto itself among the translations of $\mathscr{E}$ then the set of translations of $\mathscr{E}$ is closed with respect to function composition. In Part H you saw that the set of translations of $\mathscr{E}$, as we then understood 'translation', is closed with respect to inversing. Is it still? You also know that function composition is an associative operation—so, in particular, composition of translations is associative. Finally, you know that function composition is not commutative—but, it may be well to check up on composition of translations. It *might* be commutative. [Explain.] We shall investigate this possibility in the following exercises.

Part J

1. Suppose that f and g are translations. We already know that $g \circ f$ and $f \circ g$ are translations. Because of another property which we know that translations have, if we choose any point P then $g \circ f$ and $f \circ g$ will be the same if it turns out that $[f \circ g](P) = [g \circ f](P)$. Explain.
2. Mark a point P and draw an arrow from there to describe a translation f. From your mark for $f(P)$ draw an arrow to describe a translation g in a direction different from that of f. Mark and label the point $[g \circ f](P)$.
3. Continue with your drawing for Exercise 2 and, using your parallel ruler, find $g(P)$. Now, mark and label the image of $g(P)$ under the translation f. Label it appropriately.
4. Is it the case that $f(g(P)) = g(f(P))$?

we already know. Since we are given an arrow describing f_1 we do have such a point. If we choose for P the point marked by the tail of this arrow then $f_1(P)$ is the point marked by its head. Now, all that is left to do is to find the image of $f_1(P)$ under the translation f_2. This can be done either with one shift of the tracing sheet or by using the parallel line construction. An even easier way is to use a parallel ruler to make a copy of the arrow describing f_2 so that its tail coincides with the head of the arrow which describes f_1. Then, the arrow from the tail of the f_1-arrow to the head of this new f_2-arrow describes $f_2 \circ f_1$. Students will have much use for this triangle construction for finding resultants of translations.]

*

We know that function composition is not a commutative operation because we can produce functions f and g such that $f \circ g \neq g \circ f$. This does not mean that function composition restricted to a particular class of functions must turn out to be noncommutative. Consider, for example, the set of all linear functions with intercept 0 — that is, all functions f such that the domain of f is $\mathcal{R}$ and, for each $x \in \mathcal{R}$, $f(x) = mx$, for some nonzero m. If f_1 and f_2 are such functions then for some nonzero real numbers m_1 and m_2, $f_1(x) = m_1 x$ and $f_2(x) = m_2 x$. Clearly $f_1 \circ f_2 = f_2 \circ f_1$, for $[f_1 \circ f_2](a) = f_1(f_2(a)) = m_1(m_2 a) = m_2(m_1 a) = f_2(f_1(a)) = [f_2 \circ f_1](a)$.

*

Answers for Part J

1. We know that a translation is determined once we know the image under it of some given point. Since each of $f \circ g$ and $g \circ f$ is a translation it follows that if P has the same image under both then they must be the same translation.
2. [f(P) is, of course, at the point of the first arrow and g(f(P)) is at the point of the second arrow.]
3. [Since g is described by the arrow from f(P) to g(f(P)), g(P) is the intersection of the line through P parallel to $\overleftrightarrow{f(P)g(f(P))}$ and the line through g(f(P)) parallel to $\overleftrightarrow{Pf(P)}$ (Exercise 4(b), Part E). Similarly, f(g(P)) is the intersection of the line through g(P) parallel to $\overleftrightarrow{Pf(P)}$ and the line through f(P) parallel to $\overleftrightarrow{Pg(P)}$.]
4. Yes.

5. (a) In working Exercise 3 you used the arrow from $f(P)$ to $g(f(P))$ as a description of g. [On your paper, draw the line containing this arrow and label it 'l'.] You found $g(P)$ as the intersection of lines m and n, where m is the line through P parallel to l and n is the line through $[g \circ f](P)$ parallel to $\overrightarrow{Pf(P)}$. [On your paper, draw the lines m and n.]

   (b) Having found $g(P)$ you next found its image under the translation f as the intersection of two lines. One of these is the line through $g(P)$ which is parallel to $\overrightarrow{Pf(P)}$; the other is the line through $f(P)$ which is parallel to $\overrightarrow{Pg(P)}$. What two lines are these?

   (c) Relate your answer for part (b) to your answer for Exercise 4.

6. If you had chosen for g a translation with the same direction as that of f, would it have been the case that $f(g(P)) = g(f(P))$?

*

In the preceding exercises you have discovered several things about the mappings of $\mathcal{E}$ into itself which we call *translations*. For review and for future reference we shall now summarize some of these things.

(1) A translation is completely determined when one is given any point and its image under the translation.

[In fact, if the given point and its image are different points, you know how to find the image of any other point by using your parallel ruler. If the given point and its image are the same point then the translation must be the identity mapping of $\mathcal{E}$ onto $\mathcal{E}$.]

(2) For any point A and any point B, there is a translation which maps A on B.

[Because of (1) and (2), there is a unique translation which maps a given point A on a given point B.]

(3) The set of translations is closed under function composition.

(4) Composition of translations is commutative.

(5) A translation which leaves any point fixed leaves each point fixed. [As a matter of fact, (5) follows from (2) and (4).]

(6) The identity mapping of $\mathcal{E}$ onto itself is a translation.

[This follows from (2) and (5).]

(7) The converse of a translation is a translation.

[As a matter of fact, (7) follows from (2), (3), and (5).]

(8) A translation maps each line onto a parallel line.

(9) A translation other than the identity mapping moves each point the same nonzero distance in the same sense.

(10) A translation preserves distance between points.

In this course, our aim is to make a formal study of geometry. We shall, in the main, be concerned with geometric figures which we will

5. (a) [$\ell = \overleftrightarrow{f(P)g(f(P))}$, $m = \overleftrightarrow{Pg(P)}$, $n = \overleftrightarrow{g(f(P))g(P)}$]

   (b) Lines n and ℓ. The line through g(P) parallel to $\overleftrightarrow{Pf(P)}$ is n; the line through f(P) parallel to $\overleftrightarrow{Pg(P)}$ is ℓ.

   (c) By part (b) f(g(P)) is the point at which n and ℓ intersect. But, by part (a), ℓ and n both contain the point g(f(P)). So, f(g(P)) = g(f(P)). [Referring to part (b), f(g(P)) is on the line through g(P) which is parallel to $\overleftrightarrow{Pf(P)}$. By part (a), this is n. Referring again to part (b), f(g(P)) is on the line through f(P) which is parallel to $\overleftrightarrow{Pg(P)}$. By part (a), $\overleftrightarrow{Pg(P)}$ is parallel to the line ℓ and ℓ contains f(P). So, ℓ is the line through f(P) which is parallel to $\overleftrightarrow{Pg(P)}$.]

6. Yes. [This is essentially because addition of real numbers is commutative.]

*

As remarked earlier, (1) and (2) on page 47 form the basis for the first two postulates adopted in Chapter 2. [The exact relationship of (1) and (2) with these postulates will come out in section 1.07.] Similarly, (3) is the basis for our third postulate. Our fourth postulate, whose evolution begins in Chapter 3, involves (4).

Note that neither (2) nor (3) would be true had we not included $i_{\mathcal{E}}$ among the translations.

It is perhaps worth mentioning here that, although (1) - (3) tell us quite a lot about translations, they do not characterize the set of all translations among sets of mappings of $\mathcal{E}$ into itself. For, the set of all constant mappings of $\mathcal{E}$ into itself has similar properties. However, this latter set does not have any of the properties which (4) - (7) impute to translations. As we are about to see, (4) is particularly important.

Up to the point at which we decided to include $i_{\mathcal{E}}$ among the translations, we might have said that no translation leaves any point fixed. [See Exercise 2(b) of Part A.] Statement (5) is the restatement of this result which is necessitated by this decision. Of course, (5) by itself does not tell us that $i_{\mathcal{E}}$ is a translation. But, since (2) implies that there is a translation which maps a point A on itself, (6) does follow from (2) and (5).

Recall that it was because of (7) that we included $i_{\mathcal{E}}$ among the translations in order to ensure the truth of (3). So, it is not surprising that (3) and (7) together imply (6). It is somewhat surprising, however, that (1) - (3) and (6), together, imply (7). To see that they do, let f be any translation, let A be any point, and suppose that f(A) = B. By (2), there is a translation — say, g — such that g(B) = A. It follows that $[g \circ f](A) = A$ and $[f \circ g](B) = B$. By (3), both $g \circ f$ and $f \circ g$ are translations. So, by (1), $g \circ f$ is the only translation which maps A on itself and $f \circ g$ is the only translation which maps B on itself. Since, by (6), $i_{\mathcal{E}}$ is a translation, and since $i_{\mathcal{E}}(A) = A$ and $i_{\mathcal{E}}(B) = B$, it follows that $g \circ f = i_{\mathcal{E}} = f \circ g$. Hence, f and g are inverses of each other. Since g is a translation it follows that the converse of f is a translation. Consequently, (7).

In place of appealing to (1) and (6) in the preceding argument in order to show that $g \circ f = i_{\mathcal{E}} = f \circ g$, we can appeal to (5). So, (7) is a consequence of (2), (3), and (5). Since, as shown earlier, (6) is a consequence of (2) and (5), one sees that (5) is a rather strong statement. The importance of (4) is, then, indicated by the fact that (2) and (4) imply (5). To see that this is so, suppose that f is a translation and A is a point such that $f(A) = A$. Consider any other point B. According to (2) there is a translation — say, g — such that $g(A) = B$. Since $f(A) = A$ it follows that $[g \circ f](A) = B$. So, by (4), $[f \circ g](A) = B$. Since $g(A) = B$ it follows that $f(B) = B$ — that is, that f maps B on itself. Consequently, (5).

Summarizing the results of the preceding arguments, we have seen that

> assuming (1) - (3), statements (6) and (7) are equivalent,
> and assuming (1) - (4) commits us to accepting (5), (6),
> and (7).

Since (1) - (4) will be among our postulates, (5) - (7) will be theorems.

As has been pointed out in the discussion of Exercise 2 of Part C, (5) and (8) characterize translations other than the identity translation. [In other words, all properties of translations follow from "conventional" postulates for geometry supplemented by (5), (6), and (8).]

It should be recalled that (7) implies that each translation is a mapping of $\mathcal{E}$ onto itself [see Exercise 6 of Part H] and that, in view of (7), the word 'into' in (8) may be replaced by 'onto' [see the discussion of Exercise 1 of Part B].

In our formal development, (8) will be an immediate consequence of the definitions we shall adopt for 'line' and 'parallel' and will, in fact, form part of the motivation for the latter definition. Similarly, (9) and (10) will be immediate consequences of our formal definitions of 'sense' and 'distance'; and the former will furnish motivation for these definitions.

In section 1.07 we introduce a very convenient algebraic notation which we shall use throughout this course. We continue to use uppercase letters, from the beginning of the alphabet, as variables whose domain is $\mathcal{E}$ [the set of all points] and lower-case letters, from the beginning of the alphabet, surmounted by arrows as variables whose domain is $\mathcal{T}$ [the set of all translations]. [We shall as at the beginning of this chapter, use lower-case letters without arrows as variables whose domain is $\mathcal{R}$.] Instead of '$\vec{a}(A)$' for 'the image of the point A under the translation $\vec{a}$', we shall use '$A + \vec{a}$'. [Read '$A + \vec{a}$' as 'A plus arrow a'.] As an abbreviation for 'the translation from A to B' we shall use '$B - A$'. In accordance with these conventions, the fact that, for any points A and B, the translation from A to B maps A on B is formulated in:

Postulate 2(a). $\quad A + (B - A) = B$

[see (9) on page 51]. To be able to prove that, for any points A and B, there is a translation which maps A on B we need to complete the formulation of our convention as to the meaning of '$B - A$' by:

Postulate 1(a). $\quad B - A \in \mathcal{T}$

[By Postulate 1(a), Postulate 2(a) is an instance of the existential generalization

$$\exists_{\vec{x}}\; A + \vec{x} = B \qquad [(2) \text{ on page } 47].$$

So, the existential generalization is a consequence of Postulates 1(a) and 2(a).]

That a translation is determined once one knows the image under it of some given point may be formulated in:

$$(\star) \qquad A + \vec{a} = B \Longrightarrow \vec{a} = B - A$$

which, by our notational conventions says that if the image of A under $\vec{a}$ is B then $\vec{a}$ is the translation from A to B. Alternatively, one might formulate the same property of translations [the property stated in (1) on page 47] in:

$$(\star\star) \qquad A + \vec{b} = A + \vec{a} \Longrightarrow \vec{b} = \vec{a}$$

which says that if some point A has the same image under a translation $\vec{b}$ and under a translation $\vec{a}$ then $\vec{b}$ is $\vec{a}$. These two formulations can be shown to be equivalent once we adopt:

Postulate 1(b). $\quad A + \vec{a} \in \mathcal{E}$

which, according to our conventions, formulates the fact that a translation is a mapping of $\mathcal{E}$ into itself.

In fact, by Postulate 1(b), $(\star)$ has as a consequence:

$$A + \vec{a} = A + \vec{b} \Longrightarrow \vec{a} = (A + \vec{b}) - A$$

In particular, since $A + \vec{a} = A + \vec{a}$, it follows that $\vec{a} = (A + \vec{a}) - A$ and [switching '$\vec{a}$' and '$\vec{b}$'] that if $A + \vec{b} = A + \vec{a}$ then $\vec{b} = (A + \vec{a}) - A$. Consequently, if $A + \vec{b} = A + \vec{a}$ then $\vec{b} = \vec{a}$. So, $(\star\star)$ is a consequence of $(\star)$ and Postulate 1(b).

think of as subsets of the space $\mathscr{E}$ of points. Using our intuitions about geometric figures, we saw in this chapter that the mappings we called *translations* "preserve" important geometric properties such as length, parallelism, and sense.

We want to organize and extend our knowledge of geometry. To do this, we shall make use of notions which are based on the properties of translations. One advantage of doing this is that it will enable you to solve problems in geometry by using techniques very much like those you learned in your study of algebra. Another advantage is that you will become familiar with some of the basic techniques by which new mathematics is being discovered at the present time.

Before we begin to formalize our notions about points and translations, it will be convenient to introduce some useful and suggestive notation. We do this in the next section.

1.07 A New Kind of Algebra

In stating generalities about points we have used capital letters – 'A', 'B', etc. – as variables whose domain is $\mathscr{E}$. For example, we might have written:

For any two points A and B, there is a line which contains A and B. More shortly, we might write:

If $A \neq B$ then there is a line which contains A and B.

or:

$$A \neq B \Longrightarrow \exists_l (l \text{ is a line containing } A \text{ and } B)$$

In stating generalities about mappings we have used a similar device, choosing lower case letters – for example, 'f' and 'g' – as variables whose domain is the set of all mappings. For example, we might write:

$$Rf \subseteq Dg \Longrightarrow D[g \circ f] = Df$$

In what follows we shall have frequent need to state generalities concerning the special mappings which we have called 'translations of $\mathscr{E}$'. We might, for example, write:

$$f \text{ and } g \text{ are translations of } \mathscr{E} \Longrightarrow f \circ g = g \circ f$$

Much space and effort will be saved if, instead of using 'f' and 'g', we introduce special variables whose domain is the set of all translations of $\mathscr{E}$ rather than the set of all mappings. Since translations can be represented pictorially by arrows it seems natural to use '$\vec{a}$', '$\vec{b}$', etc. as

On the other hand, from (☆☆) it follows [switching '$\vec{a}$' and '$\vec{b}$'] that if $A + \vec{a} = A + \vec{b}$ then $\vec{a} = \vec{b}$. So, by Postulate 1(a), if $A + \vec{a} = A + (B - A)$ then $\vec{a} = B - A$. Consequently, by Postulate 2(a), if $A + \vec{a} = B$ then $\vec{a} = B - A$. So, (☆) is a consequence of (☆☆) and Postulates 1(a) and 2(a).

Instead of adopting either the "transposition" principle (☆) or the "cancellation" principle (☆☆) as a postulate, we shall adopt the simpler:

Postulate 2(b). $\vec{a} = (A + \vec{a}) - A$

This, by itself, implies that if $A + \vec{a} = B$ then $\vec{a} = B - A$. On the other hand, it has already been shown in an earlier paragraph that (☆) and 1(b) imply that $\vec{a} = (A + \vec{a}) - A$.

Summarizing, Postulates 1 and 2 formulate the fact that translations are mappings of $\mathscr{E}$ into itself which have the properties that, for any point A and any point B, there is a unique translation which maps A on B.

The two parts of Postulate 2, as well as (☆) and (☆☆) exemplify the close similarity which exists between this algebra of points and translations and the algebra of real numbers.

Our aim in the preceding discussion has been to give you an introduction to the algebra of points and translations which would clarify the content of the first two postulates which will be adopted in the next chapter and relate this content to (1) and (2) on page 47. The aim of the discussion in the text is slightly different. There we introduce students to more of the algebra but concentrate on motivating the choice of symbolism, the prime motive being on obtaining an algebra of points and translations which will be as similar as possible to the familiar algebra of the real numbers.

such variables. [Read '$\overrightarrow{a}$' as 'arrow a'.] Doing this, we can simplify the last of the sentences displayed above to:

$$\overrightarrow{a} \circ \overrightarrow{b} = \overrightarrow{b} \circ \overrightarrow{a} \tag{1}$$

Another generality we wish to be able to state easily is that any resultant of translations is a translation. This we can now do by writing '$\overrightarrow{a} \circ \overrightarrow{b}$ is a translation'. It will be a great time-saver in this and similar situations if we adopt '$\mathcal{T}$' as a name for the set of all translations. Then, we can say that any resultant of translations is a translation by writing merely:

$$\overrightarrow{a} \circ \overrightarrow{b} \in \mathcal{T} \tag{2}$$

In short, because of our convention about arrow-letters, we shall accept any sentences obtained from:

$$\begin{gathered} \square \circ \triangle \in \mathcal{T} \\ \square \circ \triangle = \triangle \circ \square \end{gathered}$$

by substituting for the frames expressions whose values are translations. An example of such a sentence is:

$$\overrightarrow{b} \circ \overrightarrow{c} \in \mathcal{T} \tag{3}$$

Two other examples are:

$$\begin{gathered} \overrightarrow{a} \circ (\overrightarrow{b} \circ \overrightarrow{c}) \in \mathcal{T} \\ (\overrightarrow{b} \circ \overrightarrow{c}) \circ \overrightarrow{b} = \overrightarrow{b} \circ (\overrightarrow{b} \circ \overrightarrow{c}) \end{gathered}$$

(Give substitutions for the frames in each of the three examples. How does (3) help you in justifying some of these substitutions?)

Suppose, now, that you are given a translation $\overrightarrow{a}$ and a point P, as illustrated in the diagram below. Copy this picture on a sheet of paper

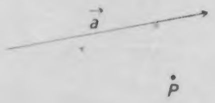


Fig. 1-20

and use your parallel ruler to locate the point Q which is the image of P under $\overrightarrow{a}$. Since $\overrightarrow{a}$ is a mapping we may use ordinary function notation to describe Q:

$$Q = \overrightarrow{a}(P) \tag{4}$$

After introducing "arrow-letters" as variables whose domain is the set of all translations and 'T' as a name for this domain we give a few illustrations [for example, (1) and (2) on page 49] of how we can say things in terms of this notation. We also point out at the top of the page how one can infer other sentences like these by substitution. The use of '$P + \vec{a}$' in place of '$\vec{a}(P)$' is then introduced. The immediate motivation for this is that in order to show the image of P under the translation $\vec{a}$ one begins by marking the point P on one's paper and then, from it, draws an arrow which describes the translation $\vec{a}$.

*

It is important that students learn to recognize logical connections among sentences and to discover and verify such connections by deriving desired conclusions from given premisses. Ultimately, they should be able to put such arguments [i.e. derivations] in the "paragraph form" which has been illustrated frequently in this commentary. Experience has shown that this latter skill is gained more surely and rapidly if students practice giving derivations in a "column form" where each "line" of the derivation follows from preceding lines by virtue of explicitly stated rules of logic. In turn, such derivations are more readily understood when analyzed into "trees". Consequently, rules of logic, and derivations in tree form and in column form are dealt with later in the text and, more explicitly, in the commentary. These matters are also treated in *High School Mathematics*, Courses 1 - 3 and students who have studied Course 1 — or better, Courses 1 and 3 — will already have a good understanding of much which they would otherwise have to learn from the present course.

It cannot be over-emphasized that the explicit rules of logic and the formal procedures for writing derivations which we shall introduce are, for us, crutches. Their purpose is to aid students to learn to reason logically and to formulate their reasoning in acceptable prose. You can easily bog down your students and distract their attention from the mathematical subject matter of this course by insisting on too many formal proofs. [A proof is a derivation whose premisses are postulates, definitions, or previously established theorems.] Students should be encouraged to give acceptable paragraph proofs. When they fail, formal proofs — or parts of such — can be used in showing them how they might have succeeded. Doing so will teach them to write bits of formal or semi-formal proofs as scratch-work aids to formulating paragraphs.

The following remarks deal briefly with the rules of logic which are implicit in section 1.07. These rules are discussed in the text in sections 2.03 and 2.04. Introducing them in your discussion of the present section will ''reduce the load'' when you take up Chapter 2.

The rule of logic which formulates our decision to express universal generalities by open sentences is:

> *The Substitution Rule*
>
> Any sentence implies each of its substitution-instances.

— for short, '(Subst)'. [Since we shall also use open sentences in another way — as ''assumptions'' in derivations — this rule will be modified later. The form just given is, however, suitable for the present context.] Before stating this rule you should, of course, point out the basis for it. Our decision as to the meaning of an open sentence is that, when we assert such a sentence we are to be understood as saying something about all values of the variables which occur in the sentence. For example, what we mean when we assert '$A + \vec{a} \in \mathcal{E}$' is expressed in English by:

> The result of adding *any* translation to *any* point is a point.

This being so, any sentence obtained from the given one by substituting for 'A' an expression whose values are points, and for '$\vec{a}$' an expression whose values are translations, is a sentence which expresses no more than does the given sentence. For this reason, we say that such a sentence is implied by the given sentence. For example, the sentence '$(A + \vec{a}) + \vec{b} \in \mathcal{E}$' is implied by — or, is a consequence of — or, follows from — the sentence '$A + \vec{a} \in \mathcal{E}$'.

There are two points which you should check your students' knowledge of. First, *substitution* of a given expression for a variable is replacing *each* occurrence of the variable by an occurrence [or: copy] of the given expression. Second, ''given expression'' is not, in the preceding, to be taken quite literally. For example, if we were being very precise, we would have said, on TC 48(1), ''Instead of '$\vec{a}(A)$' ..., we shall use '$(A + \vec{a})$'.'' It is common practice to omit the second pair of parenthesis and we shall follow this practice. As a result, when a student is asked to ''substitute '$A + \vec{a}$' for ...'' what he should do is substitute '$(A + \vec{a})$'.

Continuing, now, with a possible preamble to stating the substitution rule, the sentences which follow from a given one because of our decision to use open sentences to express universal generalities are just those which can be obtained from it by substituting for some, or all, of its variables expressions whose values belong to the domains of the variables for which they are substituted. Thus, '$(A + \vec{a}) + \vec{b} \in \mathcal{E}$' is implied by '$A + \vec{a} \in \mathcal{E}$', but '$A + (A + \vec{b}) \in \mathcal{E}$' is not. [In fact, speaking strictly, the latter is not even a sentence.] The sentences which can be obtained from a given one by substitutions which are ''appropriate'' ones are, by definition, the *substitution-instances* of the given sentence.

After this, the substitution rule, as stated above, should make sense.

A minor embellishment which you may wish to make when discussing substitution concerns our later use of the word 'term'. For us, 'term' is analogous to 'algebraic expression' — rather than to, say, the more specific phrase 'indicated product'. A term, then, is an expression which ''refers'' to objects, in the sense in which nouns and pronouns do. The objects to which a term refers are its values. ['term' contrasts with 'sentence'— terms are ''nominative'', sentences are ''declarative''.] Variables are terms, and expressions which are constructed out of variables and operators [with grouping symbols supplied where needed] are terms — if the construction is in accord with the rules of the language. [In our language, the expression '(A + B)', for example, is not a term.] Terms may be distinguished from one another by what kind of objects they have as values. For example, '$A + \vec{a}$' is a point-term and 'B - A' is a translation-term.

With this ''terminology'' available, the notion of substitution-instance can be described by saying that a substitution-instance of a given sentence is any sentence which can be obtained from it by substitutions of point-terms for point-variables, translation-terms for translation-variables, or — eventually — real number-terms for real number-variables.

The substitution rule tells you that you may [or: that it is legitimate to] *infer* from a sentence any of its substitution-instances. The act of doing so is an act of ''making an inference''. It is very convenient to stretch this last colloquialism a bit and refer to a *figure* such as:

$$\frac{A + (B - A) = B}{A + ((A + \vec{a}) - A) = A + \vec{a}}$$

as an *inference*. [You may read the horizontal bar as 'therefore'.] Since the sentence below the bar is a substitution-instance of the one above it, the substitution rule asserts that this particular inference is *valid*. Inferences of this type will be called *substitution inferences*. In Chapter 2 we shall require students to insert another *premiss* in such an inference to show why the *conclusion* of the inference is a substitution-instance of the first premiss:

$$\frac{A + (B - A) = B \qquad A + \vec{a} \in \mathcal{E}}{A + ((A + \vec{a}) - A) = A + \vec{a}}$$

If several substitutions are involved, several such auxillary premisses are required. This requirement is made purely for reasons of pedagogy. Its purpose is to direct the student's attention to the fact that there are terms of different kinds and to be a partial safeguard against ''inappropriate'' substitutions. It also presents you with a point of attack in cases when inappropriate substitutions have been made. The requirement is dropped after Chapter 2. The use of such auxillary premisses does correspond with the verbal translation of the inference which might occur in a paragraph proof:

> Since $A + \vec{a} \in \mathcal{E}$ and $A + (B - A) = B$ it follows that
> $A + ((A + \vec{a}) - A) = A + \vec{a}$.

As an example of the need for thought as to what kind of term a given expression is [or whether it is any kind of term] you might ask whether '$(A + \vec{b}) - ((C + \vec{b}) + \vec{a})$' refers to points or to translations.

One answer is that, for the same reason that '$C + \vec{b}$' refers to points, '$(C + \vec{b}) + \vec{a}$' refers to points. Since, also, '$A + \vec{b}$' refers to points and since a difference of points is a translation, the term in question refers to translations. As an aid to giving such answers, suggest the adoption of Postulate 1:

(a) $B - A \in \mathcal{T}$ (b) $A + \vec{a} \in \mathcal{E}$

With these assertions to refer to one can substitute the following for the answer suggested above:

> From Postulate 1(b) it follows that $C + \vec{b} \in \mathcal{E}$ and, so, by the same postulate, $(C + \vec{b}) + \vec{a} \in \mathcal{E}$. Also, from Postulate 1(b), $A + \vec{b} \in \mathcal{E}$. Consequently, by Postulate 1(a), $(A + \vec{b}) - ((C + \vec{b}) + \vec{a}) \in \mathcal{T}$.

[As motivation for rapid development of skill, assure your students that they will need to write this sort of thing only until they have convinced you that they know what is going on.] In case of need, or as an introduction to tree-form derivations, the preceding verbal argument can be set forth as:

$$\dfrac{\dfrac{B - A \in \mathcal{T} \qquad \dfrac{A + \vec{a} \in \mathcal{E} \qquad \dfrac{A + \vec{a} \in \mathcal{E}}{C + \vec{b} \in \mathcal{E}}}{(C + \vec{b}) + \vec{a} \in \mathcal{E}}}{B - ((C + \vec{b}) + \vec{a}) \in \mathcal{T}} \qquad \dfrac{A + \vec{a} \in \mathcal{E}}{A + \vec{b} \in \mathcal{E}}}{(A + \vec{b}) - ((C + \vec{b}) + \vec{a}) \in \mathcal{T}}$$

If students ask why you don't write:

$$\dfrac{A + \vec{a} \in \mathcal{E} \qquad \vec{b} \in \mathcal{T}}{A + \vec{b} \in \mathcal{E}} \quad \text{instead of:} \quad \dfrac{A + \vec{a} \in \mathcal{E}}{A + \vec{b} \in \mathcal{E}}$$

tell them that you are willing to assume that they know which variables have points as values and which have translations as values.

As another example, take '$(A + \vec{a}) + (B + \vec{b})$' and ask whether this expression refers to points or to translations. Answer: Since '$A + \vec{a}$' and '$B + \vec{b}$' refer to points, and we have not defined addition for points, the expression in question is nonsense.

The other rules of logic whose discussion is pertinent here are those for dealing with equations. Briefly, since, for us, '=' is to mean what 'is the same as' does, an equation such as '2 + 3 = 5' says that 2 + 3 is 5. An equation such as 'A + (B - A) = B' says that any value of its left side is the corresponding value of its right side. Assuming that a first object is a second object, anything one says about the first object one may, just as properly, say about the second object. For example, each of the inferences in the figure:

$$\dfrac{[R + (Q - P)] - R = Q - P \qquad \dfrac{S = R + (Q - P) \qquad S - R = S - R}{S - R = [R + (Q - P)] - R}}{S - R = Q - P}$$

is valid. [For the source of this example, see page 51.] In words:

> Assuming that S = R + (Q - P) it follows, since $S - R = \underline{S} - R$, that $S - R = [\underline{R + (Q - P)}] - R$. [Under our assumption, what '$S - R = \underline{S} - R$' says about S, we may, equally well, say about R + (Q - P).] Furthermore, since [by Postulate 2(a)] [R + (Q - P)] - R = Q - P it follows that $S - R = \underline{Q - P}$. What has been said about [R + (Q - P)] - R we may, equally well, say about Q - P.]

[On the chalkboard, colored arrows running from the leftmost 'S' in the top line of the figure to the rightmost one, and from the 'R + (Q - P)' in the top line to the rightmost one in the second line, help to show what is going on in the first inference. Similar arrows can be used in explaining the second inference.]

The same tree-form derivation gives an opportunity to point out that the premiss 'S - R = S - R' is somewhat special. Unlike the other sentences, this one is true merely because of what '=' means [''A thing is the same as itself.''] and because 'S - R' is a term — that is, is meaningful. Sentences of this nature are said to be valid.

Our two remarks about '=' motivate our two rules of logic for dealing with equations:

The Replacement Rule for Equations

Given an equation and a second sentence, if either side of the equation is replaced, somewhere in the second sentence, by its other side then the resulting sentence is implied by the given equation and sentence.

The Introduction Rule for Equations

Each of the sentences 'A = A', $\vec{a} = \vec{a}$, and 'a = a' is valid.

[Note that, by the substitution rule, the sentence 'S - R = S - R' of our example is a consequence of the sentence '$\vec{a} = \vec{a}$'. So, as we shall argue later, it follows from the introduction rule for equations and the substitution rule that the former sentence is also valid.]

For additional material on these rules, see section 2.05 and its commentary.

Although this notation is excellent for many purposes it will be to our advantage, as well as more suggestive, to write '$P + \overrightarrow{a}$' instead of '$\overrightarrow{a}(P)$'. [You may see why this is more suggestive if you draw on your paper the arrow from P to Q. How should you label this arrow?] Using this notation we would write:

(5) $$Q = P + \overrightarrow{a}$$

instead of '$Q = \overrightarrow{a}(P)$'. [Read '$P + \overrightarrow{a}$' either as 'the image of P under arrow a' or as 'P plus arrow a'.] Notice that "adding" a translation to a point gives a point.

Since, as we know, the converse of a translation is a translation, it follows that any translation has an inverse. We may, of course, use the usual notation '$\overrightarrow{a}^{-1}$' to refer to the inverse of the translation $\overrightarrow{a}$. On your paper, draw an arrow which describes the translation which is the inverse of $\overrightarrow{a}$, and label it '$\overrightarrow{a}^{-1}$'. Locate the point $Q + \overrightarrow{a}^{-1}$. On your paper, complete the following sentence:

(6) $$Q + \overrightarrow{a}^{-1} = (P + \overrightarrow{a}) + \overrightarrow{a}^{-1} = \underline{\qquad}$$

This sentence may suggest to you that, since we are using '+' to indicate application of a translation to a point, it would be natural to use '$-\overrightarrow{a}$', instead of '$\overrightarrow{a}^{-1}$', to refer to the inverse of $\overrightarrow{a}$. [Read '$-\overrightarrow{a}$' as 'the inverse of arrow a', or as 'the opposite of arrow a'.] We shall adopt this notation and, instead of (6), we shall write:

$$Q + -\overrightarrow{a} = (P + \overrightarrow{a}) + -\overrightarrow{a} = P$$

You may already begin to see that, with our new meanings for the addition and oppositing signs, some sentences we can write about points and translations look very much like sentences you are familiar with from the ordinary algebra of real numbers. This will be more apparent – and helpful – as we go on.

On a fresh sheet of paper, mark three noncollinear points P, Q, and R. Draw an arrow which represents a translation which maps P on Q. How do we know there is such a translation? Is there more than one such translation? [Explain.] So, we may speak of

(7) *the* translation which maps P on Q.

Because of our convention as to the meaning of '+' we might write:

(8) P + (the translation which maps P on Q) = Q

The additive notation for function application suggests the use of an oppositing sign to indicate function inversion. The resulting identity:

$$(P + \vec{a}) + -\vec{a} = P$$

is similar to a familiar identity for real numbers. Since a translation is the inverse of its inverse, you may also suggest at this point the identity '$--\vec{a} = \vec{a}$', as well as '$(P + -\vec{a}) + \vec{a} = P$. Since the inverse of a translation is a translation, '$-\vec{a} \in T$' is an identity. At this point students may anticipate the answer to a later question and suggest using '$Q - \vec{a}$', say, in place of '$Q + -\vec{a}$'. If so, you may agree that this seems like a good idea, and note the resulting identities:

$$(P + \vec{a}) - \vec{a} = P \qquad \text{and:} \qquad (P - \vec{a}) + \vec{a} = P$$

Should students suggest '$P + (\vec{a} + -\vec{a}) = P$' or '$P + (\vec{a} - \vec{a}) = P$' call their attention to the fact that we have not yet defined addition [or subtraction] of translations. If you wish to carry the matter further at this time you could ask what '$\vec{a} + -\vec{a}$' would have to mean if the first of these sentences is an identity. [Answer: $i_{\mathcal{E}}$] Students may then suggest that "addition" of translations must mean function composition. If they do, tell them that there is another possibility, but leave what this is to their imaginations. [We shall use '$\vec{a} + \vec{b}$' as an abbreviation for '$\vec{b} \circ \vec{a}$' rather than for '$\vec{a} \circ \vec{b}$'.]

Can you think of an "algebraic expression" we might use in place of (7) which would make (8) look familiar?

You may have already guessed what we shall write instead of (7). It is:

$$Q - P$$

[Read '$Q - P$' as you would (7) or as 'Q minus P'.] Notice that "subtracting" a point from a point gives a translation. Using this notation we shall write:

$$P + (Q - P) = Q \tag{9}$$

rather than (8).

On your paper, label the arrow you have drawn to describe the translation from P to Q with '$Q - P$'. [How many arrows could you draw which would describe the translation $Q - P$? Draw several such arrows.] Be sure you realize that, while Q and P are points, $Q - P$ is a translation. On your paper complete the sentence '$Q \in \mathscr{E}$ and $P \in \mathscr{E}$ and $Q - P \in$ ____'.

Since $Q - P$ is a translation it maps the point R on the point

$$R + (Q - P).$$

Locate this point on your paper and label it 'S'. So,

$$S = R + (Q - P).$$

This sentence tells us that $Q - P$ is a translation which maps R on S. Since there is only one translation which does this, $Q - P$ is *the* translation which maps R on S. So, by our convention concerning the use of the subtraction sign, it follows that

$$S - R = Q - P.$$

Notice, again, how our "algebra of points and translations" parallels the algebra of real numbers.

As another example, note that if $\vec{a}$ is the translation which maps P on Q then, by our convention as to the use of '+',

$$P + \vec{a} = Q \tag{10}$$

and, by our convention as to the use of '−',

$$\vec{a} = Q - P \tag{11}$$

The next step is to introduce "subtraction" of points. Again the motivation is that of algebraic elegance. In view of one's familiarity with the algebra of real numbers there is some aesthetic satisfaction to be gained from the fact that '$P + (Q - P) = Q$' is an identity. Call attention to the fact that, for any P and Q, $Q - P \in T$, and remind students that, for any P and any $\vec{a}$, $P + \vec{a} \in \mathscr{E}$.

The argument given in the text to show that '$S = R + (Q - P)$' implies '$S - R = Q - P$' appeals, of course, to the meanings of the terms occurring in these equations. Once we have adopted Postulates 1(a) and 2(b) we shall be able to derive the second of these sentences from the first and these postulates. For, assuming that $S = R + (Q - P)$ it follows that $S - R = [R + (Q - P)] - R$. Since, by Postulate 2(b), $[R + (Q - P)] - R = Q - P$ it follows that $S - R = Q - P$. [Strictly speaking, Postulate 1(a) is needed at two points in the preceding argument.] Similarly, using Postulates 1 and 2(a) and the second of the two sentences as premisses we can derive the first sentence.

Remarks similar to those in the preceding paragraph apply to (10) and (11). Specifically, (11) can be derived from (10) and (☆) on TC 1.07(1); and that (11) implies (10) is essentially the content of Postulate 2(a). 'Substituting from (11) into (10)' means replacing the '$\vec{a}$' in (10) by '$Q - P$'. So, the result is (9). 'Substituting from (10) into (11)' means replacing the 'Q' in (11) by '$P + \vec{a}$'. The result is:

$$\vec{a} = (P + \vec{a}) - P$$

which is, essentially, Postulate 2(b).

Substituting from (11) into (10) you get (9). What sentence do you get if you substitute from (10) into (11)?

Finally, it follows from (11) that

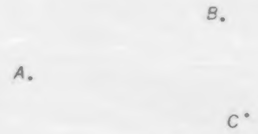
$$(12) \qquad -\vec{a} = -(Q - P).$$

Since $\vec{a}$ is the translation which maps P on Q, its inverse, $-\vec{a}$, maps Q on P. Since $-\vec{a}$ is a translation, it follows that it is the translation which maps Q on P. So,

$$\left.\begin{array}{lr} (13) & -\vec{a} = \underline{\qquad} \\ \text{and [from (12) and (13)]} & \\ & -(Q - P) = \underline{\qquad}. \end{array}\right\} \text{Complete these sentences.}$$

Exercises

1. Suppose that A, B, and C are points of $\mathscr{E}$ as shown in the diagram below.



Make a diagram similar to this one on your paper.

(a) Draw an arrow to describe the translation $B - A$.

(b) Draw an arrow to describe the inverse of $B - A$.

(c) Complete these sentences:

   (i) $A + (B - A) = \underline{\qquad}$

   (ii) The inverse of $B - A$ maps ____ on ____

   (iii) $-(B - A) = \underline{\qquad} - \underline{\qquad}$

   (iv) $B + -(B - A) = B + (\underline{\qquad}) = \underline{\qquad}$

(d) Locate the point D, such that $D = C + (B - A)$. Locate the point $D + (A - B)$.

(e) Complete: $D + (A - B) = \underline{\qquad}$

(f) $B - A$ maps C on ____. Also $D - C$ maps C on ____. So, $B - A = \underline{\qquad}$.

2. Let A be any point of $\mathscr{E}$ and let $\vec{a}$ be any translation. Then $A + \vec{a}$ is the image of ____ under ____. Also, $(A + \vec{a}) - A$ is the translation that maps ____ on ____. Therefore,

$$(A + \vec{a}) - A = \underline{\qquad}.$$

3. Recall that for any point A, the [only] translation which maps A on A is the identity translation, $i_{\mathscr{E}}$. Hence,

$$A - \underline{\qquad} = i_{\mathscr{E}}.$$

Since the identity translation on $\mathscr{E}$ leaves each point fixed, it follows that for any points A and B,

$$A + (A - \underline{\qquad}) = A + i_{\mathscr{E}} = \underline{\qquad},$$

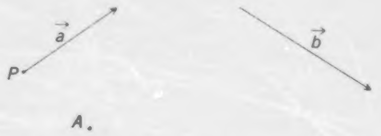
and

$$B + (\underline{\qquad} - A) = B + i_{\mathscr{E}} = \underline{\qquad}.$$

This suggests that we use '$\vec{0}$' instead of '$i_{\mathscr{E}}$' to denote the identity translation on $\mathscr{E}$. So, for any points A and B,

$$\begin{aligned} A - A &= \vec{0} = B - B \\ A + (A - A) &= A + \underline{\qquad} = \underline{\qquad} \\ B + (A - A) &= B + \underline{\qquad} = \underline{\qquad}. \end{aligned}$$

4. On your paper, mark points A and P and draw arrows to describe translations $\vec{a}$ and $\vec{b}$, as in the diagram below.



(a) Draw an arrow from P to describe the translation $\vec{b} \circ \vec{a}$.

(b) Mark points B and C such that $B = A + \vec{a}$ and $C = B + \vec{b}$. Then,

$$(A + \vec{a}) + \vec{b} = \underline{\qquad}.$$

(c) Mark the point $A + [\vec{b} \circ \vec{a}]$.

(d) The diagram shows that

$$A + [\vec{b} \circ \vec{a}] = (A + \vec{a}) + \vec{b}.$$

Explain why it ought to.

*

Suppose that $\vec{a}$ and $\vec{b}$ are any translations and that A is any point. Since we are using '+' to indicate application of a translation to a point it follows that

$$\begin{aligned} (A + \vec{a}) + \vec{b} &= \vec{b}(A + \vec{a}) \\ &= \vec{b}(\vec{a}(A)) \\ &= [\vec{b} \circ \vec{a}](A) \end{aligned}$$

and, finally,

$$(14) \qquad (A + \vec{a}) + \vec{b} = A + [\vec{b} \circ \vec{a}].$$

This result certainly does not look much like ordinary algebra. To change it to something which does, we would have to replace '$[\vec{b} \circ \vec{a}]$' by '$(\vec{a} + \vec{b})$' [or by '$(\vec{b} + \vec{a})$']. If we do so we shall be using '+' in two ways – sometimes to refer to the result of applying a function to an argument, and sometimes to refer to the result of composing functions. A little reflection will suggest that doing this would not be as confusing as one might think. We shall certainly always be able to tell at a glance whether an expression refers to a point or a translation. This being so, we shall have no trouble recognizing that a '+' which occurs after a point-expression and is followed by a translation-expression indicates function application; while a '+' which occurs between translation-expressions indicates function composition. So, we shall adopt this second meaning of '+', along with the first. We still have two choices ['$\vec{a} + \vec{b}$' or '$\vec{b} + \vec{a}$']. It is slightly more convenient to use:

$$\vec{a} + \vec{b}$$

in place of:

$$\vec{b} \circ \vec{a}$$

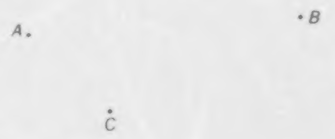
Doing so, we write:

$$(A + \vec{a}) + \vec{b} = A + (\vec{a} + \vec{b}) \tag{15}$$

instead of (14). There are four '+'s in (15). For each '+', tell whether it refers to function application or function composition.

Exercises

1. On your paper, draw points A, B, and C, as in the figure below.



(a) Draw arrows to describe the translations that map A on B, B on C, and A on C. Use the "minus sign" notation to label each of these translations.

(b) Using a '+' to indicate composition of translations, we have

$(B - A) + (C - B) =$ _______.

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Answers for Exercises

1. (a) [The easiest answer to give is the arrow from A to B.]
   (b) [Any arrow with the same length and direction as the arrow from A to B, but having the opposite sense.]
   (c) (i) B (ii) B, A (iii) A, B (iv) A - B, A
   (d) [D should be marked at the tip of an arrow from C whose length and sense are the same as those of the arrow from A to B; in short, ABDC is a parallelogram. $D + (A - B) = C$; note that $D + (A - B) = [(C + (B - A)] + -(B - A)$.]
   (e) C
   (f) D; D; D - C [The justification for the last answer is, of course, that a translation is completely determined by what it does to any one point.]
2. A, $\vec{a}$; A, $A + \vec{a}$; $\vec{a}$

TC 53

3. A; A, A; A, B; $\vec{0}$, A; $\vec{0}$, B
4. (a) [Students should first draw, from the tip of the arrow given to describe $\vec{a}$, an arrow which describes $\vec{b}$. The answer is obtained by drawing the arrow from P to the tip of this latter arrow.]
   (b) C
   (c) [The mark in question has already been labeled 'C'.]
   (d) By our convention, $A + [\vec{b} \circ \vec{a}] = [\vec{b} \circ \vec{a}](A) = \vec{b}(\vec{a}(A)) = \vec{b}(A + \vec{a}) = (A + \vec{a}) + \vec{b}$.

TC 54

In (15), each of the first three '+'s [from left to right] refers to function application; the fourth refers to function composition.

Answers for Exercises

1. (a) The arrows should be those from A to B, from B to C, and from A to C, respectively. They should be labeled 'B - A', 'C - B', and 'C - A'.
   (b) C - A [The completed sentence will be adopted as Postulate 3. As will be shown in Chapter 2 it follows from Postulates 1 - 3 that, for any translations $\vec{a}$ and $\vec{b}$, $\vec{a} + \vec{b} \in T$ and, for any point A, $A + (\vec{a} + \vec{b}) = (A + \vec{a}) + \vec{b}$. On the other hand, from these two sentences and Postulates 1 and 2, one can derive Postulate 3. So, given Postulates 1 and 2, Postulate 3 does neither more nor less than formulate our convention that '+' between translation-expressions refers to function composition, and assert that a resultant of translations is a translation.]

2. Recall that $\vec{a}^{-1} \circ \vec{a}$ is the identity translation $\vec{0}$. Using a '+' to indicate composition of translations and '$-\vec{a}$' instead of '$\vec{a}^{-1}$', we have

$$\_\_\_\_ + \_\_\_\_ = \vec{0}.$$

*

As a preparation for an algebraic study of points and translations we have introduced five conventions:

(a) $A + \vec{a} = \vec{a}(A)$; so, $A + \vec{a} \in \mathscr{E}$. [A translation is a mapping of $\mathscr{E}$ into $\mathscr{E}$.]

(b) $-\vec{a} = \vec{a}^{-1}$; so, $-\vec{a} \in \mathscr{T}$. [The inverse of a translation is a translation.]

(c) $B - A$ = the translation from A to B; so, $B - A \in \mathscr{T}$.

(d) $\vec{0} = i_{\mathscr{E}}$; so, $\vec{0} \in \mathscr{T}$. [The identity mapping of $\mathscr{E}$ onto itself is a translation.]

(e) $\vec{a} + \vec{b} = \vec{b} \circ \vec{a}$; so, $\vec{a} + \vec{b} \in \mathscr{T}$. [A resultant of translations is a translation.]

Later we shall find it convenient to adopt two conventions which will give meaning to expressions like '$A - \vec{a}$' and '$\vec{a} - \vec{b}$'. Perhaps you can guess what these conventions will be.

1.08 What Comes Next?

In earlier courses you have probably had some practice in showing that if a given operation has certain properties then it must have certain others. For a very simple example, you may have proved that because addition of real numbers is commutative and associative it must have another property which may be expressed by saying that, for any real numbers a, b, and c,

$$(a + b) + c = (a + c) + b.$$

[Given any real numbers a, b, and c, it follows by commutativity that $(a + b) + c = c + (a + b)$. By associativity, $c + (a + b) = (c + a) + b$. By commutativity, $(c + a) + b = (a + c) + b$. Consequently, for any real numbers a, b, and c, $(a + b) + c = (a + c) + b$.] This new property we shall call *the switch property*. Can you think of another operation on real numbers which has the switch property? As you have learned, function composition is an operation on the set $\mathscr{T}$ of translations of $\mathscr{E}$. Can you tell whether composition of translations has the switch property? Explain.

Although most of your experience along the lines illustrated in the preceding paragraph has probably been gained in developing the algebra of real numbers, you will find that geometry can be developed

2. $\vec{a}$, $-\vec{a}$

Here is a sample quiz. It consists of some completion and true-false items which are typical of the kinds of questions you should expect your students to be able to answer. Answers are given in brackets.

Sample Quiz

1. $P + \vec{p}$ is a ___; $P - Q$ is a ___. [point; translation]
2. The translation which maps A on Q is ___. [Q - A]
3. Composing a translation $\vec{a}$ with the identity translation gives ___. [$\vec{a}$]
4. Composing $\vec{a}$ with its inverse gives ___. [the identity translation (or: $i_{\mathscr{E}}$ or: $\vec{0}$)]
5. If $P - Q = R - Q$ then ___. [$P = R$]
6. If $P + \vec{p} = P + \vec{r}$ then ___. [$\vec{p} = \vec{r}$]

True-False.

7. A resultant of translations is a translation. [True.]
8. Each translation has an inverse which is a translation. [True.]
9. Each point determines a proper translation. [False.]
10. Any translation maps a line onto a parallel line. [True.]
11. The identity translation is the inverse of each translation. [False.]
12. Any translation composed with itself is the identity translation. [False.]
13. Each two points determine a different translation. [False.]
14. Any translation is determined by a pair of points. [True.]

in a similar manner. In the next chapter we shall adopt three *postulates* [or: axioms, or: basic principles] which say that translations have certain of the properties which your work in Section 1.06 has convinced you that they do have. In later chapters we shall postulate a few other properties of translations, some with which you are already acquainted, and some which you will discover on the basis of experiments like those you carried out while studying Sections 1.05 and 1.06. From these postulates we can derive other theorems which tell us of other properties of translations. Some of these properties will be familiar to you, but others will be quite new. You will also discover that it is possible to define other geometric notions, like those of line, plane, angle, triangle, parallelism, perpendicularity, etc., in terms of just the notions of point and translation. These definitions of kinds of geometric figures and of geometric relations will be quite different from the descriptions which you used in Section 1.05, but the work you have done in Sections 1.05 and 1.06 will help you to see that the new definitions are appropriate. From these definitions and our postulates concerning translations you will be able to deduce theorems which tell you about properties of geometric figures and relations among such figures. In this way you will be able to explore the geometry of space and so enlarge your knowledge of geometry.

In our study of geometry, we shall state our postulates in the algebraic language which you were introduced to in Section 1.07, and we shall also use this language in stating many of our definitions and theorems. This has the advantage that you will be able to apply to the study of geometry many of the techniques which you have already learned while studying the algebra of real numbers. It has the disadvantage, however, that you might come to carry out the algebraic manipulations without thinking about why you are doing so. To avoid this, draw pictures to illustrate what your equations and other sentences say. Such pictures will serve other purposes besides helping you to keep in mind the geometry you should be thinking about. Sometimes such a picture will suggest a procedure which may help you to prove a theorem or to solve a problem. Sometimes it will suggest a conjecture which you will be able to justify on the basis of the postulates and definitions. If so, you will have discovered a new theorem.

1.09 Chapter Summary

Vocabulary Summary

function
value
domain
linear function
translation of $\mathscr{R}$
function composition
translation of $\mathscr{E}$

mapping
image
range
permutable functions
converse of a function
inverse
identity translation

Definitions

A *function* is a set of ordered pairs no two of which have the same first component [and any such set is a function].

A function is *linear* if and only if (i) Df is the set of all real numbers, and (ii) for some nonzero real number m and some real number b, $f(x) = mx + b$ for each real number x.

A *translation of* $\mathscr{R}$ is a linear mapping with slope 1.

For any functions f and g, (i) $D[g \circ f] = \{x \in Df \colon f(x) \in Dg\}$, and (ii) for each $x \in D[g \circ f]$, $[g \circ f](x) = g(f(x))$.

Other Theorems

Any subset of a function is a function.

For any function f and any function g such that $g \subseteq f$, (a) $Dg \subseteq Df$, and (b) $g = f$ if and only if $Dg = Df$.

Function composition is associative.

For any function f which has an inverse, $f^{-1} \circ f = i_{Df}$ and $f \circ f^{-1} = i_{Rf}$.

For any function f, f has an inverse if and only if there is a function g such that $g \circ f = i_{Df}$.

For any functions f and g, g is the inverse of f if and only if (i) $g \circ f = i_{Df}$ and (ii) $Dg = Rf$.

For any functions f and g, g is the inverse of f if and only if (i) $g \circ f = i_{Df}$ and (ii) $f \circ g = i_{Dg}$.

Conventions Concerning Points and Translations

(a) $A + \overrightarrow{a} = \overrightarrow{a}(A)$; so, $A + \overrightarrow{a} \in \mathscr{E}$. [A translation is a mapping of $\mathscr{E}$ into $\mathscr{E}$.]

(b) $-\overrightarrow{a} = \overrightarrow{a}^{-1}$; so, $-\overrightarrow{a} \in \mathscr{T}$. [The inverse of a translation is a translation.]

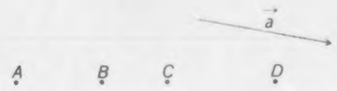
(c) $B - A =$ the translation from A to B; so, $B - A \in \mathscr{T}$.

(d) $\overrightarrow{0} = i_{\mathscr{E}}$; so, $\overrightarrow{0} \in \mathscr{T}$. [The identity mapping of $\mathscr{E}$ onto itself is a translation.]

(e) $\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{b} \circ \overrightarrow{a}$; so, $\overrightarrow{a} + \overrightarrow{b} \in \mathscr{T}$. [A resultant of translations is a translation.]

Chapter Test

1. Suppose that g and h are linear functions such that, for each x, $g(x) = 3x - 1$ and $h(x) = x + 2$. Complete the following.
   - **(a)** g maps -3 on ____.
   - **(b)** 5 is the image of ____ under h.
   - **(c)** $g(h(a)) =$ ________.
   - **(d)** $h(g($____$)) = 1$.
   - **(e)** $g(h($____$)) = a$.
   - **(f)** g maps $\{x: -2 \leq x \leq 3\}$ on $\{x:$ ________$\}$.
2. On your paper, draw a picture like the one below.

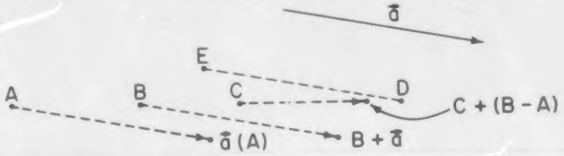


   - **(a)** Locate $\vec{a}(A)$.
   - **(b)** Locate $B + \vec{a}$.
   - **(c)** Locate $C + (B - A)$.
   - **(d)** Locate E such that $E + \vec{a} = D$.
3. Complete these sentences.
   - **(a)** The translation ____________ maps A on $A + \vec{b}$.
   - **(b)** The translation which maps $B + \vec{c}$ on B is ____________.
   - **(c)** The translation which maps $C + \vec{d}$ on $C + \vec{a}$ is ____________.
   - **(d)** The sides of a triangle ABC have lengths 5 in., 12 in., and 13 in. The image of triangle ABC under a translation which moves points 6 in. to the north is a triangle with sides of lengths ____, ____, and ____.
4. Each of the following expressions is either a point-expression, a translation-expression, or nonsense. In each case, tell which.
   - **(a)** $C - B$
   - **(b)** $(A - B) + \vec{c}$
   - **(c)** $A - (B + \vec{c})$
   - **(d)** $\vec{a} + B$
   - **(e)** $(A + \vec{c}) + B$
   - **(f)** $C + -(C - B)$

Answers for Chapter Test

1. (a) -10 (b) 3
   (c) $3a + 5$ (d) 0
   (e) $\frac{a - 5}{3}$ (f) $-7 \leq x \leq 8$

2. Your students should have a picture something like this:
   (a)-(d)



3. (a) $(A + \vec{b}) - A$ [or: $\vec{b}$]
   (b) $B - (B + \vec{c})$ [or: $-\vec{c}$]
   (c) $(C + \vec{a}) - (C + \vec{d})$ [or: $\vec{a} - \vec{d}$]
   (d) 5 in., 12 in., 13 in.

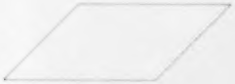
4. (a), (b) and (c) are translation-expressions
   (d) and (e) are nonsense
   (f) is a point-expression

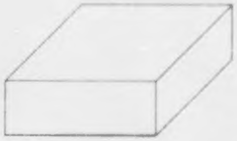
Chapter Two

A Start at Formalizing Our Intuitions

2.01 The Need for Postulates

In Chapter 1, we listed some properties of translations of $\mathcal{E}$, that is, translations of the set of points of space. You became aware of some of these properties by making use of arrows to represent translations. Although we cannot rely completely on diagrams and intuitive notions in our study of geometry, we shall quite often refer to diagrams and make use of our intuitions to help us as we proceed in our formal development of geometry. As we mentioned at the close of Chapter 1, diagrams can be quite useful in helping us to develop a deeper understanding and insight into the geometric ideas that are discussed. There are, however, serious limitations involved with deriving information from diagrams, and it is important for you to be aware of these limitations. For example, in the figure below, we cannot assume that the angles are not all right angles for it just may be the case that we are looking at a picture of the top of a rectangular box:





Discuss some of the reasons why we should not rely completely on pictures in our study of geometry. Some reasons you might consider are

(a) optical illusions,
(b) inaccurate measurements and drawings,
(c) danger of accepting a general result on the basis of a single drawing [or even on the basis of many drawings],
(d) lack of precise definitions and proofs.

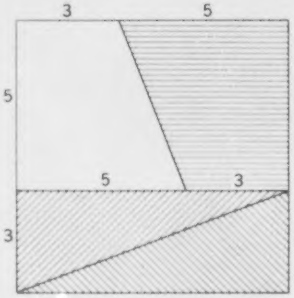
In this chapter we introduce our first three postulates [see pages 64, 65, and 105], and we prove five theorems. The theorems are of less importance than the methods used in proving them, and the bulk of the chapter deals with the rules of logic which [together with the postulates and theorems] are summarized on pages 111-113. It is worth emphasizing that, for this course, such rules are not an end in themselves — for example, we place no value on students' memorization of them. Experience has shown, however, that understanding such rules, and applying them occasionally to justify the steps in a formal proof, enables students to acquire — surprisingly rapidly — the ability to construct acceptable "paragraph proofs". The development of this ability is one of the purposes of this course.

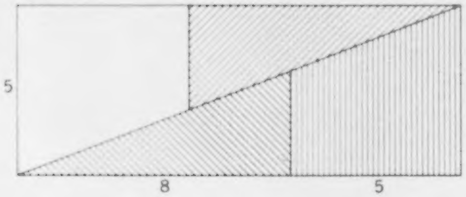
Up to this point in the course we have tried to develop the student's intuitions about how translations act on points by making use of tracing sheets, parallel rulers, and diagrams. What we want to point out here is that a formal treatment of a mathematical topic requires that postulates be established and that accepted results be those which are logical consequences of the postulates. Diagrams may suggest theorems but diagrams do not establish theorems.

The point to be emphasized in discussing the drawings on page 59 is that pictures require interpretation by the viewer. We must be wary as to what information we read into a picture in supplementing what we read out of it. Just as the communicative power of language rests on conventions as to the meanings of words, so does the communicative power of pictures rest on conventions. It is, for example, because of a convention which most of us accept that the second figure is interpreted as representing a closed box or a block rather than three coplanar parallelograms.

Exercise

Here is a square 8 centimeters by 8 centimeters. The area of the square is ____ square centimeters.





This square may be cut along the lines shown and then reassembled like this. This rectangular boundary is 13 cm. long and 5 cm. high. The area of the region bounded by the rectangle is ____ sq. cm. [What is wrong?]

As was mentioned in Chapter 1, we are going to study geometry by developing an algebra of points and translations. There will be many similarities between this study of geometry and your study of the algebra of real numbers, but there will also be some important differences. Before discussing some of the differences, let us recall the general procedure you may have used in studying the real numbers.

You probably expressed some properties of real numbers in sentences which you called *postulates* and from these you derived other sentences expressing further properties of real numbers. The postulates together with the sentences which can be derived from them are called *theorems.*

We plan to follow much the same procedure in studying geometry. We shall adopt some basic principles which we call *postulates.* The first few postulates will be suggested by what we learned from our intuitive discussion about translations of $\mathscr{E}$. Later we shall consider other properties of translations of $\mathscr{E}$ and add to our list of postulates.

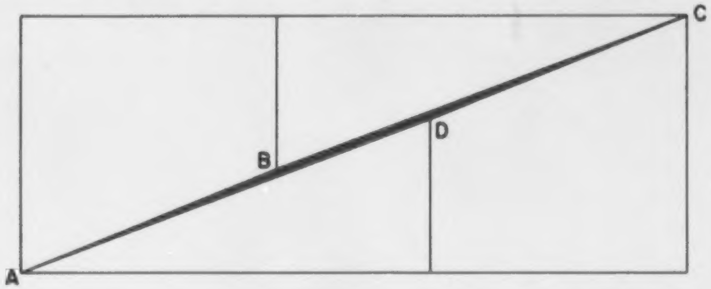
Just as in the study of real numbers, the postulates will serve as a foundation upon which we can organize our knowledge and as a starting point from which we can proceed to derive other theorems.

We now discuss some of the important differences between this study of geometry and your study of real numbers. In the case of real

Answers to Exercise

The area of the square region which is pictured on page 60 is $\underline{64}$ square centimeters; that of the region bounded by the rectangle is $\underline{65}$ square centimeters.

The students should have an intuitive feeling for the notion that if a geometric figure is dissected, the sum of the areas of the resulting parts is equal to the area of the original figure. With this notion, they will be able to see that there is "something more" in the rectangular region than just the "parts" of the original square region. Of course, it is not difficult to prove that the edges which appear to line up along the diagonal of the rectangular region really do not do so, for the slopes of the lines involved are different. With a careful diagram, one can illustrate where the "extra" 1 square centimeter comes from. It is the area of the region bounded by the parallelogram ABCD shown in the diagram below.



If you were to make a cardboard model of the given square region and then dissect it as indicated, you would find that it is very difficult to locate or to see the parallelogram ABCD. It turns out that the measure of the angle $\angle BCD$ is approximately $1° \, 15'$.

This puzzle is related to the Fibonacci sequence,

$$1,\ 1,\ 2,\ \underline{3},\ \underline{5},\ \underline{8},\ 13,\ \ldots,$$

each of whose terms [after the second] is the sum of the two which precede it. One property of this sequence is that the square of any term [except the first] differs from the product of the adjacent terms by -1 or 1. [$1^2 - 1 \cdot 2 = -1$, $2^2 - 1 \cdot 3 = 1$, $3^2 - 2 \cdot 5 = -1$, etc.] So, any square whose side-measure is a Fibonacci number can be dissected so that the parts can be fitted together in a rectangle in such a way that a parallelogram with area-measure 1 either is not covered or is covered twice.

The word 'theorem' is often used with the intent of excluding postulates from "theoremhood". Since any sentence is a logical consequence of itself, one who wishes to use 'theorem' in this way must be careful in his definition of the word — it will not do, for his purposes, to say that a theorem is a sentence which is a consequence of his postulates. Since, however, it is just these sentences which make up whatever theory is being developed, this definition seems to us a good one. [The statement in the text is equivalent to it, though redundant.] Postulates, then, are somewhat arbitrarily chosen theorems, from which the remaining theorems — as well as they, themselves — can be derived.

* * *

To develop this point a bit further, let's define a deductive theory to be any set of sentences which is such that each sentence which is a logical consequence of members of the set is also a member of the set. In short, a deductive theory is a set of sentences which is closed with respect to deduction. We choose to refer to the members of a deductive theory as the theorems of that theory. For example, the set of all true sentences concerning some subject matter — say, geometry — which can be formulated in a given language is a deductive theory. This is so because, since logical consequences of true sentences are true, this set of sentences is deductively closed. For another example, if one chooses certain of these sentences then the set of all consequences of the chosen sentences is also a deductive theory. This is so because a sentence which is a consequence of sentences which can be derived from the chosen ones is also a consequence of the chosen sentences. The chosen sentences are called 'postulates', and any one of many sets of postulates will engender, through deduction, the same deductive theory.

We have, then, two ways [at least] of obtaining deductive theories. We may take all true sentences concerning the subject which interests us, or we may take the "deductive closure" of some set of these sentences. A common procedure — and the one upon which we are now embarking — is to adopt the latter alternative, and to try to choose our postulates in such a way that the resulting deductive theory will include among its theorems a large number and variety of the theorems of the deductive theory which consists of all true sentences. To this end, we begin by choosing a few promising postulates and investigating some of their consequences. Having done so, we adopt additional postulates — defining in this way a more inclusive deductive theory — and see what additional sentences become theorems. We may hope by this means to arrive at last at a set of postulates whose deductive closure is the set of all true sentences. At any rate, we shall by this method obtain a better understanding, and a more nearly complete knowledge, of the subject.

Another procedure — aside from merely adopting additional postulates — for enlarging a deductive theory is that of adding new words to our language. This automatically enlarges the set of true sentences. In order to catch up, we must, of course, adopt some of these new true sentences as postulates. Sometimes the new words merely allow us to say more simply things which we could say without using them. In such a case we can catch up by adopting a postulate of the particularly simple kind called definition. As has been implied, definitions [in a deductive development] are postulates. They differ from other postulates only in that we could, in theory, get along without them. [In practice, we couldn't.] Note that any true sentence of a certain form can serve as a definition. Whether it is adopted as a definition is a matter of choice, just as it is a matter of choice with regard to any true sentence whether that sentence is adopted as a postulate.

TC 61 (1)

It seems convenient to use abbreviations — like 'CPA' for 'the commutative principle for addition' — for common descriptive phrases. Some people, however, find it a nuisance. Use them or not, as you like.

Some students will have used quantifiers, '∀' and '∃', in their study of algebra, and some will not. Those who have not will learn how to use them when studying later chapters. For the present, and in Chapter 3 as well, we shall have no use for them.

Even though students have used generalizations in their work in algebra, some may have difficulty in making or in recognizing instances of generalizations about points and translations. One reason for this is that they must now begin to concern themselves about the domains of the variables involved in the generalizations they work with. In their work in algebra, all of the generalizations were about real numbers. In this work, the generalizations are about points, or translations, or real numbers, or even about sets of points. For this reason, special sorts of variables will be used for points, for translations, and for real numbers, and, later for certain kinds of sets of points such as lines, rays, planes, etc. It may take some time for all of the students to get thoroughly familiar with this notion.

One device which has proved to be quite helpful in checking instances of generalizations [especially on a chalkboard] involves the use of frames and underlines, instead of letter variables, in pattern sentences. For example, consider the sentence:

(1) $$A + (B - A) = B$$

To show that the sentence:

(2) $$(P + \vec{r}) + [(Q + \vec{s}) - (P + \vec{r})] = Q + \vec{s}$$

is an instance of (1), one might establish a convention for drawing loops around expressions for translations and underlining expressions for points. [Colored chalk works very nicely here.] Making use of this convention, we see that (1) has the pattern:

$$\underline{\underline{\quad A \quad} + \boxed{\underline{\quad B \quad} - \underline{\quad A \quad}}} = \underline{\quad B \quad}$$

and that (2) fits this pattern precisely.

$$\underline{\underline{\underline{P} + \boxed{\vec{r}}} + \boxed{\underline{\underline{Q} + \textcircled{$\vec{s}$}} - \underline{\underline{P} + \boxed{\vec{r}}}}} = \underline{\underline{Q} + \textcircled{$\vec{s}$}}$$

Of course, we must know that '$P + \vec{r}$' and '$Q + \vec{s}$' refer to points in order to see that (2) fits the pattern for (1). It is this feature of the generalizations that we will be dealing with in this course that is different from the generalizations met in elementary algebra.

numbers, you dealt with only one kind of thing, namely, with real numbers. So, only one kind of variable [lower case letters] was needed. Also, since the operations dealt with always led from real numbers to real numbers, there was never any need to say that the result of an operation was a real number. For example, you could be sure that every value of '$a - b$' was a real number [There was nothing else it could be!]. Consequently, you could be sure that:

$$(a - b) + c = c + (a - b)$$

was an instance of the commutative principle for addition [the CPA]. For these reasons you did not need a name [such as '$\mathscr{R}$'] for the set of real numbers, nor did you need a postulate such as:

$$\forall_x \forall_y\, x - y \in \mathscr{R}$$

[Read '$\forall_x$', for example, as 'for *each* real number x'.]

On the other hand, in our proposed study of geometry we shall be dealing with three kinds of things—points, translations, and [later] real numbers. So, in order to deal easily with these things, we need three kinds of variables. We shall use

upper case letters [$A, B, \ldots$] as variables for points,
lower case letters with arrows [$\vec{a}, \vec{b}, \ldots$] as variables for translations, and
lower case letters [$a, b, \ldots$] as variables for real numbers.

In addition to dealing with three kinds of things, it is also the case, as you saw in Section 1.07, that the operations we shall deal with "mix up" these different kinds of things. For example, the result of adding a translation to a point is a point, and the result of subtracting a point from a point is a translation. Consequently, we must be more particular when getting instances from generalizations than we were in our study of real numbers. For example, consider the generalization:

$$(*) \qquad \forall_X \forall_{\vec{x}}\, (X + \vec{x}) - X = \vec{x}$$

or the "free variable generalization":

$$(**) \qquad (A + \vec{a}) - A = \vec{a} \qquad \text{[Exercise 2 on page 52]}$$

Some sentences which are instances of both (*) and (**) are:

$$[P + (Q - P)] - P = Q - P$$

and:

$$[(P + \vec{m}) + \vec{s}] - (P + \vec{m}) = \vec{s}$$

As an example of a sentence which is algebraically correct but is not an instance of the generalization (1) is the following:

$$\vec{a} + (B - \vec{a}) = B$$

This is not an instance of (1) because '$\vec{a}$' has values which are translations and only expressions whose values are points may be substituted for 'A' in (1).

In addition to keeping straight which kinds of substitutions are legitimate, it is also necessary to pay attention to the placement of grouping symbols. For example, the expressions:

$$(A + B) - A$$

and:

$$A + B - A$$

are meaningless in our system.

Give the substitutions for 'A' and '$\vec{a}$' which you would make in (**) to obtain these instances.

In order to know that the given sentences are instances of (**), we need to know that the values of '$Q - P$' are translations, and that those of '$P + \vec{m}$' are points. For this reason we need postulates which tell us that this is the case. To state these postulates, we shall use the names '$\mathcal{T}$' for the set of translations and '$\mathcal{E}$' for the set of points. The postulates we need are as follows:

(a) $B - A \in \mathcal{T}$

(b) $A + \vec{a} \in \mathcal{E}$

[Read (a) as '$B - A$ belongs to $\mathcal{T}$' or as '$B - A$ is a translation'; read (b) as '$A + \vec{a}$ belongs to $\mathcal{E}$' or as '$A + \vec{a}$ is a point'.]

In case you are used to using quantifiers [as in (*)] to formulate generalizations and are accustomed to think of "open sentences" like (**), (a), and (b) as merely showing forms which statements may have, you will need to remember that, in this course, when you "assert" such a sentence – say, (a) – you are using this sentence just as you are accustomed to use a quantified sentence – for example, the sentence:

(a′) $\forall_X \forall_Y X - Y \in \mathcal{T}$

In particular, adopting (a) as a postulate has the same effect as adopting (a′) would have. An open sentence which is adopted as a postulate or definition, or one which is proved to be a theorem, has the same meaning as does a corresponding quantified sentence. What this amounts to is that such a sentence implies each of its instances.

Exercises

Part A

1. Here is a quantified sentence which can be used to state the associative principle for addition:

(*) $\forall_x \forall_y \forall_z (x + y) + z = x + (y + z)$

(a) Write a corresponding "free variable generalization" sentence which has the same meaning as (*).

(b) Does the sentence you wrote in (a) have any false instances?

2. Suppose that somebody told you that the following sentence is true:

(**) $\forall_x \forall_y \forall_z (x - y) + z = x - (y + z)$

(a) Show that this person is mistaken.

To obtain the first of the exhibited instances of the sentence (☆☆) on page 61, substitute 'P' for 'A' and 'Q - P' for '$\vec{a}$'. [The first substitution is legitimate because 'P' is a variable with the same domain, $\mathcal{E}$, as the variable 'A'. The second is legitimate because 'Q - P' refers to translations, and these constitute the domain of the variable '$\vec{a}$'.]

To obtain the second of the instances, substitute '$P + \vec{m}$' for 'A' and '$\vec{s}$' for '$\vec{a}$'. [The first substitution is legitimate because '$P + \vec{m}$' is a point-term and the values of 'A' are points. The second is legitimate because both '$\vec{s}$' and '$\vec{a}$' are translation-variables.]

Making use of the convention of drawing loops around expressions for translations and underlining expressions for points, we see that (☆☆) has the pattern:

$$\boxed{\underline{(\underline{A} + \boxed{\vec{a}})} - \underline{A}} = \boxed{\vec{a}}$$

and that each of the given sentences fits this pattern precisely:

$$\boxed{\underline{(\underline{P} + \boxed{Q - P})} - \underline{P}} = \boxed{Q - P}$$

$$\boxed{\underline{(\underline{(P + \vec{m})} + \boxed{\vec{s}})} - \underline{(P + \vec{m})}} = \boxed{\vec{s}}$$

The universal quantifier '∀' is used occasionally in this section only for the benefit of students who are unaccustomed to using "open sentences" to express universal generalities. If you have no such students, pass over the occurrences of '∀' as lightly as possible. A brief discussion of quantification is given on TC 67(1) and TC 67(2) but, if feasible, the subject should be ignored until it is introduced in the text.

Answers for Part A

1. (a) [Answers may vary here. They should all be of the form '(a + b) + c = a + (b + c)' and should be composed of variables — that is, lower case letters from near the beginning of the alphabet.]

   (b) No.

2. (a) [The students should give counter-examples to (**) in order to show that one who thinks that (**) is a true generalization is mistaken.]

(b) Write a corresponding free variable generalization sentence which has the same meaning as does (**).

(c) Show that the sentence you wrote in (b) is a false generalization.

3. Suppose that somebody told you that the following free variable generalization about points and translations is a true one:

$$(\psi) \qquad A - (A + \vec{a}) = \vec{a}$$

(a) Write a corresponding quantified sentence which has the same meaning as does (ψ).

(b) Is (ψ) a true generalization or not? Explain your answer.

(c) On the basis of your answer in (b), what can you say about the quantified sentence you wrote in (a)?

Part B

For the generalization listed in each exercise, make an instance as suggested. (There may be several correct answers.)

Sample: Make an instance of:

$$\forall_x \forall_y\, x + y = y + x$$

concerning the numbers $^+2$ and $^-8$.

Solution: Two correct instances we could make concerning $^+2$ and $^-8$ are:

$$^+2 + {^-8} = {^-8} + {^+2}$$

and: $$^-8 + {^+2} = {^+2} + {^-8}$$

1. Make an instance of:

$$\forall_x \forall_y \forall_z\, x(y + z) = xy + xz$$

concerning $^-3$, $^+1/2$, and 0.

2. Make an instance of:

$$\forall_x\, x + 0 = x$$

concerning $^+3$ and $^-2$.

3. Make an instance of:

$$B - A \in \mathcal{T}$$

(a) concerning P and $P + \vec{a}$; (b) concerning $A + \vec{a}$ and C;
(c) concerning $A + (B - A)$ and B; (d) concerning $\vec{a}$ and $A + \vec{a}$.

4. Make an instance of:

$$A + \vec{a} \in \mathcal{E}$$

(a) concerning P and $Q - P$; (b) concerning $B + \vec{c}$ and $\vec{b}$;
(c) concerning $A + \vec{a}$ and $\vec{a} + \vec{b}$; (d) concerning $\vec{a}$ and B.

---

2. (b) [Answers should all be of the form '(a - b) + c = a - (b + c)'.]

(c) [The same counter-examples as were given in (a) are appropriate here.]

3. (a) There are three possible answers for this part. They are:

(i) $\forall_X \forall_{\vec{x}}\; X - (X + \vec{x}) = \vec{x}$

(ii) $\forall_X\; X - (X + \vec{a}) = \vec{a}$

(iii) $\forall_{\vec{x}}\; A - (A + \vec{x}) = \vec{x}$

(b) No. We already know that $\vec{a}$ is the translation from A to $A + \vec{a}$. If it were the case that $\vec{a}$ is also the translation from $A + \vec{a}$ to A, then $\vec{a}$ would be its own inverse. But, the only translation which is its own inverse is the identity translation. So, (ψ) is not true for any translation except $\vec{0}$. [This argument suggests a source of counter-examples, namely translations different from $\vec{0}$.]

(c) It is false.

* * *

Part B of the exercises focuses attention on a matter that has been a problem with some students in this course. As we have indicated, stating generalizations without quantification will be to our advantage in this work and is something mathematics students should learn to do. However, students who are accustomed to dealing with generalizations in contexts where quantifiers are used may have trouble at first making and recognizing instances of generalizations stated without quantification. Part of the problem is that it is sometimes necessary for letters used in the generalization to also be used, perhaps in different positions, in an instance of the generalization. We have included exercises like those in Part B in several other places in the forthcoming material. If your students seem to have trouble with this activity, it would be wise to generate work sheets with additional exercises of this type for them.

Answers for Part B

1. [There are six correct answers which use $^-3$, $^+\frac{1}{2}$, and 0 as values for the indices. There are infinitely many answers if combinations of $^-3$, $^+\frac{1}{2}$, and 0 are used. For example, '$(^+\frac{1}{2} + {^-3})(^+\frac{1}{2} + 0) = (^+\frac{1}{2} + {^-3}) \cdot {^+\frac{1}{2}} + (^+\frac{1}{2} + {^-3}) \cdot 0$' is one such instance.]

$^-3(^+\frac{1}{2} + 0) = {^-3} \cdot {^+\frac{1}{2}} + {^-3} \cdot 0$ $\qquad$ $^+\frac{1}{2}(0 + {^-3}) = {^+\frac{1}{2}} \cdot 0 + {^+\frac{1}{2}} \cdot {^-3}$

$^-3(0 + {^+\frac{1}{2}}) = {^-3} \cdot 0 + {^-3} \cdot {^+\frac{1}{2}}$ $\qquad$ $0(^-3 + {^+\frac{1}{2}}) = 0 \cdot {^-3} + 0 \cdot {^+\frac{1}{2}}$

$^+\frac{1}{2}(^-3 + 0) = {^+\frac{1}{2}} \cdot {^-3} + {^+\frac{1}{2}} \cdot 0$ $\qquad$ $0(^+\frac{1}{2} + {^-3}) = 0 \cdot {^+\frac{1}{2}} + 0 \cdot {^-3}$

2. $(^+3 + {^-2}) + 0 = {^+3} + {^-2}$ [Notice that it is sometimes clarifying to insert grouping symbols in an instance that do not appear in the generalization.]

3. (a) $P - (P + \vec{a}) \in T$ or
$(P + \vec{a}) - P \in T$

(b) $(A + \vec{a}) - C \in T$ or
$C - (A + \vec{a}) \in T$

(c) $[A + (B - A)] - B \in T$ or
$B - [A + (B - A)] \in T$

2.02 Our First Postulates and Another Theorem

As just mentioned, we shall adopt these sentences:

(a) $B - A \in \mathscr{T}$
(b) $A + \vec{a} \in \mathscr{E}$

as postulates to use in determining what terms we may substitute for our different kinds of variables. Let's review what these sentences mean.

In Section 1.07 you learned that, given a point A and a point B, there is a mapping of $\mathscr{E}$ into itself which is called *the translation from A to B*. This translation could be pictured by marking points A and B on a sheet of paper, copying this on a tracing sheet, and sliding the tracing sheet, without twisting, over the paper until the mark which was over A is over B. This translation could also be described in terms of the points A and B and parallel lines. Since this unique translation is certainly determined when we are given A and B, we may, as we decided in Section 1.07, call it '$B - A$'. In adopting (a) as a postulate we are merely giving notice that, for any points A and B, $B - A$ is a translation.

Having adopted this subtraction notation for translations it seemed, in Section 1.07, sensible to use an addition notation, '$A + \vec{a}$', to refer to the image of a point A under a translation $\vec{a}$. Since a translation is a mapping of $\mathscr{E}$ into $\mathscr{E}$ it follows that, for any translation $\vec{a}$ and any point A, $A + \vec{a}$ is a point. This is what (b) says.

We combine (a) and (b) into a single postulate, Postulate 1.

> Postulate 1 (a) $B - A \in \mathscr{T}$ (b) $A + \vec{a} \in \mathscr{E}$
>
> (a) A difference of points is a translation.
> (b) A sum of a point and a translation is a point.

Exercises

In each of the following exercises, complete the given expression so that, when the result is treated as a universal generalization, it is a true one. [If it is not possible to do so, say so.] You may use '$\mathscr{E}$', '$\mathscr{T}$', or a variable for a point or a translation.

1. $A + \vec{b} \in$ ______
2. $\vec{a} + B \in$ ______
3. $A - A \in$ ______
4. $C - B \in$ ______
5. $C +$ ______ $\in \mathscr{E}$
6. ______ $- B \in \mathscr{E}$
7. $B -$ ______ $\in \mathscr{T}$
8. ______ $- B \in \mathscr{T}$
9. $A +$ ______ $\in \mathscr{T}$

(d) We cannot make an instance concerning $\vec{a}$ by using '$\vec{a}$' as a value for 'A' or for 'B', since the values of both 'A' and 'B' are points, not translations. Of course, if '$\vec{a}$' is used in a point-valued expression then the resulting sentence is, in a way, about $\vec{a}$ and whatever else was used in that point-valued expression. For example, '$(C + \vec{a}) - (A + \vec{a}) \in T$' is an instance of '$B - A \in T$' and is about $C + \vec{a}$ — and, so, about C and $\vec{a}$ — and $A + \vec{a}$.

4. (a) $P + (Q - P) \in \mathcal{E}$
   (b) $(B + \vec{c}) + \vec{b} \in \mathcal{E}$
   (c) $(A + \vec{a}) + (\vec{a} + \vec{b}) \in \mathcal{E}$
   (d) $B + \vec{a} \in \mathcal{E}$

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As is illustrated with Postulate 1, we frequently accompany postulates and theorems by "English sentences" which approximate as closely as possible the "algebra sentences" in question. [This is not always easy; if it were, there would be less reason than there is for developing mathematical notation.]

The purpose of exercises such as those following the statement of Postulate 1 is to help the student become familiar with the kinds of expressions that are meaningful in the formal system we are developing. It may be the case that you will need to construct more such exercises in order to give individual students more practice. One practice which proved to be helpful was that of having students generate some expressions which make sense and some which are "nonsense".

We wish to call attention to the facts that in our system, (a) adding a point to a translation is not meaningful, (b) neither is subtracting a point from a translation and (c) neither is adding a point to a point. This is not to say that systems cannot be constructed in which these processes are meaningful. It is simply the case that there is no real need to deal with such expressions in this particular formal system. As things have been developed to this point, and as they will continue to be developed, each of the meaningful expressions is introduced via the postulates and is based on our intuitions about points and translations. The care that must be exercised in dealing correctly with expressions about points and translations will serve to reinforce the dependence of one's results on the postulates.

Answers to Exercises

1. $\mathcal{E}$
2. [not possible]
3. T
4. T
5. $\vec{a}$ [or, any variable for a translation]
6. [not possible]
7. A [or, any variable for a point]
8. A [or, any variable for a point]
9. [not possible]

Notice that Postulate 1 tells us absolutely nothing about how the translation $B - A$ relates to points A and B, nor does it tell us how the point $A + \vec{a}$ relates to point A and translation $\vec{a}$. For this we need additional postulates.

In Section 1.06 you saw that the translation *from A to B* maps A on B. Using our addition-subtraction notation we may formulate this as:

(c) $$A + (B - A) = B$$

You also learned that, given any point A, any translation $\vec{a}$ is determined by the image of A under $\vec{a}$. This image is $A + \vec{a}$ and, as we have just noted, A has the same image under the translation from A to $A + \vec{a}$. It follows that, however the translation $\vec{a}$ may have been chosen, $\vec{a}$ is the translation from A to $A + \vec{a}$. Using our addition-subtraction notation, this may be formulated as:

(d) $$\vec{a} = (A + \vec{a}) - A$$

We combine (c) and (d) into a single postulate, Postulate 2.

Postulate 2 (a) $A + (B - A) = B$ (b) $\vec{a} = (A + \vec{a}) - A$

(a) The translation from A to B maps A on B.

(b) $\vec{a}$ is the translation from A to $A + \vec{a}$.

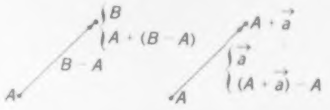

Fig. 2–3

[Notice the similarity between Postulates 2(a) and 2(b) and the real number sentences

'$a + (b - a) = b$' and '$a = (b + a) - b$'.]

Exercises

Part A

In each of the following exercises complete the expression so that when the result is treated as a universal generalization, it is a true one. [If it is not possible to do so, say so.]

1. $B + (B - B) =$ ____ **2.** ____ $= [A + (B - A)] - A$

3. $B - C = [B +$ ____ $] - B$ **4.** ____ $= [A + (A - A)] - A$

The purpose of these exercises is to give the students some practice in making use of Postulates 2(a) and 2(b). It may be that more such exercises are needed in individual cases. Since none of the exercises in Part A are impossible to complete in a meaningful way, you may wish to make up a few expressions which cannot be completed. Here are a few you might start with:

$$(A - B) + B = \underline{\qquad}$$

$$[A + (B - A)] + [B + (A - B)] = \underline{\qquad}$$

$$[(P - Q) + A] - (P - Q) = \underline{\qquad}$$

Note that while each of the completed sentences is a consequence of Postulate 2, only the first four are instances of one or the other part of this postulate. [Exercise 2 gives an instance of Postulate 2(b) which is also a consequence of Postulate 2(a).]

Answers for Part A

1. B
2. B - A
3. B - C
4. A - A [Formally, we cannot accept '$\vec{0}$' here as we have nothing in our postulates at this time which tells us what '$\vec{0}$' is.]

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5. $B + \vec{a}$
6. $B + \vec{a}$
7. A - A
8. $\vec{a}$

Answers for Part B

[There are infinitely many correct answers for each part. We give some of the obvious ones.]

1. (a) $(P + \vec{a}) + [Q - (P + \vec{a})] = Q$;
$Q + [(P + \vec{a}) - Q] = P + \vec{a}$

(b) There is no instance which uses '$\vec{a}$' as a value for 'A' or for 'B'. Of course, if '$\vec{a}$' is used in a point-valued expression then the resulting sentence is about $\vec{a}$ and whatever else was used in that expression. With this in mind, the following are among the possible answers:

$$R + ((R + \vec{a}) - R) = R + \vec{a} \qquad (R + \vec{a}) + (R - (R + \vec{a})) = R$$

There are, of course, infinitely many possible correct answers.

(c) $A + (C - A) = C$; $C + (A - C) = A$

(d) $[P + (Q - P)] + \{B - [P + (Q - P)]\} = B$;
$B + \{[P + (Q - P)] - B\} = P + (Q - P)$

2. (a) $B - A = [B + (B - A)] - B$

(b) $\vec{a} + \vec{b} = [C + (\vec{a} + \vec{b})] - C$

(c) $\vec{b} = \{[A + (B - A)] + \vec{b}\} - [A + (B - A)]$

(d) $(B - A) = ((A + \vec{a}) + (B - A)) - (A + \vec{a})$;
$(B - A) + \vec{a} = (A + ((B - A) + \vec{a})) - A$ [This assumes that $(B - A) + \vec{a}$ is a translation.]

It might be argued in connection with Theorem 2-1 that we are not yet in a position to prove a theorem since we have not established the "logical ground rules" under which we intend to operate. What we wish to do at this stage of the game is to motivate a discussion of some matters of logic. We do this by presenting proofs of both "parts" of Theorem 2-1.

5. $[A + (B - A)] + \vec{a} =$ ____ **6.** $[A + (B - A)] + [(A + \vec{a}) - A]$ $=$ ____

7. $A - [B + (A - B)] =$ ____ **8.** $[P + \vec{a}] - [A + (P - A)] =$ ____

Part B

Make an indicated instance of each generalization.

1. An instance of:

$$A + (B - A) = B$$

(**a**) concerning $P + \vec{a}$ and Q; (**b**) concerning R and $\vec{a}$;
(**c**) concerning A and C; (**d**) concerning $P + (Q - P)$ and B.

2. An instance of:

$$\vec{a} = (A + \vec{a}) - A$$

(**a**) concerning $B - A$ and B; (**b**) concerning C and $\vec{a} + \vec{b}$;
(**c**) concerning $A + (B - A)$ and $\vec{b}$; (**d**) concerning $B - A$ and $\vec{a}$.

We can now prove a theorem:

> Theorem 2-1.
> $A + \vec{a} = B$ if and only if $\vec{a} = B - A$

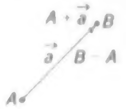

Fig. 2–4

[Recall that a theorem is a sentence which is a logical consequence of our postulates.] To *prove* a theorem is to show that it is a consequence of our postulates. Theorem 2-1 will follow if we can prove two simpler theorems:

(a) If $\vec{a} = B - A$ then $A + \vec{a} = B$.
(b) If $A + \vec{a} = B$ then $\vec{a} = B - A$.

[*STOP*. Before reading the proofs for (a) and (b), try to prove these sentences yourself. Try to derive (a) by logical reasoning from Postulates 1 and 2. Do the same for (b).]

Proof of Theorem 2-1. [We first prove (a).]

Assume that $\vec{a} = B - A$. By Postulate 2(a),

(1) $A + (B - A) = B$.

The strategy employed in these proofs will be somewhat familiar to the students from their work in algebra. Notice that we invite the students to try their hands at writing their own proofs of the theorem before reading the proofs in the text. Students who make a practice of this should gain power and confidence in their ability to use both the rules of reasoning and concepts in the formal system we are developing.

We shall, generally, use 'derivation', or 'argument' when referring to a discussion which purports to show that a given conclusion is a consequence of given premisses. [A derivation may be valid or invalid.] A *proof* is a derivation which shows that a given conclusion is a theorem — that is, is a consequence of our postulates. More loosely, a proof is a valid derivation whose premisses are sentences already known to be theorems.

The bracketed remarks inserted in the proof of Theorem 2-1 are, of course, not part of this proof. Rather, they serve to explain the proof. It is recommended that the teacher write the proof on the board or overhead as the discussion on pages 66 and 67 proceeds so that students see those sentences actually constituting this proof.

In connection with the proof of (a) you *may* wish to point out that the use of '$\vec{a} = B - A$' as an *assumption* is quite different from asserting Postulate 2(a). In writing 'Assume that $\vec{a} = B - A$.' we are in effect saying "Consider some translation $\vec{a}$, and some points A and B, such that $\vec{a} = B - A$." On the other hand, our justification for writing, in this context, 'By Postulate 2(a), $A + (B - A) = B$.' is that this postulate asserts that, for *any* points, including the points referred to in the assumption, the result of adding their difference to the first of them is the second. [The distinction might be made clearer by using 'C' instead of 'A', and 'D' instead of 'B' throughout the argument. Postulate 2(a) is unaffected, but (1) becomes a different sentence which is an instance of this postulate.]

Since the subject of the preceding paragraph will be taken up later, you may well ignore it now, unless students raise questions about it.

There are other proofs for (a) and (b). For example:

[(a)] Suppose that $\vec{a} = B - A$. Since $A + \vec{a} = A + \vec{a}$ it follows [replacing the '$\vec{a}$' on the right by '$B - A$'] that $A + \vec{a} = A + (B - A)$. Since, by Postulate 2(a), $A + (B - A) = B$ it follows that $A + \vec{a} = B$. Hence, if $\vec{a} = B - A$ then $A + \vec{a} = B$.

[(b)] Suppose that $A + \vec{a} = B$. Since $B - A = B - A$ it follows that $(A + \vec{a}) - A = B - A$. Since, by Postulate 2(b), $\vec{a} = (A + \vec{a}) - A$ it follows that $\vec{a} = B - A$. Hence, if $A + \vec{a} = B$ then $\vec{a} = B - A$.

These proofs make use of the logically valid sentences '$A + \vec{a} = A + \vec{a}$' and '$B - A = B - A$' as premisses. The second of these sentences is also used in the derivation of Postulate 2(a) from sentence (a) and Postulate 1(a) which is given on page 64. As is pointed out in the discussion of this derivation, '$B - A = B - A$' is valid because "a thing is itself" *and* Postulate 1(a) assures us that the expression '$B - A$' refers to *things* of some kind. [A similar remark applies to '$A + \vec{a} = A + \vec{a}$'. On the other hand, '$A + B = A + B$', for example, is not a sentence, let alone a valid one, because '$A + B$' is nonsense.]

Since [by our assumption] $B - A$ is the same as $\vec{a}$ it follows that

$$(2) \qquad A + \vec{a} = B.$$

Hence [it follows from (1), above, that],

$$(3) \qquad \text{if } \vec{a} = B - A \text{ then } A + \vec{a} = B.$$

[*STOP*. As you see, our procedure for showing that (a) is a consequence of our postulates was to show that '$A + \vec{a} = B$' is a consequence of our postulates together with the assumption '$\vec{a} = B - A$'. From what we mean when we say 'if . . . then ___', it follows that 'if $\vec{a} = B - A$ then $A + \vec{a} = B$' is implied by our postulates alone.

In deriving (2) from our assumption and (1) we made use of the fact that by '=' we mean what 'is the same as' does. So, it is logical that, under our assumption, we may replace any occurrence of '$\vec{a}$' or of '$B - A$' by the other.

Now, before reading the proof we give for (b), try to write your own.]

Assume that $A + \vec{a} = B$. Since, by Postulate 2(b), $\vec{a} = (A + \vec{a}) - A$ it follows that $\vec{a} = B - A$. Hence, if $A + \vec{a} = B$ then $\vec{a} = B - A$. [To complete the proof of Theorem 2-1 it remains to be shown that this sentence is a consequence of (a) and (b). Briefly, this is so because of what we mean when we say '*if* and *only if*'. We shall go into this later in this chapter. Granted this point, our proof shows that Theorem 2-1 is a consequence of just Postulate 2.]

In carrying out the preceding proofs we have used several rules of reasoning. [The use of these rules was pointed out in the three bracketed remarks following the proofs.] Familiarity with the rules we have used is important if one is to learn to understand proofs and to make them up. So, in the next few sections, we shall discuss these rules.

2.03 Substitution

We are using sentences which contain variables as one way of making general statements about all values of these variables. For an example concerning real numbers, consider the sentence:

$$(1) \qquad \text{If } a + c = b + c \text{ then } a = b.$$

Here is another way of saying what (1) says:

> If the sum of a first number and a third number is the same as the sum of a second number and the third number then the first number is the same as the second number.

Sentence (1) illustrates one way of making a general assertion about all real numbers. Another way of making the same assertion is:

$$(1') \qquad \forall_x \forall_y \forall_z \ [\text{if } x + z = y + z \text{ then } x = y]$$

The use of the universal quantifier '∀' and indices [letters from the ends of our three alphabets] is discussed later in this course. There is, also, a relevant discussion in High School Mathematics, Course 1, pages 116 - 123. This latter discussion needs some modification to make it appropriate to the language developed in this text. Here, open sentences are used to make assertions about all values of the variables which occur in them; there, variables are merely place-holders — differing only in form from frames and underlinings — and serve only to show some of the forms which statements may have. What are, in the reference cited above, called 'open sentences' can, here, better be referred to as 'predicates'.

Although you will probably have no need to discuss the universal quantifier at this time, the following discussion may be helpful should the need arise. In our present language we can say that all real numbers have the property which is expressed by the predicate:

$$\square \cdot 1 = \square$$

by filling the places held by the place-holder '□' by occurrences of a variable:

$$a \cdot 1 = a$$

Another way [see the reference cited] of expressing the same generality is to write:

$$\text{Always } \boxtimes \cdot 1 = \boxtimes$$

or, more shortly:

$$\forall \boxtimes \cdot 1 = \boxtimes$$

where the place-holders have been crossed out to show that they no longer serve their function of holding places. Instead of crossing out the place-holders we might omit them, but indicate that the spaces which remain are not a new kind of place-holder by linking them with the quantifier:

$$\underbrace{\forall \quad \cdot 1 = \quad}$$

Next, we note that it is more economical merely to indicate the upward prongs of the horizontal bracket by some arbitrarily chosen symbol:

$$\underset{*}{\forall} \; \underset{*}{} \cdot 1 = \underset{*}{}$$

Finally, since letters are easier to write and since it is desirable not to use two lines for one sentence, we adopt the format:

$$\forall_x \ x \cdot 1 = x$$

But, since the function of letters when used as indices is different from their function when used [as in the present text] as variables, we take care not to use the same letter both as a variable and as an index. Specifically, variables are chosen from beginnings of alphabets, indices from the ends.

This sentence tells us many things about numbers. For instance, since we know that −9 and 11 are real numbers it tells us that

> if the sum of a first number and the number 11 is the same as the sum of −9 and 11 then the first number is −9.

Using variables we can express this by writing:

(2) If $a + 11 = -9 + 11$ then $a = -9$.

Since what (2) says is part of what (1) says, we shall say that (2) follows from (1), or that (2) is a *consequence* of (1), or that (1) *implies* (2). [Note, however, that to know that (1) implies (2) we had to know that both −9 and 11 are real numbers.]

Sentence (2) is obtained from sentence (1) by substituting the numeral '−9' for the variable '*b*' and the numeral '11' for the variable '*c*'. [Recall that to *substitute* an expression for a variable you must replace *each* occurrence of the variable by the given expression.] Here is another sentence which can be obtained from (1) by substitution:

(3) If $a + b = (2a - b) + b$ then $a = 2a - b$.

[What expression should you substitute for '*b*' in (1), and what expression should you substitute for '*c*', in order to obtain (3)?] Again, what (3) says is part of what (1) says [but, to know this one must know that $2a - b \in \mathscr{R}$ – that is, that the expression '$2a - b$' refers to real numbers].

For another example, consider Postulate 1(b):

(4) $A + \vec{a} \in \mathscr{E}$

This says that the result of adding any translation to any point is a point – for short, any point-translation sum is a point. Since we know by Postulate 1(a) that any point difference is a translation, it follows, for instance, that

(5) $A + (B - A) \in \mathscr{E}$.

Other "consequences-by-substitution" of Postulate 1(b) are:

$C + (B - A) \in \mathscr{E}$ and: $(A + \vec{b}) + (B - A) \in \mathscr{E}$

[For each of these sentences, tell what substitutions you must make in Postulate 1(b) to get it, and why these substitutions are legitimate.]

As our work with sentences (1) and (4) illustrates, one way to obtain

To obtain (3) by substitution in (1), substitute '2a - b' for 'b' and 'b' for 'c'.

To obtain '$C + (B - A) \in \mathscr{E}$' [on page 68] by substitution in (4) one must substitute 'C' for 'A' and 'B - A' for '$\vec{a}$'. The first of these substitutions is legitimate because 'C' is a variable whose domain is the same as that of 'A'. The second is legitimate because the values of 'B - A' belong [by Postulate 1(a)] to the domain of the variable '$\vec{a}$'.

To obtain '$(A + \vec{b}) + (B - A) \in \mathscr{E}$' from (4) one must substitute '$A + \vec{b}$' for 'A' and 'B - A' for '$\vec{a}$'. The first substitution is legitimate because the values of '$A + \vec{b}$' belong [by Postulate 1(b)] to the domain of 'A'. The reason for the legitimacy of the second substitution has been given in the preceding paragraph. [Note, as another example, that these instances of (4) are also instances of (5), and that the second of them is also an instance of the first.]

In discussing the preceding examples it may be well to point out that 'B - A' is, when considered as a term, an abbreviation of '(B - A)'. So, 'substitute 'B - A'' is always a short way of saying 'substitute '(B - A)''. While making the substitution one decides whether or not the resulting expression will be ambiguous if the parentheses are omitted. If not, one may — but need not — omit them. In ordinary algebra one usually adopts a number of rules for omitting grouping symbols. Because the algebra of points and translations with its two kinds of terms, is more complex, the only such rule we adopt is that for omitting outermost grouping symbols. [For example, we write (4) rather than '$(A + \vec{a}) \in \mathscr{E}$'.] One who feels quite "at home" in this algebra might, without danger, omit the parentheses about 'B - A' from each of the two instances of (4) which we have been discussing. For, since indicated additions of point-terms are meaningless, the omitted parentheses could be reintroduced in only one way. On the other hand, the expression '$A + \vec{b} + (B - A)$' is — or, will be, once we introduce addition of translations — ambiguous. It might be taken for an abbreviation of either '$(A + \vec{b}) + (B - A)$' or '$A + [\vec{b} + (B - A)]$'.

As noted in our discussion of the proof of (a) on page 66, variables sometimes occur in assumptions, and such sentences are not intended to be understood as assertions about all values of the variables which occur in them. This other use of open sentences requires a modification of the substitution rule. For this, see page 111.

a sentence which is a consequence of a given sentence is to substitute expressions for variables. Be careful to see that the expression substituted for a variable is one which refers to members of the domain of that variable. Sentences which can be obtained in this way are called *substitution-instances* [for short: instances] of the given sentence. Our first rule of logic is:

> **The Substitution Rule**
> Any sentence implies each of its substitution-instances.

As pointed out in connection with the preceding examples, this rule is reasonable because of the way we intend to use variables in sentences to express general properties of points, translations, and real numbers.

Although you should have little trouble in forming instances of a given sentence, you may need some practice before you find it easy to tell whether a given sentence is an instance of another. For example, the sentence:

(6) $Q + (C - D) = P$ if and only if $C - D = P - Q$

is an instance of a sentence which you should recall. [In case you don't, look for it on page 66.] Show that (6) is an instance of the sentence in question.

One way to find a "simpler" sentence of which (6) is an instance is to notice that the variables 'C' and 'D' occur in (6) only in the combination '$C - D$'. Since '$C - D$' refers to translations, (6) is an instance of a similar sentence which has a translation-variable—say, '$\vec{a}$'—where (6) has '$C - D$' [or '$(C - D)$']:

(7) $Q + \vec{a} = P$ if and only if $\vec{a} = P - Q$

The sentence (7) is "simpler" than (6) in that '$\vec{a}$' is "shorter" than '$C - D$'. Because of this it is easier to understand what (7) says than what (6) says, and it may be easier to recognize that it is similar to some sentence you have seen before.

As another example, consider:

(8) $B - A = [(B + \vec{a}) + (B - A)] - (B + \vec{a})$

This might be obtained from another sentence by substituting '$B - A$' for some translation-variable which does not occur in (8). For example, (8) is an instance of:

(9) $\vec{s} = [(B + \vec{a}) + \vec{b}] - (B + \vec{a})$

(6) is an instance of Theorem 2-1. To obtain (6) from Theorem 2-1, substitute 'Q' for 'A', 'C - D' for '$\vec{a}$', and 'P' for 'B'. These substitutions are legitimate, for 'Q' and 'P' are variables whose domain is the same as that of 'A' and of 'B', and the values of 'C - D' belong to the domain of '$\vec{a}$'.

Sample Quiz

Each of the following is either a point-expression, a translation-expression, a real number-expression, or nonsense. Tell which.

1. $(R + \vec{r}) - (P + \vec{r})$ [translation-expression]
2. $(P + \vec{p}) + (R - P)$ [point-expression]
3. $(\vec{p} + P) - (Q + \vec{p})$ [nonsense]
4. $(a + 3) - (b + 3)$ [real number-expression]
5. $(P + \vec{p}) + (Q + \vec{p})$ [nonsense]
6. $((P - Q) + Q) - P$ [nonsense]
7. $(\vec{p} + \vec{q}) + (P - Q)$ [translation-expression]
8. $(\vec{p} + 5) + (\vec{q} - 5)$ [nonsense]
9. $(P + (Q - R)) - (R + \vec{r})$ [translation-expression]
10. $(P + \vec{p}) + ((P + \vec{q}) - R)$ [point-expression]

Sentence (9) is an instance of, say:

$$\vec{b} = (C + \vec{b}) - C$$

and this last is an instance of Postulate 2(b):

$$\text{(10)} \qquad \vec{a} = (A + \vec{a}) - A$$

Evidently, (8) can be obtained from (10) by substituting '$B - A$' for '$\vec{a}$' and '$B + \vec{a}$' for 'A'. [Note that in order to obtain (8) the substitutions must be made "simultaneously". If one first substitutes '$B - A$' for '$\vec{a}$' in (10) one gets '$B - A = (A + (B - A)) - A$'. If one then substitutes '$B + \vec{a}$' for 'A' in this sentence, the result is a different instance of (10) than (8). Can you tell what it is?]

The importance of being able to see that (8) is an instance of (10) is due to the fact that (10) is a theorem. Since this is the case, and since sentences imply their instances, it follows that (8) is also a theorem. In brief, by showing that (8) is an instance of (10) [and recognizing that (10) is a theorem] we have *proved* (8). As this example illustrates, in looking for sentences of which a given sentence is an instance, we shall usually be looking for such sentences which we also know to be theorems [postulates included]. For example, although '$A + (A - A) = A$' is an instance of the very simple sentence '$B = A$', this fact is not very important. [Of what *postulate* is the given sentence an instance?]

Exercises

Part A

For each sentence, decide whether it is an instance of any of our [four] postulates or of Theorem 2-1. Justify your answers.

1. $B - A = [A + (B - A)] - A$
2. $A - B = A - [A + (B - A)]$
3. $A + \vec{a} = A + \vec{a}$ if and only if $\vec{a} = (A + \vec{a}) - A$
4. $A + (B - A) = B$ if and only if $B - A = B - A$
5. $\vec{a} + A \in \mathscr{E}$
6. $(P + \vec{p}) - (Q + \vec{q}) \in \mathscr{T}$
7. $B + A \in \mathscr{T}$
8. $A + (B - A) \in \mathscr{E}$
9. $(B - A) + A = B$
10. $B + \vec{a} = A + \vec{a}$ if and only if $\vec{a} = B - (A + \vec{a})$

Part B

We shall sometimes indicate an application of the substitution rule by writing a substitution-instance of a given sentence below a horizontal line and writing above the line the given sentence and [to the right of it] the sentences which assert that the substitutions performed are legitimate. For example, the diagram:

$$\frac{\text{If } a + c = b + c \text{ then } a = b. \qquad 2 \in \mathscr{R} \qquad 3 \in \mathscr{R}}{\text{If } a + 3 = 2 + 3 \text{ then } a = 2.} \text{ (Subst)}$$

The result of first substituting 'B - A' for '$\vec{a}$' in (10) and then substituting 'B + $\vec{a}$' for 'A' in the resulting sentence is:

$$B - (B + \vec{a}) = [(B + \vec{a}) + [B - (B + \vec{a})]] - (B + \vec{a})$$

The result of "simultaneous substitutions" can, however, always be obtained by successive substitutions. For example, to obtain (8) from (10), begin by substituting 'C' for 'A' to obtain:

$$\vec{a} = (C + \vec{a}) - C$$

Now, substitute 'B - A' for '$\vec{a}$' and, having done so, substitute 'B + $\vec{a}$' for 'C'. Another procedure is to begin by substituting 'B - C' for '$\vec{a}$' in (10). Then, substitute 'B + $\vec{a}$' for 'A' and, finally, 'A' for 'C'.

The sentence 'A + (A - A) = A' is an instance of Postulate 2(a). To obtain this particular instance, substitute 'A' for 'B'. This substitution is legitimate, for 'A' and 'B' are variables whose domain is the same.

Answers for Part A

1. Postulate 2(b); 'B - A' for '$\vec{a}$'
2. This sentence is not an instance of any of our postulates [or of Theorem 2-1]. Since it is an equation whose sides are translation-terms, the only postulate it could be an instance of is Postulate 2(b). And, substitution of 'A - B' for '$\vec{a}$' in this postulate does not yield the sentence in question. The latter sentence is, however, a consequence of Postulate 2(a), since it may be obtained from the valid sentence 'A - B = A - B' by replacing the second 'B' by 'A + (B - A)'. This kind of inference is discussed in section 2.04.
3. Theorem 2-1; 'A + $\vec{a}$' for 'B'
4. Theorem 2-1; 'B - A' for '$\vec{a}$'
5. Not an instance of any postulate or theorem, since '$\vec{a}$ + A' is nonsense.
6. Postulate 1(a); 'P + $\vec{p}$' for 'B', 'Q + $\vec{q}$' for 'A'
7. Like 5.
8. Postulate 1(b); 'B - A' for '$\vec{a}$'
9. Like 5.
10. From its form, this sentence could be an instance only of Theorem 2-1. It isn't.

"says" that the conclusion 'If $a + 3 = 2 + 3$ then $a = 2$.' is a consequence of the premisses 'If $a + c = b + c$ then $a = b$.', '$2 \in \mathscr{R}$', and '$3 \in \mathscr{R}$'. [Later we shall—as we already do in the statement of the substitution rule—omit the premisses '$2 \in \mathscr{R}$' and '$3 \in \mathscr{R}$'. For the present, they serve the useful purpose of reminding us that the expression we substitute for a variable must refer to things which are values of that variable.]

It is sometimes helpful to read the horizontal line in diagrams of this kind as 'therefore'. As another example, the diagram:

$$\frac{B - A \in \mathscr{T} \qquad A + \vec{a} \in \mathscr{E}}{C - (A + \vec{a}) \in \mathscr{T}} \text{ (Subst)}$$

indicates the application of the substitution rule to infer '$C - (A + \vec{a}) \in \mathscr{T}$' from Postulate 1(a). Postulate 1(b) is written to the right of Postulate 1(a) to indicate that the substitution of '$A + \vec{a}$' for 'A' is a legitimate one. The substitution of 'C' for 'B' is legitimate because 'C' is a variable of the same kind as 'B'.

1. Complete this diagram:

$$\frac{\vec{a} = \vec{a}}{B - A = B - A} \text{ (Subst)}$$

2. For each of the sentences in Part A which is an instance of a postulate or of Theorem 2-1, make a diagram which shows the appropriate application of the substitution rule.

2.04 Equations

Our simplest sentences are equations and membership-sentences. For example,

'$A + (B - A) = B$' is an equation, and
'$B - A \in \mathscr{T}$' is a membership-sentence.

In English, confusingly enough, one can use 'is' in saying what either of these says. The equation, for example, can be "translated" into:

$A + (B - A)$ is B.

The membership-sentence can be read as:

$B - A$ is a translation.

In the first of these two sentences the meaning of 'is' is that of 'is the same as' or of 'is identical with'. When 'is' is used with this meaning it is called *the 'is' of identity*. Evidently, the 'is' in the second sentence does not have this meaning. When 'is' is followed by a phrase of the

Answers for Part B

[Diagrams like those displayed here, and asked for as answers, are called inferences. Although an inference "says" that its conclusion follows logically from its premisses it may, of course, lie in saying so — that is an inference may be invalid. Most of our rules of logic will be assertions that inferences of one or another form are valid. The substitution rule may be interpreted as saying that inferences of the form discussed here are valid.

It seems well at this point to insist on students writing the "auxillary premisses" which indicate the legitimacy of the substitutions. We shall continue doing so throughout the present chapter. After this, there should be no danger in omitting them.]

1. $B - A \in T$

2. $$\frac{\vec{a} = (A + \vec{a}) - A \qquad B - A \in T}{B - A = [A + (B - A)] - A}$$

3. $$\frac{A + \vec{a} = B \text{ if and only if } \vec{a} = B - A \qquad A + \vec{a} \in \mathcal{E}}{A + \vec{a} = A + \vec{a} \text{ if and only if } \vec{a} = (A + \vec{a}) - A}$$

4. $$\frac{A + \vec{a} = B \text{ if and only if } \vec{a} = B - A \qquad B - A \in T}{A + (B - A) = B \text{ if and only if } B - A = B - A}$$

6. $$\frac{B - A \in T \qquad Q + \vec{q} \in \mathcal{E} \qquad P + \vec{p} \in \mathcal{E}}{(P + \vec{p}) - (Q + \vec{q}) \in T}$$

8. $$\frac{A + \vec{a} \in \mathcal{E} \qquad B - A \in T}{A + (B - A) \in \mathcal{E}}$$

We shall treat the relation of equality [better: identity] as pertaining to logic. What relation it is must, then, be explained by rules of logic. The requisite rules are the replacement rule for equations [page 74] and the introduction rule for equations [page 75]. An alternative procedure — adopted in, for example, some algebra texts — is to treat '=' as a mathematical predicate which is, then, characterized by postulates. In the case of our language — as so far developed — we would need ten such postulates:

$$\begin{cases} A = A \\ A = B \implies B = A \\ (A = B \text{ and } B = C) \implies A = C \end{cases} \qquad \begin{cases} \vec{a} = \vec{a} \\ \vec{a} = \vec{b} \implies \vec{b} = \vec{a} \\ (\vec{a} = \vec{b} \text{ and } \vec{b} = \vec{c}) \implies \vec{a} = \vec{c} \end{cases}$$

$$\begin{cases} A = B \implies A - C = B - C \\ A = B \implies C - A = C - B \end{cases} \qquad \begin{cases} A = B \implies A + \vec{a} = B + \vec{a} \\ \vec{a} = \vec{b} \implies A + \vec{a} = A + \vec{b} \end{cases}$$

Each enlargement of our symbolism would require the adoption of additional postulates to further specify the meaning of '='.

Our treatment of '=' as a logical predicate has the advantage that all such sentences — which would, otherwise, have to be adopted as postulates — become, essentially, logically valid sentences. More precisely, they can be derived from "closure postulates" like Postulate 1 by the use of our rules of logic. [See, for example, the discussion of (4) on page 84, as well as TC 84 and TC 85(1).]

TC 72 (1)

Note that our use of the word 'term' for expressions which have "things" as values — as opposed, for example, to expressions which are sentences — is broader than the usual use of the word in elementary algebra. The broader usage is convenient and is customary in discussions of logic.

form 'a so-and-so', it is called *the 'is' of membership.* It is in order to avoid this somewhat ambiguous use of 'is' that we use '=' in place of the 'is' of identity and 'ϵ' instead of the 'is' of membership. In particular, '=' means just what 'is the same as' does.

For an equation to make sense, the expressions which flank the '=' [the "sides" of the equation] should refer to the same kind of thing—numbers, points, or translations. Expressions which refer to things are called *terms*—for example, '$a + 2$' is a real number-term, '$A + \vec{a}$' is a point-term, and '$B - A$' is a translation-term. [Also, 'a' and '2' are real number-terms, 'A' is a point-term, and '$\vec{a}$' is a translation-term.] Formally, an *equation* is a sentence obtained by flanking an '=' with terms which refer to the same kind of things. Note that, by Postulate 1, both parts of Postulate 2 are equations.

Here is an example of how an equation may be used in a proof. Our problem is to prove:

$$\text{If } a + {}^-4 = 9 \text{ then } a = 13.$$

This desired conclusion may suggest the cancellation principle used as an example in the preceding section:

(1) $$\text{If } a + c = b + c \text{ then } a = b.$$

We know by the substitution rule that (1) implies the instance:

(2) $$\text{If } a + {}^-4 = 13 + {}^-4 \text{ then } a = 13.$$

Now, (2) is very much like the sentence we wish to prove. All we need do to get our conclusion is to replace '$13 + {}^-4$' in (2) by '9'. If we know that $13 + {}^-4$ *is* 9 then what (2) says about $13 + {}^-4$ must be "equally true" of 9. More formally, the equation:

(3) $$13 + {}^-4 = 9$$

and (2) together imply:

(4) $$\text{If } a + {}^-4 = 9 \text{ then } a = 13.$$

So, since (2) is a consequence of (1), (4) is a consequence of (1) and (3) together.

An argument like the preceding is equally valid in case the equation corresponding to (3) contains variables. To illustrate this we prove a cancellation principle for our algebra of points and translations:

$$\text{If } A + \vec{a} = A + \vec{b} \text{ then } \vec{a} = \vec{b}.$$

Note, also, that an equation is, in this text, not merely an '=' flanked by terms. The terms must be of the same "kind". For example, '$B - A = C + \vec{a}$' is not a false sentence; it is, rather, a meaningless expression.

The intuitive justification of the replacement rule as it is applied in the argument involving sentences (1) - (4) is very simple. [We shall discuss (b) - (7) later.] This justification rests on the insight that from a sentence — say, (3) — which says that a first object is a second object, and a sentence which asserts something about the first object, one may reasonably infer a sentence which says the same thing about the second object. From 'John is the boy next door.', and any sentence which is explicitly about John [i.e., which contains 'John'], it is legitimate to infer the corresponding sentence which is explicitly about the boy next door.

Note that the last-mentioned inference is valid whether or not John actually is the boy next door, and whether or not what is said about him is true. The validity of an inference depends in no way on the truth of its premisses. What the validity of an inference does depend on is the forms of its premisses and its conclusion. What the validity of an inference guarantees is that its conclusion is no whit more dubious than the most dubious of its premisses. As limiting cases, if the premisses of a valid inference are true — that is, are indubitable — then so must be its conclusion; and if the conclusion of a valid inference is false — utterly dubious — then so must be at least one of its premisses. This latter remark justifies the common kind of argument in which one establishes the falsity of a sentence by adopting it as a premiss and deriving, from it and other premisses which are true, a conclusion which is false. [Example: "You say that John is the boy next door. Well, we both know that John has a bicycle. But, it's not true that the boy next door has a bicycle." (As often happens, the speaker here assumes that his listener will supply the conclusion 'The boy next door has a bicycle.' which is implied by his assumptions.)] Of more moment for us is the type of proof — proof by contradiction — in which one shows that a sentence is a theorem by deriving a contradiction from premisses one of which is the denial of the sentence in question while the others are theorems. The justification of this kind of proof depends on rules of logic which will be discussed later. We mention it here, however, to emphasize the importance of recognizing that the validity of an inference is independent of the "acceptability", in any sense, of its premisses. [Incidentally, proof by contradiction bears no very close analogy to arguments like that in our example concerning John and his bicycle. Arguments of this latter type are of the form: If it were the case that p then it would be the case that q; but, it isn't. An attempt to give a similar form to proofs by contradiction leads to: If '~p' were a theorem then this resulting contradiction would be a theorem; but no contradiction is a theorem. Even if one is convinced that one's postulates are consistent [so that no contradiction is a theorem], all that an argument of this form shows is that '~p' is not a theorem. It is seldom the case that one can argue from this that 'p' is a theorem. Proof by contradiction requires, for its justification, stronger rules of logic than those which are sufficient to justify arguments of the previous kind.]

[This theorem says that, for any point A, if the image of A under a translation $\vec{a}$ is the same as the image of A under a translation $\vec{b}$ then $\vec{a}$ is the same translation as $\vec{b}$. In other words, there is at most one translation under which a given point has a given image.] We have already proved a theorem which looks something like this. It is (b) on page 66:

$$\text{(b)} \qquad \text{If } A + \vec{a} = B \text{ then } \vec{a} = B - A.$$

We could get a start toward the desired conclusion by substituting '$A + \vec{b}$' for 'B'. [Postulate 1(b) assures us that this is a legitimate substitution.] So, we know by the substitution rule that (b) implies its instance:

$$\text{(5)} \qquad \text{If } A + \vec{a} = A + \vec{b} \text{ then } \vec{a} = (A + \vec{b}) - A.$$

Just as in the preceding argument, the equation:

$$\text{(6)} \qquad \vec{b} = (A + \vec{b}) - A$$

and (5) together imply:

$$\text{(7)} \qquad \text{If } A + \vec{a} = A + \vec{b} \text{ then } \vec{a} = \vec{b}.$$

So, since (5) is a consequence of (b), (7) is a consequence of (b) and (6) together. [In fact, (6) is an instance of Postulate 2(b) and, as we have seen earlier, (b) is also a consequence of this postulate. So, (7) is a consequence of Postulate 2(b).]

The justification we gave for inferring (4) from (3) and (2) and for inferring (7) from (6) and (5) can be formulated as a rule of logic. Before doing so, it may be well to note a slight difference between these two inferences. In the first case we replaced an occurrence of the left side of the equation (3) by its right side in sentence (2). This is illustrated in the following diagram:

$$\frac{\mathbf{13 + {}^{-}4 = 9} \quad \text{If } a + {}^{-}4 = 13 + {}^{-}4 \text{ then } a = 13}{\text{If } a + {}^{-}4 = 9 \text{ then } a = 13}$$

In the second case we replaced an occurrence of the right side of the equation (6) by its left side in sentence (5).

$$\frac{\vec{\mathbf{b}} = \mathbf{(A + \vec{b}) - A} \quad \text{If } A + \vec{a} = A + \vec{b} \text{ then } \vec{a} = \mathbf{(A + \vec{b}) - A}}{\text{If } A + \vec{a} = A + \vec{b} \text{ then } \vec{a} = \vec{\mathbf{b}}}$$

Although your students are not likely to raise this issue, you should be aware that the justification for the use of the replacement rule in the argument involving the sentences (b) - (7) is somewhat more complicated. Although the sides '2 + 3' and '5' of equation (3) are "names for the same thing", this is not the case for the sides '$\vec{b}$' and '$(A + \vec{b}) - A$' of equation (6). Neither of these expressions is a name for anything. [The common explanation of equality — in which explanation, for clarity, we insert semi-quotes — is:

'a = b' means that [or, is true if and only if]
'a' and 'b' are names for the same thing.

With or without semi-quotes, this is clearly nonsense.] Nevertheless, if we treat (6) as an assertion then what it asserts is that whatever values are chosen for '$\vec{b}$' and 'A', the corresponding values of '$\vec{b}$' and '$(A + \vec{b}) - A$' are the same. So, from (6) and any sentence — say, (5) — which asserts something about all values of one of these terms, we may infer the sentence — in this case, (7) — which says the same thing about all values of the other.

As before, the validity of the inference from (6) and (5) to (7) is independent of the "acceptability" of its premisses. Since in this particular argument, the premisses are theorems, the conclusion is also a theorem. And, for our purposes, theorems happen to be acceptable. In "most" applications of the replacement rule it will not, however, be the case that the equation premiss is an "acceptable" sentence. For example, recall the argument given as a proof of the sentence (b) on page 87:

> Assume that $A + \vec{a} = B$. Since, by Postulate 2(b), $\vec{a} = (A + \vec{a}) - A$ it follows that $\vec{a} = B - A$. Hence, if $A + \vec{a} = B$ then $\vec{a} = B - A$.

The use of the replacement rule occurs in the second sentence of this argument. The validity of the inference from '$A + \vec{a} = B$' and Postulate 2(b) to '$\vec{a} = B - A$' may be justified intuitively just as, in the preceding paragraph, we justified the inference from (6) and (5) to (7). Here, however, the "acceptability" of the equation premiss and of the conclusion are a different matter. We indicate our unwillingness to accept everything that the equation '$A + \vec{a} = B$' says by using the phrase 'Assume that'. Since, however, the inference is valid even in the case of points and translations which do not satisfy the premiss '$A + \vec{a} = B$' we are intuitively justified in claiming that the final conclusion:

$$\text{If } A + \vec{a} = B \text{ then } \vec{a} = B - A.$$

is satisfied by *all* points and translations [strictly, by all which satisfy Postulate 2(b); but, this *is* all]. In short, granted the validity of the replacement inference, the acceptability of the conclusion (b) derives solely from the acceptability of Postulate 2(b). [*Formally*, the validity of the inference from Postulate 2(b) to (b) would be justified by the deduction rule of section 2.06. The preceding may be interpreted as an intuitive justification for this — fairly representative — application of the deduction rule.]

[In each case, we "chose sides" in such a way as to arrive at the desired result.] Since '=' has the meaning of 'is the same as', one procedure is just as logical as the other.

Another point of great importance is that with this kind of argument we do not need to replace one side of the equation by the other *everywhere* it occurs in the second sentence. For example, using left-by-right replacement, it follows from (6) and (5) that

$$\text{if } A + \vec{a} = A + \vec{b} \text{ then } \vec{a} = (A + [(A + \vec{b}) - A]) - A.$$

Note that we chose to replace only the second occurrence of '$\vec{b}$' in (5) by the right side of (6). Which occurrences we replace is determined by where we wish to arrive.

We can now state the first of two rules dealing with equations:

The Replacement Rule for Equations

Given an equation and a second sentence, if either side of the equation is replaced by the other side somewhere in the second sentence, the resulting sentence is a consequence of the given equation and sentence.

[When, in applying this rule, we replace the left side of the equation by its right side we say that we have used the *left-by-right replacement rule*.]

To arrive at our second rule of logic for dealing with equations we need to look at the kinds of reasons we may have for believing what a given equation says. As an example, consider Postulate 2(a):

$$A + (B - A) = B$$

This equation is our way of saying that, for any points A and B, the image of A under the translation from A to B is B. Our belief in this is based on our knowledge of translations. No one who did not know at least a little about what translations are could believe this. On the other hand, consider the equation:

$$B - A = B - A \tag{8}$$

which says that, for any points A and B, the translation from A to B is the translation from A to B. In order to believe this all one needs to know is that there is *something* which is called 'the translation from A to B'—he doesn't need to know any properties of translation, or even what a translation is. In short, we are willing to accept (8) just because we know that '$B - A$' is a term and that '=' means what 'is the same as' does.

A sentence which is even simpler than (8) is:

(9) $\overrightarrow{a} = \overrightarrow{a}$

Our grounds for accepting this are purely logical: '$\overrightarrow{a}$' is a variable and '=' means what 'is the same as' does. Sentences which, like (8) and (9), are acceptable on purely logical grounds are called *valid sentences*. We can now state our second rule for equations:

> **The Introduction Rule for Equations**
> The equations '$A = A$', '$\overrightarrow{a} = \overrightarrow{a}$', and '$a = a$' are valid sentences.

[The name of this rule of logic is meant to remind you that the rule tells you of equations which you may introduce into any argument without making any extra-logical commitments.]

You will learn more about valid sentences in the next section.

Exercises

In order to say that a sentence follows logically from one or more sentences we shall sometimes write the first sentence underneath a horizontal line and write the others above the line. For example, the diagram:

$$\frac{2 + 3 = 5 \quad \text{If } a + 3 = 2 + 3 \text{ then } a = 2.}{\text{If } a + 3 = 5 \text{ then } a = 2.}\ \text{(RRE)}$$

says that the sentences above the horizontal line imply the sentence below that line. [You may think of the horizontal line as standing for the word 'therefore'.] The replacement rule for equations tells you that what the above diagram says is correct. [When such a diagram is meant to illustrate the replacement rule for equations we write the "equation" on the left and the "second sentence"—which may, in special cases, also be an equation—on the right.]

You are to complete the diagrams in the following exercises correctly by using the replacement rule for equations.

Sample 1. $\underline{a - b = c \qquad c > c - 2}$

Discussion. Since the right side of '$a - b = c$' occurs twice in '$c > c - 2$' we can apply the right-by-left replacement rule in either of three ways to arrive at a *conclusion* which is a consequence of the *premisses* '$a - b = c$' and '$c > c - 2$'. We may replace, in '$c > c - 2$',

the first 'c' by '$a - b$', or
the second 'c' by '$a - b$', or
both 'c's by '$a - b$'.

Answers.

[replacing the first 'c'] $\dfrac{\mathbf{a - b = c \qquad c > c - 2}}{\mathbf{a - b > c - 2}}$

The "more" that is to be learned about valid sentences is that such sentences may be used as premisses of an argument and then "forgotten" when reporting the results. More explicitly,

> a sentence which is a consequence of given premisses, of which some are valid sentences is also a consequence of the other ["invalid"] premisses, alone.

* * *

The rule "for ignoring valid premisses" which has just been stated can be justified on the grounds of some underlying rules of logic which we do not discuss in the text. For completeness we state them here, and illustrate their use:

(C_1) A sentence is a consequence of any set of sentences to which it belongs.

(C_2) A sentence which is a consequence of a set of sentences, each of whose members is a consequence of a second set of sentences, is, itself, a consequence of this second set of sentences.

In explanation of these, "consequence" is a relation between sentences and sets of sentences — a given sentence may [or may not] be a consequence of a given set of sentences. For example, a sentence is a theorem if and only if it is a consequence of the set of postulates — whatever this set may be. (C_1) and (C_2) are statements concerning this relation. When supplemented by more special rules, such as the replacement rule for equations, they constitute a definition of the word 'consequence'.

If we take seriously the definition according to which a theorem is a sentence which is a consequence of the set whose members are our postulates, it is (C_1) which tells us that each postulate is a theorem; and it is (C_2) which tells us that any consequence of given theorems is, itself, a consequence of the postulates and, so, is a theorem.

For convenience, we define, in terms of 'consequence' two uses of the word 'valid':

(V_1) A valid sentence is one which is a consequence of the empty set [of sentences].

(V_2) A valid inference is one whose conclusion is a consequence of the set consisting of its premisses.

It follows, for example, that

1. Any consequence of [a set of] valid sentences is a valid sentence.

and

2. Any inference of the form '$\frac{p}{p}$' is a valid inference.

[The first follows from (V_1) and (C_2); the second, from (V_2) and (C_1).]

To obtain the "rule for ignoring valid premisses" we begin by noting:

3. A sentence which is a consequence of a given set of sentences is also a consequence of any set of sentences which has the given set as a subset.

[This follows from (C_1) and (C_2). By (C_1), each member of the given set is a consequence of the "larger" set. So, by (C_2), any sentence which is a consequence of the given set is, itself, a consequence of the "larger" set.] From this and (V_1) it follows at once that

4. A valid sentence is a consequence of any set of sentences.

[For, the empty set is a subset of any set.] Finally,

5. A sentence which is a consequence of a given set of sentences is also a consequence of the subset consisting of the members of this set which are not valid sentences.

[For, by (C_1), each of the non-valid members of the set is a consequence, of the subset of all such members; and by 4., each of the valid members of the set is also a consequence of this subset. So, by (C_2), since the given sentence is a consequence of the set in question, and each member of the set is a consequence of the subset, the given sentence is, itself, a consequence of the subset.]

[replacing the second 'c'] $\dfrac{\mathbf{a} - \mathbf{b} = \mathbf{c} \qquad c > \mathbf{c} - 2}{c > (\mathbf{a} - \mathbf{b}) - 2}$

[replacing both 'c's] $\dfrac{\mathbf{a} - \mathbf{b} = \mathbf{c} \qquad \mathbf{c} > \mathbf{c} - 2}{\mathbf{a} - \mathbf{b} > (\mathbf{a} - \mathbf{b}) - 2}$

Remark. Diagrams like these and those of Part B on page 70 are called *inferences*. An inference whose conclusion is a consequence of its premisses is said to be *valid*.

Sample 2. $\dfrac{B = A + \vec{a}}{C - (A + \vec{a}) = -(B - C)}$

Discussion. The desired conclusion contains both the left side 'B' of the equation, as well as its right side '$A + \vec{a}$'. So the conclusion can be obtained from the given equation and either one of these two possible second sentences:

(a) $C - B = -(B - C)$
(b) $C - (A + \vec{a}) = -((A + \vec{a}) - C)$

[If you use (a) you will use the left-by-right replacement rule to derive the conclusion, and if you use (b) you will use the right-by-left replacement rule to derive the conclusion.]

Answers. $\dfrac{B = A + \vec{a} \qquad C - B = -(B - C)}{C - (A + \vec{a}) = -(B - C)}$ [using (a) as a premiss]

$\dfrac{B = A + \vec{a} \qquad C - (A + \vec{a}) = -((A + \vec{a}) - C)}{C - (A + \vec{a}) = -(B - C)}$ [using (b) as a premiss]

1. $\dfrac{A + \vec{a} = B \qquad (A + \vec{a}) - (B + \vec{a}) \in \mathscr{T}}{}$

2. $\dfrac{b + c = a + c \qquad b + c = c + b}{}$

3. $\dfrac{B - A = \vec{a}}{A + \vec{a} = A + (B - A)}$

4. $\dfrac{M = N \qquad M \text{ is the midpoint of } \overleftrightarrow{AB}.}{}$

5. $\dfrac{M \text{ is the midpoint of } \overleftrightarrow{AB}. \qquad \text{The midpoint of } \overleftrightarrow{AB} \text{ belongs to } \overleftrightarrow{AB}.}{}$

6. $\dfrac{B = A + \vec{a} \qquad \text{If } A + \vec{a} = A + \vec{a} \text{ then } A = (A + \vec{a}) - \vec{a}.}{}$

7. $\dfrac{B - A = \vec{a} \qquad A + (B - A) = B \text{ only if } B - A = B - A.}{}$

8. $\dfrac{(A + \vec{a}) - A = \vec{a} \qquad \text{If } (A + \vec{a}) - A = B - A \text{ then } A + \vec{a} = B.}{}$

9. $\dfrac{a + b = c + d \qquad c + d < 0}{}$

10. $\dfrac{\qquad\qquad A + (C - A) = C}{A + (B - A) = B}$

Answers for Exercises

[The rule according to which the "equation" premiss of a replacement rule inference is to be written to the left of the "second" premiss is adopted here merely to make it easy to distinguish one premiss from the other in case both are equations. There is no logical necessity for such a rule, but you will find it a convenient one to adhere to.]

1. $(A + \vec{a}) - [(A + \vec{a}) + \vec{a}] \in T$; right-by-left replacement rule
   or: $B - (B + \vec{a}) \in T$; left-by-right replacement rule
2. $a + c = c + b$; left-by-right replacement rule
3. $A + \vec{a} = A + \vec{a}$; right-by-left replacement rule
   or: $A + (B - A) = A + (B - A)$; left-by-right replacement rule
4. N is the midpoint of $\overleftrightarrow{AB}$.; left-by-right replacement rule
5. M belongs to $\overleftrightarrow{AB}$.; right-by-left replacement rule, with the word 'is' playing the role of '=' in the first sentence.
6. There are seven [$2^3 - 1$] possibilities, each involving the use of the right-by-left replacement rule. The possible answers are:
   (i) If $B = A + \vec{a}$ then $A = (A + \vec{a}) - \vec{a}$.
   (ii) If $A + \vec{a} = B$ then $A = (A + \vec{a}) - \vec{a}$.
   (iii) If $B = B$ then $A = (A + \vec{a}) - \vec{a}$.
   (iv) If $A + \vec{a} = A + \vec{a}$ then $A = B - \vec{a}$.
   (v) If $B = A + \vec{a}$ then $A = B - \vec{a}$.
   (vi) If $A + \vec{a} = B$ then $A = B - \vec{a}$.
   (vii) If $B = B$ then $A = B - \vec{a}$.
7. There are seven possibilities, each involving the use of the left-by-right replacement rule. The possible answers are:
   (i) $A + \vec{a} = B$ only if $B - A = B - A$.
   (ii) $A + (B - A) = B$ only if $\vec{a} = B - A$.
   (iii) $A + (B - A) = B$ only if $B - A = \vec{a}$.
   (iv) $A + \vec{a} = B$ only if $\vec{a} = B - A$.
   (v) $A + \vec{a} = B$ only if $B - A = \vec{a}$.
   (vi) $A + \vec{a} = B$ only if $\vec{a} = \vec{a}$.
   (vii) $A + (B - A) = B$ only if $\vec{a} = \vec{a}$.
8. Using the left-by-right replacement rule, the answer is:
   If $\vec{a} = B - A$ then $A + \vec{a} = B$.

   Using the right-by-left replacement rule, the three possible answers are:
   (i) If $\{A + [(A + \vec{a}) - A]\} - A = B - A$ then $A + \vec{a} = B$.
   (ii) If $(A + \vec{a}) - A = B - A$ then $A + [(A + \vec{a}) - A] = B$.
   (iii) If $\{A + [(A + \vec{a}) - A]\} - A = B - A$ then $A + [(A + \vec{a}) - A] = B$.

2.05 Conditional Sentences and Modus Ponens

In order to say more complex things than we could by using only equations and membership-sentences, we need ways of constructing complex sentences. One such way is by using 'if' and 'then':

(a) If $\vec{a} = B - A$ then $A + \vec{a} = B$.

Such sentences are called *conditional sentences*. In such a sentence the sentence between 'if' and 'then' is called *the antecedent* and the sentence which follows 'then' is called *the consequent*. Here is an example from everyday speech:

(1) If Bill lives in Princeton then Bill lives in New Jersey.

Just as there are two rules for equations – the replacement rule and the introduction rule – which tell you all you need to know about '=' – so there are two rules which tell you all need to know about 'if . . . then ___'. One of these rules tells how conditional sentences are used in arguments; the other tells how a conditional sentence can be deduced from given premisses. In this section we shall discuss the first of these two rules.

To begin with, consider the following conversation:

Jack: Can you tell me in which state Bill lives?

Jim: Well, Bill lives in Princeton. And, if Bill lives in Princeton then he lives in New Jersey.

Has Jim given Jack enough information to answer his question? If your answer is 'Yes.' then you know what this section is about. You probably agree that the answer:

(2) Bill lives in New Jersey.

follows logically from 'Bill lives in Princeton.' and (1). If asked why, you might answer that it does because that's [part of] what 'if . . . then ___' means.

Notice that we do not claim that (2) is a *correct* answer to Jack's question. Jim may be mistaken in saying that Bill lives in Princeton. He might even be mistaken in asserting (1) – Bill may live in Princeton, Illinois. But, whether or not Jim is mistaken in his facts, (2) does follow logically from his two assertions, and if Jack agrees to them then he certainly should accept (2).

As the preceding argument illustrates, from any sentence and any conditional sentence which has this as its antecedent, we may validly infer the consequent of the conditional sentence. More briefly, we can

Answers for Exercises [cont.]

9. $a + b < 0$; right-by-left replacement rule

10. $C = B$; left-by-right replacement rule
or: $B = C$; rignt-by-left replacement rule

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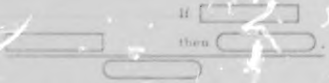
Figures used to exhibit the form of an inference may be called inference schemes. They may contain loops, like the first such figure on page 78, or "schematic letters", usually, 'p', 'q', and 'r' — like the one in the boxed statement of modus ponens. ['Schematic letters' rather than 'variables' because variables must have values. The schematic letters are place-holders for sentences but we [personally] are not aware of entities which, being denoted by sentences, could be assigned as values of these letters. Those who believe that there are entities — say, "propositions" — which are denoted by sentences may properly call our schematic letters 'variables' — say, 'propositional variables'.]

Experience reveals that students are not inherently sensitive to the sentence patterns in the inference scheme for modus ponens. This may, in part, be because the meaning of 'If . . . then ___.' in ordinary speech varies so much from one sentence to another that it is not practicable to single out a component of the meanings of such conditional sentences which is due solely to their "conditionality". In mathematical reasoning, however, there is such a component and it is just this which is specified by modus ponens and the deduction rule of section 2.06.

Although there is a sense in which carrying out an argument without paying attention to the total meanings of the sentences one utters is a sterile pursuit, it is perhaps impossible to develop a critical understanding of logic in any other way. One needs to become aware of the fact that the validity of one's arguments depends solely on the structure of one's sentences — and not at all on the meanings of the extra-logical words which, by sheer bulk, are most prominent. For validity it's the little words — 'any', 'if', 'then', 'not', etc. — which count. It is only once one has a firm grasp of this that he can argue logically without attending to the logic of his argument to an extent detrimental to its content.

The exercises are intended to focus a student's attention on the pattern.

say that any inference of the form:



is valid [that is, its conclusion follows from its premisses].

Since, in an argument, we can use a conditional sentence to go *from* the antecedent of this sentence *to* its consequent we shall write a conditional sentence by writing an '$\Longrightarrow$' between its antecedent and its consequent. For example, instead of the sentence (a) we shall write:

$$\overrightarrow{a} = B - A \Longrightarrow A + \overrightarrow{a} = B \tag{3}$$

Also, in writing about different kinds of inferences, it is easier to use letters—say '*p*' and '*q*'—as place-holders for sentences, instead of the frames which were used above.

We can now state our first rule for conditional sentences. [We give it its Latin name.]

> Modus Ponens
>
> Any inference of the form:
>
> $$\frac{p \quad p \Longrightarrow q}{q}$$
>
> is valid.

Exercises

Part A

Copy each of the following exercises and write [below the horizontal bar] the conclusion which can be inferred from the given premisses by modus ponens. [If no conclusion can be reached by modus ponens, say so.]

1. Mary has a dime. — If Mary has a dime then Mary has more money than I have.
2. Charles lives in California. — If Charles lives in California then Charles lives in the United States.
3. Charles lives in the United States. — If Charles lives in California then Charles lives in the United States.
4. $B - A = B - A \qquad B - A = B - A \Longrightarrow A + (B - A) = B$
5. $a + 3 = 5 \qquad a + 3 = 5 \Longrightarrow a = 2$
6. $3 > 5 \qquad 3 > 5 \Longrightarrow 0 > 2$

One caution is in order as to reading conditional sentences [and sentence schemes]. It is very tempting to read 'p $\Longrightarrow$ q' as 'p implies q'. Succumbing to this temptation has unfortunate consequences — especially as concerns pedagogy. To see what these are, note, first, that the '$\Longrightarrow$' is [like the phrase 'if ... then ___'] a conjunction, and is used to combine sentences to form another sentence. In contrast, 'implies' is a verb. By analogy, if sentences were names of appropriate objects — as, for example, numerals are names of numbers — then '$\Longrightarrow$' could be thought of as an operator used to refer to an operation on those objects — just as '+' is an operator used to refer to the operation of addition of numbers. In contrast, since 'implies' is a verb and, so, presumably refers to some relation, 'implies' must be analogous to a predicate — for example, to '>'.

In addition to hinting that 'if ... then ___' and 'implies' perform different functions, grammar can suggest what kind of relation it is to which 'implies' refers. As a first example note that while the sentence:

(☆) If it is raining then the sky is dark.

is at least grammatically correct, the sentence:

(☆☆) It is raining implies the sky is dark.

begs for at least one — and, better, two — 'that's:

That it is raining implies that the sky is dark.

In this usage, 'implies' refers to a relation between the "facts" which are referred to by the noun-clauses 'that it is raining' and 'that the sky is dark'. One might then argue for the existence of a relation among "facts" — whatever they are — to which relation the word 'implies' refers. For our purposes, however, it is more helpful to use 'implies' to refer to a relation — about whose existence there is no question — among sentences. With this meaning (☆☆) should be reformulated:

'It is raining.' implies 'The sky is dark.'.

[This sentence is certainly false but, like (☆), is grammatically correct.]

We can now see clearly why, in spite of one's grammatical instincts, one tends to read '$\Longrightarrow$' as 'implies'. The reason is that

a sentence of the form:

'p' implies 'q'

is true if and only if the corresponding sentence of the form:

$$p \Longrightarrow q$$

is logically valid.

[In detail, if a conditional sentence is valid then, by modus ponens and the rule for ignoring valid premisses, its antecedent, by itself, implies its consequent; conversely, if the latter is the case then, by the deduction rule, the conditional sentence is valid.]

The pedagogical consequences of reading '$\Longrightarrow$' as 'implies' are now easy to anticipate. Suppose, for example, that you illustrate modus ponens by saying something in this form [with 'p' and 'q' replaced by some well chosen sentences]:

Together, 'p' and 'p implies q' imply 'q'.

Clearly, with your two uses of 'imply' you have at least given your students something to puzzle over. Even if you are careful to stay out of this hole [and the deeper one we shall next point out], students who have come to read ' $\Longrightarrow$ ' as 'implies' are likely to stumble into it by themselves.

The "deeper hole" has to do with the other basic rules for conditional sentences — the deduction rule. A special case of this rule may be formulated correctly by saying something of the form:

If 'p' implies 'q' then 'p $\Longrightarrow$ q' is valid.

Somewhat more freely put:

We can prove 'p $\Longrightarrow$ q' by showing that 'p' implies 'q'.

Exercise: Reread the preceding sentence, saying 'implies' when you get to the ' $\Longrightarrow$ '.

Moral: Don't say 'implies' when you mean 'if ... then ___' [unless you are talking with experts who are beyond danger of being confused by this usage].

Answers for Part A

1. Mary has more money than I have.
2. Charles lives in the United States.
3. no conclusion [See discussion on page 80.]
4. $A + (B - A) = B$
5. $a = 2$
6. $0 > 2$

7. $a = b$ and $b + 5 = 7$ $(a = b \text{ and } b + 5 = 7) \longrightarrow a + 5 = 7$

8. $a = b$ $(a = b \text{ and } b = c) \longrightarrow a = c$

9. $A + \vec{a} \in \mathcal{E}$ $\vec{a} = B - A \longrightarrow A + \vec{a} = B$

10. $A + \vec{a} = B$ $\vec{a} = B - A \longrightarrow A + \vec{a} = B$

11. $\vec{a} = B - A$ $\vec{a} = B - A \longrightarrow A + \vec{a} = B$

12. f is a function f is a function $\longrightarrow$ f is a set of ordered pairs

13. f is a set of ordered pairs f is a function $\longrightarrow$ f is a set of ordered pairs

Part B

In each of the following exercises, supply the missing premiss [if possible] so that the conclusion follows from the two premisses by modus ponens. [Follow the pattern given in the statement of the rule.]

1. Harry weighs 100 pounds. ________
   Harry weighs more than Harold.

2. $a + b = 0$ ________
   $a = -b$

3. ________ $2c > 0 \longrightarrow c > 0$
   $c > 0$

4. $2 \geq 0$ ________
   $2 > 0$

5. f is a one-to-one function ________
   f has an inverse

6. ________ $a + 2 = 5 \longrightarrow a > 0$
   $3 > 0$

7. ________ $a + 2 = 5 \longrightarrow a > 0$

8. $B = A + \vec{a}$ ________
   $B - A = \vec{a}$

9. ________ $(A \subseteq B \text{ and } a \in A) \longrightarrow a \in B$

10. ________
    $\vec{a} = B - A$

Part C

1. Make an instance of:

$$\vec{a} = B - A \longrightarrow A + \vec{a} = B$$

(a) concerning $\vec{c}$, $A + \vec{a}$, and P; (b) concerning A, B, and $B - A$.

7. $a + 5 = 7$

8. no conclusion [Using another rule, "exportation" in addition to modus ponens, one may conclude '$b = c \Longrightarrow a = c$'. See Exercise 6 of Part C on page 101.]

9. no conclusion

10. no conclusion

11. $A + \vec{a} = B$

12. f is a set of ordered pairs

13. no conclusion

Answers for Part B

1. If Harry weighs more than 100 pounds then Harry weighs more than Harold.

2. $a + b = 0 \Longrightarrow a = -b$

3. $2c > 0$

4. $2 \geq 0 \Longrightarrow 2 > 0$

5. f is a one-to-one function $\Longrightarrow$ f has an inverse

6. impossible

7. $a + 2 = 5$; $a > 0$

8. $B = A + \vec{a} \Longrightarrow B - A = \vec{a}$

9. $A \subseteq B$ and $a \in A$; $a \in B$ [This may give students some momentary difficulty because it is a sentence about sets. 'A' and 'B' are variables for sets.]

10. _____; _____ $\Longrightarrow \vec{a} = B - A$ [For a correct answer, fill the blanks with copies of any sentence you choose.]

Answers for Part C

1. (a) $\vec{c} = (A + \vec{a}) - P \Longrightarrow P + \vec{c} = A + \vec{a}$ or
   $\vec{c} = P - (A + \vec{a}) \Longrightarrow (A + \vec{a}) + \vec{c} = P$
   (b) $B - A = B - A \Longrightarrow A + (B - A) = B$ or
   $B - A = A - B \Longrightarrow B + (B - A) = A$

2. Make an instance of:

$$A + \vec{a} = B \longrightarrow \vec{a} = B - A$$

(a) concerning $A + \vec{a}$, A, and $\vec{a}$; **(b)** concerning $A + \vec{a}$, B, and $\vec{a}$.

*

Consider the inference:

$$\begin{array}{ll} \text{Charles lives in the United States.} & \begin{array}{l}\text{If Charles lives in California} \\ \text{then Charles lives in the} \\ \text{United States.}\end{array} \\ \hline \multicolumn{2}{c}{\text{Charles lives in California.}} \end{array}$$

People sometimes make the mistake of thinking that inferences like this are valid. Of course, they are not. Even if the sentence 'Charles lives in California' happens to be true, it doesn't follow from the given premisses by modus ponens or by any other rule of logic. [We hope you didn't make this mistake in doing Exercise 3 of Part A. If you did, you had better check your answers to the other exercises.]

Earlier, we saw that the sentence:

(a) If $\vec{a} = B - A$ then $A + \vec{a} = B$

is a consequence of Postulate 2(a) and that:

(b) If $A + \vec{a} = B$ then $\vec{a} = B - A$

is a consequence of Postulate 2(b). Looking again at sentence (a) may suggest that it should be possible to reverse our steps and derive Postulate 2(a) from (a). The sentence (a) and Postulate 2(a) seem to say about the same thing. [If you think you see how to derive Postulate 2(a) from (a), write down your argument. Do you need any other postulate as a basis for your argument?]

Actually, it is possible to show that Postulate 2(a) is a consequence of sentence (a) and another postulate. Here's how:

Since $B - A$ is a translation it follows from (a) that

(4) if $B - A = B - A$ then $A + (B - A) = B$.

Since

(5) $B - A = B - A$ [any translation is itself]

it follows that

(6) $A + (B - A) = B$

2. (a) $(A + \vec{a}) + \vec{a} = A \Longrightarrow \vec{a} = A - (A + \vec{a})$

$A + \vec{a} = A + \vec{a} \Longrightarrow \vec{a} = (A + \vec{a}) - A$

(b) $(A + \vec{a}) + \vec{a} = B \Longrightarrow \vec{a} = B - (A + \vec{a})$

$B + \vec{a} = A + \vec{a} \Longrightarrow \vec{a} = (A + \vec{a}) - A$

*

One who acts as though some inference of the form:

$$\frac{q \qquad p \Rightarrow q}{p}$$

is valid is said to have committed the fallacy of asserting the consequent.

[Recall that asserting (a) means that you are willing to accept any instance of (a). So, to infer (4) from (a) all we needed to know was that (4) is an instance of (a). But, to know this, we had to know Postulate 1(a). It is logical that (6) follows from (5) and (4) because this is part of what we mean when we say 'if . . . then ___'. The sentence (5) is, itself, a logical assertion to make, since anything is the same as itself -- but, we need to know that $B - A$ is a "thing", and Postulate 1(a) comes in, again, here. Since the assertion 'any translation is itself' is acceptable just because of what we mean by the word 'is', it is now fair to say that we have shown that the conclusion (6) follows logically from (a) and Postulate 1(a).]

Now, show that the sentence (b), together with another postulate, implies Postulate 2(b).

We now have enough rules of logic to make it worth while to analyze the argument given above. What we did there was to show that Postulate 2(a):

$$A + (B - A) = B$$

could be derived by starting from the sentence:

(a) $$\vec{a} = B - A \Longrightarrow A + \vec{a} = B$$

Our argument for this amounted to noting that, by the substitution rule, the inference:

$$\frac{\vec{a} = B - A \Longrightarrow A + \vec{a} = B \qquad B - A \in \mathscr{T}}{B - A = B - A \Longrightarrow A + (B - A) = B}$$

is valid and that, by the same rule, the inference:

$$\frac{\vec{a} = \vec{a} \quad B - A \in \mathscr{T}}{B - A = B - A}$$

is valid. Finally, we noted that, by modus ponens, the inference:

$$\frac{B - A = B - A \quad B - A = B - A \Longrightarrow A + (B - A) = B}{A + (B - A) = B}$$

is valid. If we fit these valid inferences together we obtain:

$$\dfrac{\dfrac{\vec{a} = \vec{a} \quad B - A \in \mathscr{T}}{B - A = B - A} \qquad \dfrac{\vec{a} = B - A \Longrightarrow A + \vec{a} = B \quad B - A \in \mathscr{T}}{B - A = B - A \Longrightarrow A + (B - A) = B}}{A + (B - A) = B}$$

The expected derivation of Postulate 2(b) from sentence (b) and Postulate 1(b), asked for on page 81 is:

> Since $A + \vec{a}$ is a point it follows from (b) that if $A + \vec{a} = A + \vec{a}$ then $\vec{a} = (A + \vec{a}) - A$. Since $A + \vec{a} = A + \vec{a}$ it follows that $\vec{a} = (A + \vec{a}) - A$.

As will be pointed out later in the text, the proof of Theorem 2-1 and the two derivations just given show that we might have adopted Theorem 2-1 as a postulate, in place of Postulate 2, and still have had the same theorems.

For a discussion of a derivation like that in the text, see the discussion, below, of the answer for Exercise 3 (on page 83).

This final figure shows just how we went about it to deduce the conclusion '$A + (B - A) = B$' from the sentence (a) and two other premisses, '$\vec{a} = \vec{a}$' and '$B - A \in \mathcal{T}$'. At first sight, it shows that Postulate 2(a) is a consequence of these three premisses.

Looking at the premisses of this derivation more closely we see that the premiss '$\vec{a} = \vec{a}$' is a valid sentence. [What rule tells you this?] What this means is that we would be willing to accept the sentence '$\vec{a} = \vec{a}$' just because of what we intend to mean by '=', no matter what – translations or whatever – we were talking about. [This is the same sort of reason for which we accept the last of our three inferences – just because of what we intend to mean by '⟹'.] Since the acceptability of the premiss '$\vec{a} = \vec{a}$' is a matter of logic – rather than, say, of mathematics – we shall ignore this premiss and say that the diagram above shows that Postulate 2(a) is a consequence of (a) and Postulate 1(a), alone. If we wish to say *why* the diagram shows this, we can say that it does so by virtue of the introduction rule for equations, the substitution rule, and modus ponens.

The status of the premiss '$B - A \in \mathcal{T}$' is much like that of '$\vec{a} = \vec{a}$'. Just as, when we described the language we were going to use, we said that '$\vec{a}$' is a variable whose values are translations, we might as well have said that '$B - A$' is a term whose values are translations. Doing so would have made it unnecessary to adopt Postulate 1(a). So, we can think of '$B - A \in \mathcal{T}$' as being acceptable just because of how we intend to use '−' [and not because of any special property of translations]. This being so, we shall say that the diagram shows that Postulate 2(a) is a consequence of the sentence (a), alone.

This is as far as we can go in ignoring premisses. Sentence (a), itself, definitely says something about points and translations – that the translation from A to B maps A on B.

A diagram which, like the one we have been discussing, is built up out of inferences is called *a tree-form derivation*. It is said to be valid in case each of the inferences out of which it is built is a valid inference.

Exercises

1. Look at the argument we gave on page 73 to derive the sentence:

(7) $A + \vec{a} = A + \vec{b} \Longrightarrow \vec{a} = \vec{b}$

Construct a tree-form derivation which shows that (7) is a consequence of Postulate 2(b) and the sentence:

(b) $A + \vec{a} = B \Longrightarrow \vec{a} = B - A$

The introduction rule for equations tells us that '$\vec{a} = \vec{a}$' is a valid sentence.

The derivations in these exercises serve only for practicing the application of the rules of reasoning we have introduced to date. It is not an objective of the course that a student necessarily remember how to derive, say, (7) from Postulate 2(b) and sentence (b).

Answers for Exercises

1.

$$\dfrac{\dfrac{\overset{[\text{Post. } 2(b)]}{\vec{a} = (A+\vec{a}) - A}}{\vec{b} = (A+\vec{b}) - A}\,(\text{Subst}) \qquad \dfrac{\overset{[(b)]}{A+\vec{a} = B \Longrightarrow \vec{a} = B - A} \qquad \boxed{\dfrac{A + \vec{a} \in \mathcal{E}}{A + \vec{b} \in \mathcal{E}}\,(\text{Subst})}}{A + \vec{a} = A + \vec{b} \Longrightarrow \vec{a} = (A + \vec{b}) - A}\,(\text{Subst})}{A + \vec{a} = A + \vec{b} \Longrightarrow \vec{a} = \vec{b}}\,(\text{RRE})$$

[The portion of this derivation enclosed in the dashed box may, as previously remarked, be omitted. It serves only to show that the following substitution is legitimate. At any rate, as explained on page 82, the premiss of the boxed inference is one which we shall ignore when stating what the derivation shows concerning its conclusion. The derivation shows that its conclusion (7) is a consequence of (b) and Postulate 2(b). The remarks attached to the derivation — '[Post. 2(b)]', '[(b)]', '(Subst)', and '(RRE)' — are merely explanatory and are not parts of the derivation itself. In place of '(RRE)', one might write '(E =)' [Elimination Rule for '='], since the replacement rule for equations shows how to use [or: "use up"] an equation in a derivation, thus "eliminating" an '='.]

2. On page 67 you were asked to state an argument showing that the sentence:

$$\text{(b)} \qquad A + \vec{a} = B \longrightarrow \vec{a} = B - A$$

[and another postulate] implies Postulate 2(b). Construct a tree-form derivation which shows that Postulate 2(b) is a consequence of (b).

3. What do your answers to Exercises 1 and 2 tell you about the relation of (7) to (b)?

☆ 4. Construct a tree-form derivation which shows that (7) is a consequence of (b).

2.06 The Deduction Rule

In the preceding section you have seen how, by virtue of modus ponens, you can *use* a conditional sentence in deriving other sentences. This leaves open the question of where, in the first place, conditional sentences come from. On page 67 we have given an argument to show that the conditional sentence:

$$\text{(a)} \qquad \vec{a} = B - A \longrightarrow A + \vec{a} = B$$

is a consequence of Postulate 2(a). We shall come back to this argument shortly, after we have analyzed an even simpler one.

As our first example we shall take a conditional sentence about real numbers:

$$\text{(1)} \qquad 2 + 3 = 5 \longrightarrow (2 + 3) + 7 = 5 + 7$$

You will probably grant that this sentence is true—our task is to find a reason for thinking so. One way to start is with the valid sentence:

$$\text{(2)} \qquad (2 + 3) + 7 = (2 + 3) + 7$$

This sentence is true for a very good reason—just because of what we mean by '='. If we can show that (1) is a consequence of (2) then we will have just as good a reason for accepting (1). To do this we argue this way:

Suppose that 2 + 3 = 5. Since (2 + 3) + 7 = (2 + 3) + 7 it follows that (2 + 3) + 7 = 5 + 7. Hence, *if* 2 + 3 = 5 *then* (2 + 3) + 7 = 5 + 7.

What we have done is to note that [by the replacement rule for

Answers for Exercises [cont.]

2.

$$\dfrac{\dfrac{\boxed{A = A \quad A + \vec{a} \in \mathcal{E}}}{A + \vec{a} = A + \vec{a}} \qquad \dfrac{A + \vec{a} = B \Longrightarrow \vec{a} = B - A \quad \boxed{A + \vec{a} \in \mathcal{E}}}{A + \vec{a} = A + \vec{a} \Longrightarrow \vec{a} = (A + \vec{a}) - A}\,\text{(Subst)}}{\vec{a} = (A + \vec{a}) - A}\,\text{(MP)}$$

[As in Exercise 1, the premiss in the right-hand box might be omitted. The content of the left-hand box might also be omitted on the grounds that it is an obvious result of the introduction rule for equations and the substitution rule that any equation whose sides are the same term is a valid sentence. Whether to insist, for a while, that your students include this boxed portion is a matter for you to decide. Although we shall later omit auxillary premisses such as '$A + \vec{a} \in \mathcal{E}$' when writing substitution-inferences, we shall never omit a premiss merely because it is a valid sentence. As remarked on page 82 we shall, however, ignore both kinds of premisses when stating what a derivation shows. This derivation shows, then, that Postulate 2(b) is a consequence of sentence (b). The bracketed remark in the exercise — 'and another postulate' — is merely a hint to students that, for the present, they are to include auxillary premisses when writing substitution-inferences. Finally, instead of the comment '(MP)', one might use '(E ⟹)' [Elimination Rule for ' ⟹ '].]

3. They show that sentence (7) is a consequence of sentence (b). [Point out that by placing the second derivation above the first, so that the conclusion of the second lies over the premiss '$\vec{a} = (A + \vec{a}) - A$' of the first, one would obtain a single derivation which would show that (b) implies (7).]

☆4. [Combine the two derivations in the way that has just been described. This requires a fairly wide sheet of paper since the inference line of the modus ponens-inference should not extend over the premiss (b) of the derivation in Exercise 1.]

Class discussion of the preceding exercises can be facilitated by use of an overhead projector. If properly made [see remark for Exercise 4] two projectuals — or one with an overlay — can show the answers to Exercises 1 and 3 in such a way that, in combination, they give the answer for Exercise 4. At any rate, you should have the answers for Exercises 1 and 2 in view and, pointing to the appropriate sentences, recite the corresponding verbal arguments. For Exercise 2, for example, this might be:

Since $A + \vec{a} \in \mathcal{E}$ it follows from sentence (b)
that if $A + \vec{a} = A + \vec{a}$ then $\vec{a} = (A + \vec{a}) - A$.
So, since $A + \vec{a} = A + \vec{a}$, it follows that
$\vec{a} = (A + \vec{a}) - A$.

[When saying 'it follows that' you might point at the appropriate inference line.]

equations] the sentence:

(3) $$(2 + 3) + 7 = 5 + 7$$

follows from a premiss '2 + 3 = 5' which we assumed [perhaps you see why] and another premiss, (2). This being established, we concluded ["Hence"] that the conditional sentence (1) is a consequence of just this other premiss. We can diagram our argument as follows:

$$\cfrac{\cfrac{2 + 3 \overset{*}{=} 5 \qquad (2 + 3) + 7 = (2 + 3) + 7}{(2 + 3) + 7 = 5 + 7}\,\text{(RRE)}}{2 + 3 = 5 \longrightarrow (2 + 3) + 7 = 5 + 7}\,*$$

The '*'s in this tree-form derivation do more explicitly what the word 'Hence' did in the original form of the argument. As the replacement inference shows, the sentence (3) is a consequence of two premisses – the "assumption" '2 + 3 = 5' and the sentence (2). The purpose of the '*'s is to point out that the final conclusion – which is sentence (1) – is a consequence of (2) alone. This being so, we have just as good reason for accepting (1) as we have for accepting the premiss (2). [To drive home the point that the conclusion (1) does not depend on the premiss '2 + 3 = 5', imagine the '5's in the above derivation to be replaced by '8's. Everything would go through just as before, but the conclusion would be:

$$2 + 3 = 8 \longrightarrow (2 + 3) + 7 = 8 + 7$$

This new conclusion is just as acceptable as is (1) – if 2 + 3 were 8 then (2 + 3) + 7 certainly would be 8 + 7. Of course, the new conclusion is even less interesting than is (1).]

Although there may not seem much point to (1), you have made a lot of use [when solving equations] of a sentence which has (1) as an instance:

(4) $$a = b \longrightarrow a + c = b + c$$

The argument in favor of accepting this sentence is just like that for accepting (1). It starts off:

Suppose that $a = b$.

Here is the argument in tree-form:

$$\cfrac{\cfrac{a \overset{*}{=} b \qquad a + c = a + c}{a + c = b + c}\,\text{(RRE)}}{a = b \longrightarrow a + c = b + c}\,*$$

The proof of (4) — an equality principle for addition — may be repeated to prove similar equality principles for other operations. [For the role of such principles, see TC 71.] Using Greek letters we can give the general form of such proofs as:

$$\cfrac{\cfrac{\alpha \overset{*}{=} \beta \qquad \tau\alpha \overset{\dagger}{=} \tau\alpha}{\tau\alpha = \tau\beta}}{\alpha = \beta \Longrightarrow \tau\alpha = \tau\beta}\,*$$

[The '†' is to point out that the sentence in question is valid and, so, is "automatically discharged".] In the case of the proof of (4), 'α' refers to 'a', 'β' to 'b', 'τα' to 'a + c' and, consequently, 'τβ' refers to 'b + c'. Here is another example of this form of proof:

$$\cfrac{\cfrac{A \overset{*}{=} B \qquad C - A = C - A}{C - A = C - B}}{A = B \Longrightarrow C - A = C - B}\,*$$

Since 'C - A = C - A' is a valid sentence, this shows that the equality principle:

$$A = B \Longrightarrow C - A = C - B$$

is also valid.

Equality principles for relations — for example:

$$a = b \Longrightarrow [a < c \Longrightarrow b < c]$$

can be proved in a similar manner:

$$\cfrac{\cfrac{a \overset{*}{=} b \qquad a < c \Longrightarrow a < c}{a < c \Longrightarrow b < c}}{a = b \Longrightarrow [a < c \Longrightarrow b < c]}\,*$$

Since 'a < c ⟹ a < c' is valid, so is the conclusion. Alternatively, one may proceed as follows:

$$\cfrac{\cfrac{\cfrac{a \overset{\dagger}{=} b \qquad a \overset{*}{<} c}{b < c}}{a < c \Longrightarrow b < c}\,*}{a = b \Longrightarrow [a < c \Longrightarrow b < c]}\,\dagger$$

Clearly, the conclusion is as acceptable as is the premiss '$a + c = a + c$' which implies it.

The argument we gave on page 67 to show that Postulate 2(a) implies sentence (a) is of exactly the same form:

$$\cfrac{\cfrac{\vec{a} \overset{*}{=} B - A \quad A + (B - A) = B}{A + \vec{a} = B}\ (\text{RRE})}{\vec{a} = B - A \Longrightarrow A + \vec{a} = B}\ *$$

The general procedure we have illustrated can be formulated like this:

> If you wish to show that a certain conditional sentence is a consequence of certain premisses, show that its consequent follows from its antecedent and these premisses, together.

$$\begin{array}{c} \underbrace{\overset{*}{p}, \ldots} \\ \vdots \\ \underbrace{\quad q \quad}\ * \\ p \Longrightarrow q \end{array}$$

[This figure suggests what has just been said. When one has derived a sentence q from an *assumption* p and other premisses ('. . .'), one may infer the conditional sentence $p \Longrightarrow q$ and *discharge* the assumption p. (In Part B of the exercises we shall point out a restriction on the use of this kind of argument.)]

As a final example, let's show that:

(5) $\qquad a + c = b + c \Longrightarrow a = b$

is a consequence of the following principles for real numbers:

The Associative Principle for Addition [APA]:

$$(a + b) + c = a + (b + c),$$

The Introduction Principle for Oppositing [IPO]:

$$a + -a = 0,$$

and The Principle for Adding Zero [PA0]:

$$a + 0 = a$$

To do so, we may argue as follows:

> Suppose that $a + c = b + c$. Since $(a + c) + -c = (a + c) + -c$ it follows that $(a + c) + -c = (b + c) + -c$. Since, by the APA, $(a + c) + -c = a + (c + -c)$ and $(b + c) + -c = b + (c + -c)$ it follows that $a + (c + -c) = b + (c + -c)$. Since, by the IPO, $c + -c = 0$ it follows that $a + 0 = b + 0$. Since, by the PA0, $a + 0 = a$ and $b + 0 = b$ it follows that $a = b$. Hence, if $a + c = b + c$ then $a = b$.

The derivation of the sentence (a) from Postulate 2(a) which is given in the text fits the following form:

$$(\text{I}) \qquad \cfrac{\cfrac{\alpha \overset{*}{=} \tau \quad \phi\tau}{\phi\alpha}\ (\text{RRE})}{\alpha = \tau \Longrightarrow \phi\alpha}\ *(\text{Deduction Rule})$$

[To "make this fit", let 'τ' refer to 'B - A' and '$\phi\alpha$' to '$A + \vec{a} = B$'.] This kind of argument is reversible:

$$(\text{II}) \qquad \cfrac{\tau = \tau \quad \cfrac{\alpha = \tau \Longrightarrow \phi\alpha}{\tau = \tau \Longrightarrow \phi\tau}\ (\text{Subst})}{\phi\tau}\ (\text{MP})$$

Comparison of these two kinds of arguments shows how the deduction rule and modus ponens complement one another, as do the introduction rule for equations and the replacement rule for equations.

It results from the validity of arguments of these two kinds that

$$(\text{III}) \qquad \frac{\phi\tau}{\alpha = \tau \Longrightarrow \phi\alpha} \text{ and } \frac{\alpha = \tau \Longrightarrow \phi\alpha}{\phi\tau} \text{ are valid inferences.}$$

In other words, $\phi\tau$ and $\alpha = \tau \Longrightarrow \phi\alpha$ "say the same thing". As an example from real numbers, the sentence:

$$(a + b) - b = a$$

and the sentence:

$$c = a + b \Longrightarrow c - b = a$$

"say the same thing". [In the general rule, let 'τ' refer to 'a + b' and 'α' to 'c'.]

The deduction rule, together with the two rules for equations, also makes it possible to establish the symmetry and transitivity of equality:

$$\cfrac{\cfrac{\alpha \overset{*}{=} \beta \quad \alpha \overset{\dagger}{=} \alpha}{\beta = \alpha}}{\alpha = \beta \Longrightarrow \beta = \alpha}\ * \qquad \cfrac{\cfrac{\cfrac{\beta \overset{*}{=} \gamma \quad \alpha \overset{\dagger}{=} \beta}{\alpha = \gamma}}{\beta = \gamma \Longrightarrow \alpha = \gamma}\ *}{\alpha = \beta \Longrightarrow [\beta = \gamma \Longrightarrow \alpha = \gamma]}\ \dagger$$

[The more usual form of the transitivity principle, '$(\alpha = \beta$ and $\beta = \gamma) \Longrightarrow \alpha = \gamma$' requires for its derivation some rules for 'and'. For example, the importation rule of Exercise 5, Part C, on page 101 will do.] Note that only the left-by-right replacement rule for equations is used in these derivation schemes. So, only this part of the rule need be adopted as a "basic" rule of logic.

The cancellation principle (5) differs from the equality principle (4) in that, although (4) is a valid sentence, to derive (5) one needs premisses which are mathematical principles. This is not too surprising since (4) can be interpreted as saying that "addition of a given real number" is a function, while (5) asserts that such a function has an inverse. That the former is the case has to be recognized before one is justified in adopting the symbol '+' as an operator. Sentence (5), on the other hand, says that the operation which '+' refers to has a very special property which is not shared by all operations. For example, the equality principle for squaring real numbers:

$$a = b \implies a^2 = b^2$$

becomes a valid sentence as soon as one adopts the operator '^{2}'. But, the corresponding "cancellation principle":

$$a^2 = b^2 \implies a = b$$

is, of course, not true.

Your students will probably benefit from remarks similar to the foregoing, since confusing a conditional sentence with its converse is a not uncommon error.

Remarks concerning argument given for (5):

We begin by adopting the antecedent of (5) as an assumption.

The assertion of '$(a + c) + -c = (a + c) + -c$' in the second sentence is justified by the introduction rule for equations. We could derive the sentence in question from '$a = a$', '$a + c \in \Re$' and '$-c \in \Re$' by using the substitution rule (three times). 'Since ... it follows that ...' in this sentence refers (in this case) to an application of the replacement rule for equations.

In the third sentence, '$(a + c) + -c = a + (c + -c)$' is a consequence of the APA by virtue of the substitution rule. 'Since ... it follows that ...' refers to two applications of the replacement rule.

Remarks similar to the last apply to each of the next two sentences.

The first five sentences of the argument show that '$a = b$' is a consequence of the assumption '$a + c = b + c$', the APA, the IPO, and the PA0. 'Hence' in the final sentence indicates that, because of what precedes, (5) is a consequence of just the APA, the IPO, and the PA0.

The derivation of (5) which is given in the text may be presented in the form of a tree. To save space, we abbreviate it somewhat.

$$\dfrac{\dfrac{[\text{PA0}] \qquad \dfrac{\dfrac{a + -a = 0}{c + -c = 0} \qquad \dfrac{[\text{APA}] \qquad \dfrac{a + c \overset{*}{=} b + c \qquad (a+c) + -c = (a+c) + -c}{(a+c) + -c = (b+c) + -c}\,\text{(RRE)}}{a + (c + -c) = b + (c + -c)}\,\text{(RRE)}^2}{a + 0 = b + 0}\,\text{(RRE)}}{a = b}\,\text{(RRE)}^2}{a + c = b + c \implies a = b}\,*$$

The comments '(RRE)^2' indicate that what is presented as an inference is actually a sequence of two inferences. In one case the equation-premisses of these inferences are instances of the associative principle

for addition; in the other case they are instances of the principle for adding zero. [The first pair is shown below.] Since the premiss '$(a + c) + -c = (a + c) + -c$' is a valid sentence, the derivation does show that (5) is a consequence of the APA, the IPO, and the PA0.

Here is an expansion of the first "[RRE]^2":

$$\dfrac{\dfrac{(a+b)+c = a+(b+c)}{(b+c) + -c = b + (c + -c)} \qquad \dfrac{\dfrac{(a+b)+c = a+(b+c)}{(a+c) + -c = a + (c + -c)} \quad (a+c) + -c = (b+c) + -c}{a + (c + -c) = (b+c) + -c}\,\text{(RRE)}}{a + (c + -c) = b + (c + -c)}\,\text{(RRE)}$$

The second is similar except that, since one of the equation-premisses is the PA0 itself, only one substitution-inference will appear in the figure.

Exercises

Part A

1. **(a)** In the text (on page 85) we have given a tree-form derivation to show that the sentence:

   (a) $\overrightarrow{a} = B - A \longrightarrow A + \overrightarrow{a} = B$

   is a consequence of Postulate 2(a). Give a similar argument to show that:

   (b) $A + \overrightarrow{a} = B \longrightarrow \overrightarrow{a} = B - A$

   is a consequence of Postulate 2(b).

   (b) Rewrite the argument you gave in part (a) as a paragraph beginning 'Suppose that'.

2. **(a)** Construct a tree-form proof for the sentence:

   $\overrightarrow{a} = \overrightarrow{b} \longrightarrow A + \overrightarrow{a} = A + \overrightarrow{b}$

   (b) Rewrite this proof as a paragraph.

Part B

When we *assert* a sentence in which a variable occurs we mean to be making a general statement about all values of the variable. We recognized this by adopting the substitution rule. Now we have found a new way to use a sentence other than to assert it. Sometimes we say 'Suppose that'. When, for example, in the proof of (5) given on page 85 we began by saying '*Suppose that* $a + c = b + c$.' we certainly did not mean to assert that, whatever numbers a, b, and c are, $a + c$ is $b + c$. Consequently, it would have been wrong to use the substitution rule in this proof to infer an instance of this assumption [or to substitute for 'a', 'b', or 'c' in any sentence we arrived at by using this assumption].

1. Here is a derivation which appears to show that if $2 = 2$ then $3 = 2$. Criticize it.

$$\dfrac{\dfrac{\dfrac{a \overset{*}{=} 2 \qquad 3\epsilon\mathscr{R}}{3 = 2}\,\text{(Subst)}}{a = 2 \longrightarrow 3 = 2}\,{}^{*} \qquad 2\epsilon\mathscr{R}}{2 = 2 \longrightarrow 3 = 2}\,\text{(Subst)}$$

2. Here is another derivation whose conclusion should seem unreasonable to you. Point out where it went wrong.

$$\dfrac{\dfrac{\dfrac{A + \overrightarrow{a} \overset{*}{=} B \qquad \overrightarrow{a} = (A + \overrightarrow{a}) - A}{\overrightarrow{a} = B - A}}{\overrightarrow{a} = A - A}\,\text{(Subst)}}{A + \overrightarrow{a} = B \longrightarrow \overrightarrow{a} = A - A}\,{}^{*}$$

Answers for Part A

1. (a)

$$\dfrac{\dfrac{A + \vec{a} \overset{*}{=} B \qquad \overset{[\text{Post. } 2(\text{b})]}{\vec{a} = (A + \vec{a}) - A}}{\vec{a} = B - A}\,\text{(RRE)}}{A + \vec{a} = B \Longrightarrow \vec{a} = B - A}\,{}^{*}$$

   (b) Suppose that $A + \vec{a} = B$. Since, by Postulate 2(b), $\vec{a} = (A + \vec{a}) - A$ it follows that $\vec{a} = B - A$. Hence, if $A + \vec{a} = B$ then $\vec{a} = B - A$.

[For aesthetic reasons, we tend, when writing proofs in English, to write conditional sentences in 'if ... then ___'-form. This personal quirk should not be taken as binding on anyone else.]

2. (a)

$$\dfrac{\dfrac{\vec{a} \overset{*}{=} \vec{b} \qquad A + \vec{a} = A + \vec{a}}{A + \vec{a} = A + \vec{b}}}{\vec{a} = \vec{b} \Longrightarrow A + \vec{a} = A + \vec{b}}\,{}^{*}$$

   (b) Suppose that $\vec{a} = \vec{b}$. Since $A + \vec{a} = A + \vec{a}$ it follows that $A + \vec{a} = A + \vec{b}$. Hence, if $\vec{a} = \vec{b}$ then $A + \vec{a} = A + \vec{b}$.

   [Since '$A + \vec{a} = A + \vec{a}$' is valid, so is the identity principle which has been derived from it.]

Answers for Part B

1. The substitution rule was adopted when we were thinking of an open sentence only as a way of asserting something about all values of the variables occurring in it. When an open sentence is interpreted in this way, a substitution-inference which has it as a premiss is valid. When, as in the case of the derivation of Exercise 1, such a sentence is treated as an assumption, such an inference is not valid. Specifically, '3 = 2' does not follow from 'a = 2' if the latter is an assumption.

2. Since, in this derivation, '$A + \vec{a} = B$' is an assumption, it is being used to restrict the values of 'A', '$\vec{a}$', and 'B' under discussion [rather than to assert something about all values of these variables]. So, the values of these variables which are, under this assumption, referred to by the sentence '$\vec{a} = B - A$' are similarly restricted. Since we have no reason to believe that these restrictions allow for the values of 'A' being among those of 'B', the inference to '$\vec{a} = A - A$' is not valid.

   Briefly, the substitution-inference is not valid because, since 'B' occurs in the assumption, the sentence '$\vec{a} = B - A$' cannot be interpreted as asserting something about all values of 'B'.

3. Here is a derivation which is something like that in Exercise 2 but which is valid.

$$\cfrac{\cfrac{\cfrac{\overset{*}{a = b} \qquad a = b \Longrightarrow a + c = b + c}{a + c = b + c} \qquad 3 \in \mathcal{R}}{a + 3 = b + 3}\,\text{(Subst)}}{a = b \Longrightarrow a + 3 = b + 3}\,{}^{*}$$

Both in this derivation and in the derivation of Exercise 2, we have used the substitution rule to infer an instance of a sentence which we have derived "under the assumption". In each derivation, look at the variable which has been substituted for, and look at the assumption. What difference do you notice?

*

The exercises in Part B show that some care must be taken when one uses an assumption. This, however, is nothing to worry about. You aren't likely to say "*Suppose* that $a = 2$. It follows that $3 = 2$." Nor are you likely to say "*Suppose* that $A + \vec{a} = B$. It follows by Postulate 2(b) that $\vec{a} = B - A$. In particular, it follows that $\vec{a} = A - A$." When you are working "under an assumption" you will know it and you won't be tempted to treat a sentence which contains variables which occur in your assumption as an assertion about all values of *these* variables. [As in Exercise 3, such a sentence may assert something about all values of *other* variables which occur in it.]

It is easy enough to avoid mistakes like those in Exercises 1 and 2. Still, in stating our rule for deriving conditional sentences, we should say what needs to be avoided. Evidently, what our rule should say is that the method of deriving conditional sentences which is suggested by the figure:

$$\begin{array}{c} \overset{*}{\underbrace{p, \ldots}} \\ \vdots \\ \cfrac{q}{p \Longrightarrow q}\,{}^{*} \end{array}$$

is acceptable

(*) provided that no inference used in deriving q depends for its validity on treating the assumption, or any sentence derived by using the assumption, as an assertion about all values of any variable which occurs in the assumption.

3. Since 'c' does not occur in the assumption 'a = b', we can interpret the sentence 'a + c = b + c' as asserting something about all values of 'c'. So, from it, we may infer 'a + 3 = b + 3'. [But, substitution for 'a' or for 'b' would be improper.]

Having shown in Part B that one can — if sufficiently bullheaded — construct invalid arguments by treating a premiss sometimes as an assumption and sometimes not, we make haste to point out that this error is easily avoided by men of good will. Students need to know about the restriction to which this kind of argument is subject, but should not, by this knowledge, be frightened away from using it.

In the proviso (☆), the phrase "in treating the assumption [as an assertion]" rules out, of course, the error made in Exercise 1 of Part B. The phrase "or any sentence derived by using the assumption [as an assertion]" rules out the error made in Exercise 2. The specification "of any variable which occurs in the assumption" leaves open the possibility of using the substitution rule in the way it is used in Exercise 3.

To make the proviso operative, we need to modify the substitution rule as is done in the summary on page 111.

This will take care of things once we note, in connection with the substitution rule, that the validity of a substitution-inference depends on its premiss being interpreted as an assertion about all values of the variables for which substitutions are made. [See the statement of this rule on page 111.]

Since we shall find other rules of logic which need to contain a proviso like (*), we shall introduce a very convenient abbreviation in the statement of our rule for deriving conditional sentences:

> **The Deduction Rule**
>
> Any inference of the form
>
> $$\frac{\begin{matrix}[p]\\ q\end{matrix}}{p \Longrightarrow q}$$
>
> is valid.

The abbreviation is the '[p]'. In full, the deduction rule is:

Any inference of the form:

$$\frac{q}{p \Longrightarrow q}$$

is valid and, when it occurs in a derivation, discharges any assumption p which has been used in deriving q, provided that no inference used in deriving q depends, for its validity, on treating the assumption, or any sentence derived by using the assumption, as an assertion about all values of any variable which occurs in the assumption.

*

It is convenient to say that the conclusion of a derivation *depends* only on those premisses of the derivation which are not discharged during the course of the derivation, which are not merely valid sentences, and which are not consequences of postulates, like the parts of Postulate 1, which serve merely to tell what kind of things a term has for values. For example, the conclusion '$a = b \Longrightarrow a + c = b + c$' of the final derivation on page 84 depends on none of the premisses of the derivation [and so, since the derivation is valid, this conclusion is a valid sentence]. Similarly, the conclusion '$\vec{a} = B - A \Longrightarrow A + \vec{a} = B$' of the first derivation on page 85 depends only on the premiss '$A + (B - A) = B$' [and so, since the derivation is valid and the premiss in question is a postulate, the conclusion is a theorem].

Some question may arise as to the validity of inferences of the form:

$$(\star) \qquad \frac{q}{p \Longrightarrow q}$$

[Such an inference is sometimes called a conditionalizing inference.] Perhaps the only straightforward answer to such questions is that the English phrase 'if . . . then ___' has many meanings and, for some of these, it may be the case that inferences of the form:

$$\frac{q}{\text{If } p \text{ then } q.}$$

are not valid. For that particular meaning of this phrase which we are using '$\Longrightarrow$' to express, such inferences are valid. And it is this meaning of the phrase which is relevant in the conditional sentences which occur in mathematics.

Sometimes such bald truths as that just stated are not particularly satisfying. It may, instead, be better to point out that the "unconditional" premiss of an inference of the form ($\star$) says more than does its conditional conclusion. This is the case when '$\Longrightarrow$' has the meaning of 'if . . . then ___' which is used in mathematics, and in many other places. It also may help to say that the premiss q amounts to the redundant

q, whether or not p

and argue from this that the conclusion says less — or, at least, no more — than the premiss and, so, should be considered to be a consequence of it.

It is also well to bear in mind that a student who questions the validity of conditionalizing inferences may merely be using this as a way of indicating that he sees no point in inferring a conclusion which is so obviously weaker than the premiss. To such an objection, you can point out that conclusions usually say less than do the premisses from which they are derived — certainly, they never say more. So, his objection would apply in quality — if not in quantity — to any valid inference.

The examples given at the end of the paragraph which precedes the exercises furnish an opportunity for pointing out that valid sentences are, according to our definition of 'theorem', theorems. A theorem is a sentence which is a consequence of our postulates. As pointed out in the text, '$\vec{a} = B - A \Longrightarrow A + \vec{a} = B$' is a consequence of the single Postulate 2(a). Because of this it is a consequence of the set consisting of all our postulates [however large this may become as the course proceeds]. Hence, the sentence in question is a theorem. The point to be noted here is that any consequence of a given set of sentences is, thereby, a consequence of any more inclusive set of sentences. Turning now to valid sentences we recall that they are "true on logical grounds alone" and, so, are consequences of the least inclusive set of sentences there is — the empty set. So, a valid sentence is a consequence of any set of sentences one cares to name. In particular, such a sentence is a consequence of our postulates.

Exercises

Part A

Each of the following exercises suggests a derivation in which the conclusion is obtained by applying the deduction rule. For each exercise,

- **(a)** copy the exercise, filling in the missing sentence [or sentences],
- **(b)** circle the premisses on which the conclusion depends,
- **(c)** tell whether the derivation shows that the conclusion is a theorem and, if so, whether it is a valid sentence.

Example 1.

$$\cfrac{\cfrac{a \overset{*}{=} b \qquad a^2 = a^2}{a^2 = b^2}}{?}\ {}^{*}$$

Solution. To use the deduction rule to complete this derivation you must write a conditional sentence whose consequent is '$a^2 = b^2$' [that is, you must conditionalize '$a^2 = b^2$'] and whose antecedent is '$a = b$' [because the asterisks tell you to discharge the premiss '$a = b$']. The completed diagram is:

$$\cfrac{\cfrac{a \overset{*}{=} b \quad a^2 = a^2}{a^2 = b^2}}{a = b \Longrightarrow a^2 = b^2}\ {}^{*}$$

Since the left-hand premiss is discharged and the right-hand premiss is valid, the conclusion depends on neither premiss. So, the conclusion is, itself, a valid sentence.

Example 2.

$$\cfrac{\cfrac{\cfrac{a \overset{*}{=} b \quad ac = ac}{?} \qquad\qquad ac = bc \Longrightarrow ac - bc = 0}{ac - bc = 0}}{?}\ {}^{*}$$

Solution. By the [left-by-right] replacement rule for equations, '$ac = bc$' is a consequence of '$a = b$' and '$ac = ac$'. From '$ac = bc$' and '$ac = bc \Longrightarrow ac - bc = 0$' one can infer '$ac - bc = 0$' by modus ponens. Finally, using the deduction rule, '$ac - bc = 0$' is "conditionalized" and '$a = b$' is discharged. The conclusion is:

$$a = b \Longrightarrow ac - bc = 0$$

Here is the completed derivation [with a loop around the only premiss on which the conclusion depends]:

$$\cfrac{\cfrac{\cfrac{a \overset{*}{=} b \qquad ac = ac}{ac = bc} \qquad\qquad \boxed{ac = bc \Longrightarrow ac - bc = 0}}{ac - bc = 0}}{a = b \Longrightarrow ac - bc = 0}\ {}^{*}$$

Since the only premiss on which the conclusion depends is true, so is the conclusion.

1. $$\dfrac{\dfrac{a \overset{*}{=} b \qquad 2a = 2a}{2a = 2b}}{?}\,*$$

2. $$\dfrac{\dfrac{a \overset{*}{=} b \qquad a = a}{b = a}}{?}\,*$$

3. $$\dfrac{\dfrac{\dfrac{a \overset{*}{=} -3 \qquad -3 < 0}{?} \qquad a < 0 \longrightarrow a^3 < 0}{a^3 < 0}}{?}\,*$$

4. $$\dfrac{\dfrac{A \overset{*}{+} \vec{a} = B \qquad \vec{a} = (A + \vec{a}) - A}{?}}{?}\,*$$

5. $$\dfrac{\dfrac{\dfrac{B - \overset{*}{A} = \vec{0} \qquad B - A = \vec{0} \longrightarrow B = A + \vec{0}}{?} \qquad A + \vec{0} = A}{?}\,*}{B - A = \vec{0} \longrightarrow B = A}$$

6. $$\dfrac{\dfrac{\dfrac{\overset{*}{\text{It}}\text{ is warm.} \qquad \text{If it is warm, I get thirsty.}}{?} \qquad \text{If I get thirsty, I drink milk.}}{\text{I drink milk.}}}{?}\,*$$

7. Exercise 6, properly completed, shows that its conclusion is a consequence of its two undischarged conditional premisses. Note that the antecedent of the second of these premisses is the consequent of the first. This situation occurs so frequently that it is worth formulating the result as a rule.

The Rule of the Hypothetical Syllogism

Any inference of the form:

$$\dfrac{p \longrightarrow q \qquad q \longrightarrow r}{p \longrightarrow r}$$

is valid.

Notice that this rule can be justified on the basis of our two earlier rules for conditional sentences – modus ponens and the deduction rule. To do so, all you need do is make out a scheme, using '*p*', '*q*', and '*r*', of the same form as the completed derivation for Exercise 6. Do so.

Part B

In each of the following exercises you are given a list of sentences from which you are to infer the given conclusion. Make up a valid

Answers for Part A

[Note that these exercises are very conveniently handled with an overhead projector. An exercise on one sheet, the missing sentences on one overlay, the loops on another.]

1. (a) $a = b \Longrightarrow 2a = 2b$ (b) no loops (c) valid
2. (a) $a = b \Longrightarrow b = a$ (b) no loops (c) valid
3. (a) $a < 0,\ a = -3 \Longrightarrow a^3 < 0$
   (b) loops around each undischarged premiss
   (c) theorem

[Since we have not specified any particular deductive theory for the algebra of real numbers, there is a legitimate question as to what sentences about real numbers are to be considered theorems. In this case, if the undischarged premisses are theorems, the derivation shows that the conclusion is a theorem. At any rate, since these premisses are true, the derivation shows that the conclusion is true.]

4. (a) $\vec{a} = B - A,\ A + \vec{a} = B \Longrightarrow \vec{a} = B - A$
   (b) loop around the undischarged premiss
   (c) theorem [because the derivation shows that the conclusion is a consequence of a postulate.]
5. (a) $B = A + \vec{0},\ B = A$
   (b) loop around each undischarged premiss
   (c) not a theorem [The premiss '$A + \vec{0} = A$' is not — as yet — a theorem. In fact, the symbol '$\vec{0}$' is not yet ''officially'' a part of our language. Both these shortcomings will be remedied in Chapter 3. The conclusion of the derivation will then be a theorem. Make sure that students see that the application of the word 'theorem' changes as we adopt more postulates. In contrast, from their work in Chapter 1, students should realize that the conclusion is true.]
6. (a) I get thirsty.; If it is warm, I drink milk.
   (b) loop around each undischarged premiss
   (c) not a theorem [No deductive theory.]
7. $$\dfrac{\dfrac{\dfrac{\overset{*}{p} \qquad p \Longrightarrow q}{q} \qquad q \Longrightarrow r}{r}\,*}{p \Longrightarrow r}$$

[This is a first illustration of how, on the basis of the rules of logic which we adopt, it is possible to justify other useful rules. There is an obvious analogy between this and deriving theorems from adopted postulates.]

derivation showing that the conclusion follows from the premisses.

Sample 1. Premisses: (1) $a - b = c$
(2) $b + (a - b) = a$
Conclusion: $b + c = a$

Solution. Sentence (2) contains the left side of equation (1). The conclusion follows by using the [left-by-right] replacement rule for equations with sentence (1) as the "equation" and sentence (2) as the "second sentence".

$$\frac{a - b = c \quad b + (a - b) = a}{b + c = a}$$

Sample 2. Premiss: $b + (a - b) = a$
Conclusion: $a - b = c \longrightarrow b + c = a$

Solution. Since the conclusion is a conditional sentence, you must derive its consequent. You should also note that you have an additional premiss to use, namely, the antecedent '$a - b = c$'. This premiss is then discharged when you conditionalize '$b + c = a$'.

$$\cfrac{\cfrac{\overset{*}{a - b = c} \quad b + (a - b) = a}{b + c = a}}{a - b = c \longrightarrow b + c = a}\,{}^{*}$$

1. Premisses: (1) $A + \vec{a} = B$
(2) $B + \vec{b} = C$
Conclusion: $(A + \vec{a}) + \vec{b} = C$

2. Premiss: $\vec{a} = (A + \vec{a}) - A$
Conclusion: $A + \vec{a} = B \longrightarrow \vec{a} = B - A$

3. Premisses: (1) $B - A = A - A \longrightarrow B = A + (A - A)$
(2) $A + (A - A) = A$
Conclusion: $B - A = A - A \longrightarrow B = A$

4. Premisses: (1) $7 \neq 5$
(2) $c \neq 5 \longrightarrow c - 5 \neq 0$
Conclusion: $c = 7 \longrightarrow c - 5 \neq 0$

5. Premisses: (1) $A + (A - A) = A$
(2) $A + (A - A) = B \longrightarrow B - A = A - A$
Conclusion: $A = B \longrightarrow B - A = A - A$

6. Premiss: $\vec{r} = (P + \vec{r}) - P$
Conclusion: $P + \vec{r} = Q \longrightarrow \vec{r} = Q - P$

Part C

In each of Exercises 1-4 you are given the two premisses and the conclusion of a single substitution or replacement inference. Write the inference, tell which kind it is, and circle any premisses on which the conclusion depends.

Answers for Part B

1. $$\frac{A + \vec{a} = B \quad B + \vec{b} = C}{(A + \vec{a}) + \vec{b} = C}$$

2. $$\cfrac{\cfrac{A + \vec{a} \overset{*}{=} B \quad \vec{a} = (A + \vec{a}) - A}{\vec{a} = B - A}}{A + \vec{a} = B \Longrightarrow \vec{a} = B - A}\,{}^{*}$$

3. $$\frac{A + (A - A) = A \quad B - A = A - A \Longrightarrow B = A + (A - A)}{B - A = A - A \Longrightarrow B = A}$$

[Note that the premisses are instances of Postulate 2(a) and of a theorem:

(☆) $B - A = \vec{a} \Longrightarrow B = A + \vec{a}$

respectively. (Compare (☆) with the theorem (a) on page 83.) So, the conclusion is a theorem. Once '$\vec{0}$' has been introduced, and an appropriate postulate has been adopted, this derivation can be extended to a proof of '$B - A = \vec{0} \Longrightarrow B = A$'.

In connection with (☆) you may wish to remark that, due to the symmetry of the relation of identity, (☆) is a consequence of (a). Grant students the freedom to "adopt" theorems which are related to theorems they have actually proved as (☆) is to (a).]

4. $$\cfrac{\cfrac{\cfrac{c \overset{*}{=} 7 \quad 7 \neq 5}{c \neq 5} \quad c \neq 5 \Longrightarrow c - 5 \neq 0}{c - 5 \neq 0}}{c = 7 \Longrightarrow c - 5 \neq 0}\,{}^{*}$$

5. $$\cfrac{\cfrac{\cfrac{A = B \quad A + (A - A) = A}{A + (A - A) = B} \quad A + (A - A) = B \Longrightarrow B - A = A - A}{B - A = A - A}}{A = B \Longrightarrow B - A = A - A}\,{}^{*}$$

[The premisses are instances of Postulate 2(a) and a theorem like theorem (b) of Exercise 2 on page 82. So, the conclusion is a theorem. Compare with Exercise 3, above, and our discussion of that exercise.]

6. $$\cfrac{\cfrac{P + \vec{r} \overset{*}{=} Q \quad \vec{r} = (P + \vec{r}) - P}{\vec{r} = Q - P}}{P + \vec{r} = Q \Longrightarrow \vec{r} = Q - P}\,{}^{*}$$

Sample. Premisses: (1) $(B - A) + (C - B) = C - A$
(2) $C - A \in \mathcal{T}$
Conclusion: $(B - A) + (C - B) \in \mathcal{T}$

Answer.

$$\dfrac{\boxed{(B - A) + (C - B) = C - A} \qquad C - A \in \mathcal{T}}{(B - A) + (C - B) \in \mathcal{T}}\,\text{(RRE)}$$

1. Premisses: (1) the conclusion of the sample
(2) $B + \vec{b} \in \mathcal{E}$
Conclusion: $(B - A) + [(B + \vec{b}) - B] \in \mathcal{T}$

2. Premisses: (1) an instance of Postulate 2(b)
(2) the conclusion of Exercise 1
Conclusion: $(B - A) + \vec{b} \in \mathcal{T}$

3. Premisses: (1) the conclusion of Exercise 2
(2) Postulate 1(b)
Conclusion: $[(A + \vec{a}) - A] + \vec{b} \in \mathcal{T}$

4. Premisses: (1) a postulate
(2) the conclusion of Exercise 3
Conclusion: $\vec{a} + \vec{b} \in \mathcal{T}$

5. **(a)** As you have probably seen, the inferences of the sample and of Exercises 1-4 can be fitted together to obtain a valid derivation whose conclusion is the conclusion of Exercise 4 and whose premisses are those premisses of the five inferences which are not conclusions of earlier inferences. Write out this derivation. Circle the premisses of the derivation on which its conclusion depends.

(b) You should be able to see from your derivation that its conclusion is a consequence of two sentences, one of which is a postulate. What are these two sentences?

Part D

In these exercises you will construct another derivation. This time, instead of writing out the inferences separately as you did in Part C and then rewriting them to obtain the derivation, you will construct the derivation step by step. [Begin near the right side of your paper.]

1. Write an inference with
Premisses: (1) $(B - A) + (C - B) = C - A$
(2) $A + (C - A) = C$
Conclusion: $A + [(B - A) + (C - B)] = C$

2. Infer an instance of this conclusion by substituting '$B + \vec{b}$' for 'C'.

3. Introduce an instance of a postulate as a premiss to enable you to infer the conclusion:

$$A + [(B - A) + \vec{b}] = B + \vec{b}$$

4. Continue, by repeating Exercises 2 and 3 *with changes* which will lead you to the conclusion:

$$A + (\vec{a} + \vec{b}) = (A + \vec{a}) + \vec{b}$$

5. On what premisses of your derivation does this conclusion depend?

Answers for Part C

[As pointed out in Exercise 5, the answers for the sample and Exercises 1 - 4 can be fitted together to form a derivation whose conclusion is '$\vec{a} + \vec{b} \in T$'. (This is the second meaning of '+' — related to function composition — which was introduced in section 1.07). Later in this chapter we shall adopt the first premiss of the sample as our third postulate. Thereupon, '$\vec{a} + \vec{b} \in T$' will become a theorem. It may be better to do Part C as a class exercise so that the various pieces of the derivation are fitted in place systematically. On the chalkboard you can do Exercise 1 by extending the tree from the Sample, do Exercise 2 by extending the tree from Exercise 1, etc. This helps students see the relationship among these four exercises.]

1.

$$\dfrac{\boxed{(B - A) + (C - B) \in T} \qquad B + \vec{b} \in \mathcal{E}}{(B - A) + [(B + \vec{b}) - B] \in T}\,\text{(Subst)}$$

2.

$$\dfrac{\boxed{\vec{b} = (B + \vec{b}) - B} \qquad \boxed{(B - A) + [(B + \vec{b}) - B] \in T}}{(B - A) + \vec{b} \in T}\,\text{(RRE)}$$

3.

$$\dfrac{\boxed{(B - A) + \vec{b} \in T} \qquad A + \vec{a} \in \mathcal{E}}{[(A + \vec{a}) - A] + \vec{b} \in T}\,\text{(Subst)}$$

4.

$$\dfrac{\boxed{\vec{a} = (A + \vec{a}) - A} \qquad \boxed{[(A + \vec{a}) - A] + \vec{b} \in T}}{\vec{a} + \vec{b} \in T}\,\text{(RRE)}$$

5. (a) Here is a complete tree-form derivation of the conclusion of Exercise 4:

$$\dfrac{\boxed{\vec{a} = (A + \vec{a}) - A} \qquad \dfrac{\dfrac{\dfrac{\boxed{\vec{a} = (A + \vec{a}) - A}}{\vec{b} = (B + \vec{b}) - B}\,\text{(Subst)} \qquad \dfrac{\boxed{(B - A) + (C - B) \in T} \qquad \dfrac{A + \vec{a} \in \mathcal{E}}{B + \vec{b} \in \mathcal{E}}\,\text{(Subst)}}{(B - A) + [(B + \vec{b}) - B] \in T}\,\text{(Subst)}}{(B - A) + \vec{b} \in T}\,\text{(RRE)} \qquad A + \vec{a} \in \mathcal{E}}{[(A + \vec{a}) - A] + \vec{b} \in T}\,\text{(Subst)}}{\vec{a} + \vec{b} \in T}\,\text{(RRE)}$$

(b) '$(B - A) + (C - B) \in T$' and Postulate 2(b).

Answers for Part D

[These exercises are analogous to those of Part C in that the conclusion obtained will become a theorem when Postulate 3 is adopted.]

1. - 4.

$$\begin{array}{lll}
 & (B-A)+(C-B)=C-A \qquad A+(C-A)=C & A+\vec{a}\in\mathcal{E} \\
\cline{2-3}
\vec{a}=(A+\vec{a})-A & A+[(B-A)+(C-B)]=C & B+\vec{b}\in\mathcal{E} \\
\hline
\vec{b}=(B+\vec{b})-B & A+[(B-A)+[(B+\vec{b})-B]]=B+\vec{b} & \\
\hline
 & A+[(B-A)+\vec{b}]=B+\vec{b} & A+\vec{a}\in\mathcal{E} \\
\cline{2-3}
\vec{a}=(A+\vec{a})-A & A+[[(A+\vec{a})-A]+\vec{b}]=(A+\vec{a})+\vec{b} & \\
\hline
 & A+(\vec{a}+\vec{b})=(A+\vec{a})+\vec{b} &
\end{array}$$

5. '$(B - A) + (C - B) = C - A$', Postulate 2(a) and Postulate 2(b).

Sample Quiz

1. (a) Let P, Q, R, and S be four points. Draw a picture to illustrate the sentence:

   (*) $(Q - P) + (R - Q)$ is the inverse of $(S - P) + (R - S)$

   (b) Tell whether the sentence (*) is true or false.

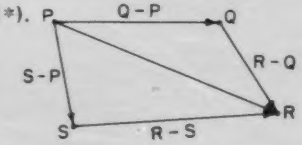
2. (a) Draw a picture to illustrate the sentence:

   (**) If $P + (Q - R) = P + \vec{a}$ then $Q = R + \vec{a}$

   (b) Prove (**). That is, derive (**) from the postulates.

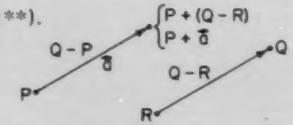
Key for Sample Quiz

1. (a) Here is a typical picture to illustrate (*).



   (b) (*) is false.

2. (a) Here is a typical picture to illustrate (**).



   (b) Proof of (**).

   Suppose that $P + (Q - R) = P + \vec{a}$. Then, $Q - R = (P + \vec{a}) - P$. Since $(P + \vec{a}) - P = \vec{a}$, it follows that $Q - R = \vec{a}$. So, $Q = R + \vec{a}$. Hence, if $P + (Q - R) = P + \vec{a}$ then $Q = R + \vec{a}$.

Students should realize that 'converse' has the same meaning — that of 'turned around' — when applied to ordered pairs, to relations, and to conditional sentences. It is applicable as well to the biconditional sentences of section 2.09 and to the conjunction sentences of Part C on page 101. Since, however, the converse of a sentence of either of these kinds is logically equivalent to the sentence itself, there are fewer occasions for the use of the word in discussing sentences of these kinds.

The notion of the converse of a conditional sentence may have come up earlier. At any rate, the remarks made on TC 85(2) may be of help here in convincing students that a sentence and its converse are not equivalent in meaning.

Although it is exceptional for a conditional sentence to imply its converse, this can happen. For example, any sentence of the form '$p \Longrightarrow p$' certainly implies its converse, and the same is true of any conditional sentence whose antecedent is a valid sentence [or whose consequent has a valid denial]. [These cases — and many others — are included in the triviality that a conditional sentence implies its converse if the latter is a valid sentence.] Consequently, it is not correct to say that inferences of the form:

$$\frac{p \Longrightarrow q}{q \Longrightarrow p}$$

are invalid. It is correct to say that if such an inference is valid it is so for some other reason than that its conclusion is the converse of its premiss.

There are cases which will be familiar to you in which a conditional sentence is helpful in proving its converse. One such case concerns the sentence:

(☆) If $\angle B$ [in $\triangle ABC$] is larger than $\angle A$ then $\overleftrightarrow{CA}$ is longer than $\overleftrightarrow{BC}$.

and its converse. Once either of these two sentences has been shown to be a theorem, the other follows readily from it together with a theorem about isosceles triangles and some properties of the order relations larger than [for angles] and longer than [for segments]. For example, suppose that (☆) is known to be a theorem, and assume that $\overleftrightarrow{CA}$ is longer than $\overleftrightarrow{BC}$. In any case either $\angle B \cong \angle A$ or $\angle A$ is larger than $\angle B$ or $\angle B$ is larger than $\angle A$. In case $\angle B \cong \angle A$ then [by a theorem an isosceles triangles] $\overleftrightarrow{CA} \cong \overleftrightarrow{BC}$ and so [by a theorem about longer than] $\overleftrightarrow{CA}$ is not longer than $\overleftrightarrow{BC}$. In case $\angle A$ is larger than $\angle B$ it follows [from an instance of (☆)] that $\overleftrightarrow{BC}$ is longer than $\overleftrightarrow{CA}$ and so [by a theorem about longer than] that $\overleftrightarrow{CA}$ is not longer than $\overleftrightarrow{BC}$. Since [by assumption] $\overleftrightarrow{CA}$ is longer than $\overleftrightarrow{BC}$ it follows that neither is $\angle B$ congruent to $\angle A$ nor is $\angle A$ larger than $\angle B$. So, $\angle B$ is larger than $\angle A$. Hence, if $\overleftrightarrow{CA}$ is longer than $\overleftrightarrow{BC}$ then $\angle B$ is larger than $\angle A$.

2.07 The Converse of a Conditional Sentence

You learned that a sentence of the form:

If p then q.

is a conditional sentence. Its *converse* is the conditional sentence which is formed by interchanging the antecedent and the consequent:

If q then p.

For example, the converse of the sentence:

(*) If Jack lives in Baltimore then Jack lives in Maryland.

is the sentence:

(**) If Jack lives in Maryland then Jack lives in Baltimore.

Notice that although (*) is true, (**) may be false [for Jack may live in Annapolis]. This illustrates the fact that one of a pair of converse conditional sentences may be true and the other false. So, a conditional sentence may not imply its converse. [There are, however, many important cases in which a conditional sentence follows from its converse *together with other premisses*.]

A conditional sentence and its converse may both be true, and as you have seen [where?], may both be theorems. But, showing that a conditional sentence and its converse are both theorems *usually* requires two proofs.

Exercises

Part A

In each of the following exercises you are given a conditional sentence. In each case, write the converse of the given sentence.

Sample. $p \longrightarrow q$
Answer. $q \longrightarrow p$

1. $a = b \longrightarrow a - b = 0$
2. $a > 0 \longrightarrow a^2 > 0$
3. $A \in \overline{BC} \longrightarrow A \neq B$
4. $A \notin \overrightarrow{BC} \longrightarrow A \in \overleftrightarrow{CB}$
5. $A + \vec{a} = B \longrightarrow \vec{a} = B - A$
6. $A = B \longrightarrow A - C = B - C$
7. $\vec{a} = \vec{b} \longrightarrow A + \vec{a} = A + \vec{b}$
8. $A = B \longrightarrow A + \vec{a} = B + \vec{a}$
9. f is a translation ⟶ f has an inverse
10. $(A + \vec{a} = B \text{ and } B + \vec{b} = C) \longrightarrow A + (\vec{a} + \vec{b}) = C$
11. Given the sentence you wrote in answer to Exercise 1, write the converse of it. What is the relation of this sentence to that of Exercise 1? Would you get similar results in Exercises 2–10?

Note, in the last sentence, the appeal to the deduction rule. The use made of (☆) in this proof depends — as one would anticipate — on modus ponens. The formal validity of the proof depends as well on other rules, concerning the meanings of 'or' and 'not', which will be discussed in a later chapter.

Answer to query: Sentences (a) and (b) [see answer for Exercise 5, below] are both theorems, and either is the converse of the other.

Answers for Part A

1. $a - b = 0 \Longrightarrow a = b$
2. $a^2 > 0 \Longrightarrow a > 0$
3. $A \neq B \Longrightarrow A \in \overline{BC}$
4. $A \in \overleftrightarrow{CB} \Longrightarrow A \notin \overrightarrow{BC}$
5. $\vec{a} = B - A \Longrightarrow A + \vec{a} = B$
6. $A - C = B - C \Longrightarrow A = B$
7. $A + \vec{a} = A + \vec{b} \Longrightarrow \vec{a} = \vec{b}$
8. $A + \vec{a} = B + \vec{a} \Longrightarrow A = B$
9. f has an inverse ⟹ f is a translation
   Has an inverse
10. $A + (\vec{a} + \vec{b}) = C \Longrightarrow (A + \vec{a} = B \text{ and } B + \vec{b} = C)$,
    $(A + \vec{a} = B \text{ and } B + \vec{b} = C) \Longrightarrow A + (\vec{a} + \vec{b}) = C$
11. $a = b \Longrightarrow a - b = 0$, the converse of the converse of a sentence is that sentence.

Part B

1. (a) Which of the sentences given in Part A are true?
   (b) Which of these sentences have true converses?
   (c) Which of the sentences in Exercises 5–10 of Part A are theorems?
   (d) Which of these sentences have theorems as converses?
2. Write three conditional sentences about real numbers which are true *and* have true converses.
3. Write three conditional sentences about real numbers which are true *and* have false converses.

Part C

If a sentence which contains a variable is false, you may be able to show that this is the case by giving a *counter-example*. For instance, as you probably decided, the sentence:

$$(*) \qquad a^2 > 0 \Longrightarrow a > 0$$

is false because there exist numbers whose squares are positive but which are not themselves positive. One such number is -2. Since $(-2)^2 > 0$ but $-2 \not> 0$, the instance:

$$(-2)^2 > 0 \Longrightarrow -2 > 0$$

of (*) is false. Consequently, (*) is false.

The same procedure which works for (*) also works for false sentences of our algebra of points and translations. To give a counter-example for such a sentence you must – instead of naming a number – draw a picture showing points and translations. For example, consider the sentence:

$$(**) \qquad (A + \vec{a}) + \vec{b} = (A + \vec{b}) + \vec{a} \Longrightarrow A + \vec{a} = A + \vec{b}$$

In trying to guess whether (**) is true, we draw pictures. If we can picture a point A and translations $\vec{a}$ and $\vec{b}$ such that $A + \vec{a} \neq A + \vec{b}$ but $(A + \vec{a}) + \vec{b} = (A + \vec{b}) + \vec{a}$ then we will know that (**) is false.

1. Show that (**) is false.
2. Can you show that the converse of (**) is false?
3. Do you think that the sentence:

$$(***) \qquad A = B \Longrightarrow (A + \vec{a}) + \vec{b} = (B + \vec{b}) + \vec{a}$$

   is true? Do you think that its converse is true?
4. What simpler [non-conditional] sentence says what (***) says?

Answers for Part B

1. (a) The sentences of Exercises 1, 2, 3, 5, 6, 7, 8, 9, and 10 are true.
   (b) The sentences of Exercises 1, 5, 6, 7, and 8 have true converses.
   (c) The sentences of Exercises 5, 6, 7, and 8 are theorems.
   (d) The sentences of Exercises 5, 6, and 7 have theorems as converses.

2., 3. [Various answers are possible, and discussion of these exercises might serve a useful purpose in reviewing algebra.]

Here follows some discussion of Exercise 1.

We expect that students will obtain answers for parts (a) and (b) without worrying about the "truth-conditions" for conditional sentences. Presumably they will understand the sentences and label 'true' those with which they agree. There is some point in knowing how to use counter-examples to establish the falsity of open conditional sentences, and this matter is taken up in the exercises of Part C which follow. Students may, also, in working part (a), recognize that the sentences of Exercises 5, 6, 7, and 8 are theorems and grant their truth on the grounds that theorems, generally, are true. [Our postulates are true, and consequences of true statements are true.] If you are interested in the source of truth-conditions for conditional sentences, there is a discussion of this in the commentary for page 265 of High School Mathematics, Course 1. There is a more complete discussion at the end of the commentary for the appendix on logic which appears both in Course 2 and in Course 3. [In these references it is argued that it is merely by definition that consequences of true sentences are true.]

In this course our interest is more in theorem-hood than in truth. We take care that our postulates are true, and this assures the truth of our theorems. The only use we make of this fact is that it allows us to show that certain sentences are not theorems by showing that these sentences are false. Hence, Part C.

The answers for parts (c) and (d) of Exercise 1 merit some discussion here. The sentence of Exercise 5 is the only too familiar theorem (b), and its converse is the theorem (a). By now students should recognize equality principles such as those in Exercises 6, 7, and 8, and realize that any such sentence is not only a theorem but is a valid sentence. Whether or not their converses are theorems is another question. We shall take it up shortly.

The sentence of Exercise 9 expresses one of the properties of translations discovered in Chapter 1 and included, implicitly, in the summary on page 47. So, the sentence is true. Students should, however, be sceptical as to whether it is a consequence of the postulates at hand. Since these postulates merely tell us that translations are mappings of $\mathcal{E}$ into itself, and that, given A and B, there is a unique translation which maps A on B, it is unlikely that they imply that translations are one-to-one mappings. That they don't is proved in Part D on page 110.

The sentence of Exercise 10 is not yet a theorem. It will become so once Postulate 3 is adopted. For, as pointed out on TC 92(2), in connection with Part D, it will then be the case that '$A + (\vec{a} + \vec{b}) =$

$(A + \vec{a}) + \vec{b}$' is a theorem. And it is not difficult to show that '$(A + \vec{a} = B$ and $B + \vec{b} = C) \Longrightarrow (A + \vec{a}) + \vec{b} = C$' is, already, a theorem. [All that is needed is to infer from the antecedent of this sentence each of the equations '$A + \vec{a} = B$' and '$B + \vec{b} = C$', perform a replacement, and apply the deduction rule.]

The converse of the sentences of Exercise 10 is not a theorem because it is false. [An assumption that $A + (\vec{a} + \vec{b}) = C$ cannot yield any information concerning a "new" point B.] The converse of the sentence of Exercise 9 is, for the same reason, not a theorem. [Lots of mappings which are not translations have inverses.]

We now come to the converses of the equality principles stated in Exercises 6, 7, and 8. The converse of the last of these principles says just that any translation has an inverse. Our discussion, above, of Exercise 9 shows that it is unlikely that this converse is a theorem. The converse of the sentence of Exercise 7 is a theorem which students have already proved. [See Exercise 1 on page 82.] The converse, '$A - C = B - C \Longrightarrow A = B$', of the sentence in Exercise 6 is also a theorem. You might ask students to prove it. [Suppose that $A - C = B - C$. It follows that $C + (A - C) = C + (B - C)$ and so, by Postulate 2(a), that $A = B$. Hence, if $A - C = B - C$ then $A = B$.] [In contrast, the sentence '$C - A = C - B \Longrightarrow A = B$', while true, is not a theorem.]

*

The discussion preceding the exercises in Part C may be amplified somewhat as follows. Since consequences of true sentences are true, one can show a sentence to be false by showing that it implies some false sentence. In particular, in view of the substitution rule, one can show that a sentence is false by exhibiting a false instance of that sentence. So, the sentence (☆) will be shown false if we can, for example, show that its instance:

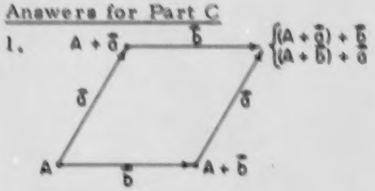
$$(-2)^2 > 0 \Longrightarrow -2 > 0$$

is false. To do so, we might look for a consequence of this sentence which is false. This looks difficult. Reconsidering, we note that it would be sufficient to derive a false consequence from this conditional sentence and some true sentences. Using modus ponens, we can infer the false sentence '$-2 > 0$' from the conditional sentence and the true sentence '$(-2)^2 > 0$'. So, finally, (☆) is false. [This is not the place to raise questions as to why '$-2 > 0$' is false and '$(-2)^2 > 0$' is true.]

Generalizing on the preceding example it is clear that the same procedure can be used for any conditional sentence. In attempting to show that such a sentence is false, we look for a situation in which its antecedent is true but its consequent is false. In dealing with sentences about real numbers we must, to describe such a situation, describe one or more numbers. Usually, we name them. In dealing with sentences about points and translations we describe appropriate situations by drawing pictures.

Returning to (☆), note that not only is (☆) false but so are its antecedent '$a^2 > 0$' [0 is a counter-example] and its consequent '$a > 0$' [any nonpositive number is a counter-example]. This may seem surprising in view of the usual truth-condition according to which conditional sentences with false antecedents and consequents are true. But, as remarked earlier, such conditions do not apply to open sentences.

Answers for Part C

1. 

[Nearly any drawing such as this will show (☆☆) to be false. The only requirement is that $\vec{a}$ and $\vec{b}$ be different translations.]

2. No. [We know that if $A + \vec{a} = A + \vec{b}$ then $\vec{a} = \vec{b}$. And, if $\vec{a} = \vec{b}$ then $(A + \vec{a}) + \vec{b} = (A + \vec{b}) + \vec{a}$. In fact, the converse of (☆☆) is a theorem.]

3. (☆☆☆) is true; so is its converse. [Attempts to draw a counter-example for (☆☆☆) will fail, as will similar attempts for its converse. Such failure is, of course, not sufficient evidence on which to assert the truth of (☆☆☆) and of its converse. [We just may not have been clever enough in conducting our search for counter-examples.) It is, however, enough evidence to justify the given answer to the question "Do you think . . . ?".]

4. $(A + \vec{a}) + \vec{b} = (A + \vec{b}) + \vec{a}$ [This follows from (☆☆☆) by using modus ponens and the validity of '$A = A$'. (☆☆☆) follows from it by using the replacement rule for equations and the deduction rule. In view of the truth of '$(A + \vec{a}) + \vec{b} = A + (\vec{a} + \vec{b})$', (☆☆☆) and this simpler sentence say, essentially, that composition of translations is commutative.]

TC 95, 96

The text may add one more technique to your arsenal of methods for getting students to accept the conventional meanings of 'if' and 'only if'. Actually, the only reason for bringing the matter up is to point out to students that people who read '⟺' as 'if and only if' are not complete idiots for doing so. The meaning of '⟺' is expressed — completely — in terms of the meaning of '⟹', by the rules for biconditional sentences given on page 98. Just as modus ponens and the deduction rule formulate that meaning of 'if . . . then ___' in which we are interested, so do the rules on page 98 formulate that meaning of 'if and only if' which interests us.

Incidentally, we introduce the symbol '⟸' just because a few people like it. We shall make no use of it after page 97.

2.08 Equivalent Forms of Conditional Sentences

Here is a conversation in which a conditional sentence is stated in three equivalent ways:

Mrs. Smith: Remember Jack Jones who used to live next door to us before he joined the army? Well, his mother told me that Jack is only 15 years old.
Mr. Smith: That's impossible. He is in the army – and if he is in the army, he is over 18 years old.
Mrs. Smith: But Mrs. Jones told me . . . . . .
Mr. Smith: I don't care what Mrs. Jones told you. You know he is in the army – and he is in the army *only if* he is over 18 years old.
Mrs. Smith: But . . . . . .
Mr. Smith: Don't tell me "but" – there is nothing more to be said. He is in the army – and he is over 18 years old if he is in the army.

Although Mr. Smith never explicitly said so, it is clear that Mr. Smith is insisting to Mrs. Smith that:

Jack Jones is over 18 years old.

He is basing his argument on the premises:

Jack is in the army. If Jack is in the army then Jack is over 18 years old.

Each time he presented his argument he merely stated his conditional sentence in a different, yet equivalent, form. The equivalent forms of the conditional premiss that Mr. Smith used are:

If Jack is in the army *then* Jack is over eighteen years old.
Jack is in the army *only if* Jack is over eighteen years old.
Jack is over eighteen years old *if* Jack is in the army.

Here is another conversation in which a conditional sentence is stated in three equivalent ways:

Mr. Smith: Does Mr. Jones live in New Jersey?
Mrs. Smith: Of course! You know he lives in Princeton – and if he lives in Princeton then he lives in New Jersey.
Mr. Smith: But I thought . . . . . .
Mrs. Smith: Don't tell me what you thought. He lives in Princeton – and he lives in Princeton only if he lives in New Jersey.

Mr. Smith: But . . . . . .
Mrs. Smith: Don't tell me "but". There is nothing more to be said. He lives in Princeton – and he lives in New Jersey if he lives in Princeton.

Poor Mr. and Mrs. Smith do not get along too well, do they? Mrs. Smith never gave Mr. Smith a chance to explain why he may not agree with her. Of course, if he accepts Mrs. Smith's premisses he must accept her conclusion. [If Mr. Smith doesn't accept her premisses he need not accept her conclusion no matter how many times she repeats her argument.] Mrs. Smith is trying to have Mr. Smith conclude that he (Mr. Jones) lives in New Jersey. She is basing her argument on the premises:

He lives in Princeton. If he lives in Princeton then he lives in New Jersey.

Here are the equivalent conditional premisses that Mrs. Smith used in the above conversation:

If he lives in Princeton *then* he lives in New Jersey.
He lives in Princeton *only if* he lives in New Jersey.
He lives in New Jersey *if* he lives in Princeton.

These imaginary conversations suggest – correctly – that a sentence of the form:

$$p \longrightarrow q$$

can be translated in any of three ways:

If p then q.
p only if q.
q if p.

Either of the first two ways gives a convenient way to read '$\longrightarrow$'. The third way suggests introducing the symbol '$\longleftarrow$' to be read as 'if'. Read:

$$q \longleftarrow p$$

as 'q if p'.

Exercises

Part A

1. In each of the following exercises you are given a sentence of the form '$p \longrightarrow q$' or '$p \longleftarrow q$'. In each case, write three equivalent

sentences using 'If . . . then . . .', '. . . only if . . .', and '. . . if . . .'.

(a) $a = b \longrightarrow a - b = 0$ (b) $\vec{a} = B - A \longrightarrow A + \vec{a} = B$

(c) $A + \vec{a} = B \longleftarrow \vec{a} = B - A$ (d) $\vec{a} = B - A \longleftarrow A + \vec{a} = B$

2. For each of the following sentences, write two equivalent sentences – one using '$\longrightarrow$', the other using '$\longleftarrow$'.

(a) $a - b = 0$ only if $a = b$ (b) $a - b = 0$ if $a = b$

(c) $\vec{a} = B - A$ if $A + \vec{a} = B$ (d) $A + \vec{a} = B$ only if $\vec{a} = B - A$

Part B

In each of the following, complete the inference so that it is an example of modus ponens.

1. $$\frac{A = A + \vec{a} \qquad A = A + \vec{a} \text{ only if } A - A = \vec{a}}{?}$$

2. $$\frac{? \qquad D - C = B - A \text{ only if } D = C + (B - A)}{?}$$

3. $$\frac{? \qquad D - C = B - A \text{ if } D = C + (B - A)}{?}$$

4. $$\frac{? \qquad D - C = D - C \longrightarrow D = C + (D - C)}{?}$$

5. $$\frac{? \qquad D - C = D - C \longleftarrow D = C + (D - C)}{?}$$

6. $$\frac{\vec{a} = \vec{b} \qquad ? \text{ only if } ?}{A + \vec{a} = A + \vec{b}}$$

7. $$\frac{\vec{a} = \vec{b} \qquad ? \text{ if } ?}{A + \vec{a} = A + \vec{b}}$$

8. $$\frac{(A + \vec{a}) + \vec{b} = B \qquad ? \text{ only if } ?}{B - (A + \vec{a}) = \vec{b}}$$

2.09 Biconditional Sentences

Consider the two conditional sentences:

(1) If $a - b = 0$ then $a = b$.
(2) If $a = b$ then $a - b = 0$.

Notice that (2) is the converse of (1). We can rewrite these sentences as follows:

(1′) $a = b$ if $a - b = 0$.
(2′) $a = b$ only if $a - b = 0$.

If we wish to write one sentence which states what (1) and (2) state together, we may write:

[$a = b$ *if* $a - b = 0$] *and* [$a = b$ *only if* $a - b = 0$]

Answers for Part A

1. (a) If $a = b$ then $a - b = 0$.
$a = b$ only if $a - b = 0$.
$a - b = 0$ if $a = b$.

(b) If $\vec{a} = B - A$ then $A + \vec{a} = B$.
$\vec{a} = B - A$ only if $A + \vec{a} = B$.
$A + \vec{a} = B$ if $\vec{a} = B - A$.

(c) [Same answers as for (b).]

(d) $\vec{a} = B - A$ if $A + \vec{a} = B$.
$A + \vec{a} = B$ only if $\vec{a} = B - A$.
If $A + \vec{a} = B$ then $\vec{a} = B - A$.

2. (a) $a - b = 0 \Longrightarrow a = b$
$a = b \Longleftarrow a - b = 0$

(b) $a - b = 0 \Longleftarrow a = b$
$a = b \Longrightarrow a - b = 0$

(c) $\vec{a} = B - A \Longleftarrow A + \vec{a} = B$
$A + \vec{a} = B \Longrightarrow \vec{a} = B - A$

(d) $A + \vec{a} = B \Longrightarrow \vec{a} = B - A$
$\vec{a} = B - A \Longleftarrow A + \vec{a} = B$

Answers for Part B

1. $A - A = \vec{a}$

2. $D - C = B - A$
$D = C + (B - A)$

3. $D = C + (B - A)$
$D - C = B - A$

4. $D - C = D - C$
$D = C + (D - C)$

5. $D = C + (D - C)$
$D - C = D - C$

6. $\underline{\vec{a} = \vec{b}}$ only if $\underline{A + \vec{a} = A + \vec{b}}$.

7. $\underline{A + \vec{a} = A + \vec{b}}$ if $\underline{\vec{a} = \vec{b}}$.

8. $\underline{(A + \vec{a}) + \vec{b} = B}$ only if $\underline{B - (A + \vec{a}) = B}$.

For these exercises, we must concentrate on precisely what modus ponens tells us, as well as on the various equivalent forms for conditional sentences. In terms of these various equivalent forms for conditional sentences, we can illustrate the inference scheme for modus ponens in the following ways:

(1) $$\frac{p \qquad \text{If } p \text{ then } q.}{q}$$

(2) $$\frac{p \qquad p \text{ only if } q.}{q}$$

(3) $$\frac{p \qquad q \text{ if } p.}{q}$$

(4) $$\frac{p \qquad p \Longrightarrow q}{q}$$

(5) $$\frac{p \qquad q \Longleftarrow p}{q}$$

A shorter, yet equivalent, way of expressing the same idea is:

$$a = b \text{ if and only if } a - b = 0.$$

A sentence such as this one is called a *biconditional sentence.*

Another way to write one sentence which states what (1) and (2) state together is the following:

$$[a = b \Longleftarrow a - b = 0] \text{ and } [a = b \Longrightarrow a - b = 0]$$

This suggests another convenient form in which to write a biconditional sentence that says what (1) and (2) say together, namely:

$$a = b \Longleftrightarrow a - b = 0$$

In general, a biconditional sentence:

(*) p if and only if q

is equivalent to the sentence:

(**) $[p$ if $q]$ and $[p$ only if $q]$

This means that from a biconditional sentence of the form of (*), we can infer either one of the conditional sentences in (**). It also means that from *both* of the conditional sentences in (**) [taken together] we can infer the biconditional sentence (*). We summarize these notions in the following:

Rules for Biconditional Sentences

Inferences of any of the forms:

$$\text{(a)}\ \frac{p \Longleftrightarrow q}{q \Longrightarrow p} \qquad \text{(b)}\ \frac{p \Longleftrightarrow q}{p \Longrightarrow q}$$

$$\text{(c)}\ \frac{q \Longrightarrow p \quad p \Longrightarrow q}{p \Longleftrightarrow q}$$

are valid.

Exercise

For each of the sentences in Exercise 1 of Part A on page 96, write its converse, using '$\Longrightarrow$', and decide whether the converse is a theorem. If you believe that the given sentence and its converse are both theorems, write the inference of type (c), above:

$$\frac{[\text{given sentence}] \quad [\text{converse}]}{[\text{conclusion}]}$$

Although the identification of '$\Longleftrightarrow$' with 'if and only if' suggests the possibility of treating a biconditional sentence as an abbreviation for the conjunction of two conditional sentences, we prefer the alternative approach, formulated in the rules given on page 126, of treating biconditional sentences, formally, as a new kind of sentence. With the adoption, in Part C on page 129, of rules for conjunction sentences it becomes possible to show that the two approaches actually are equivalent. [In particular, see Exercises 1 and 2 of Part C.]

Note that in part (c) of the rule on page 101, the premisses of the inference scheme are in this order: if-part (p if q), only if-part (p only if q).

Answers for Exercise

[For convenience we write each of the four sentences in Part A, using '$\Longrightarrow$', followed by its converse. If both are theorems, we complete the inference as required.]

$$\text{(a)}\ \frac{a = b \Longrightarrow a - b = 0 \qquad a - b = 0 \Longrightarrow a = b}{a - b = 0 \Longleftrightarrow a = b}$$

$$\text{(b)}\ \frac{\vec{a} = B - A \Longrightarrow A + \vec{a} = B \qquad A + \vec{a} = B \Longrightarrow \vec{a} = B - A}{A + \vec{a} = B \Longleftrightarrow \vec{a} = B - A}$$

(c) [Answer is same as for (b).]

$$\text{(d)}\ \frac{A + \vec{a} = B \Longrightarrow \vec{a} = B - A \qquad \vec{a} = B - A \Longrightarrow A + \vec{a} = B}{\vec{a} = B - A \Longleftrightarrow A + \vec{a} = B}$$

[Note that although the conclusions of the inferences for (b) and (d) are equivalent, they are not the same sentence. Each is the converse of the other, and the if-part of either is the only if-part of the other.]

*

Consider the following inferences:

$$\frac{a = b \Longleftrightarrow a - b = 0 \quad a = b \Longrightarrow a^2 = b^2}{a - b = 0 \Longrightarrow a^2 = b^2}$$

and:

$$\frac{a > b \Longleftrightarrow a - b \text{ is positive} \quad [a > b \text{ and } b > c] \Longrightarrow a > c}{[a - b \text{ is positive and } b > c] \Longrightarrow a > c}$$

The first of these inferences may be shown to be valid by using a biconditional inference of type (a) above and an inference of the form.

$$\frac{p \Longrightarrow q \quad q \Longrightarrow r}{p \Longrightarrow r}$$

[See Exercise 7 on page 90.]

$$\begin{array}{l} \underline{a = b \Longleftrightarrow a - b = 0} \\ \underline{a - b = 0 \Longrightarrow a = b \quad a = b \Longrightarrow a^2 = b^2} \text{ (Syllogism)} \\ \qquad a - b = 0 \Longrightarrow a^2 = b^2 \end{array}$$

The second may also be shown to be valid, but in order to do so one needs rules for 'and'. [See Part C, below.]

These inferences illustrate a rule of logic which is analogous to the replacement rule for equations.

The Replacement Rule for Biconditional Sentences

Given a biconditional sentence and a second sentence, if either "side" of the biconditional sentence is replaced, somewhere in the second sentence, by the other side, the resulting sentence is a consequence of the given sentences.

This rule should be in accord with what you feel when you say 'if and only if'. At any rate, it can be justified on the basis of the other rules we have adopted for conditional and biconditional sentences and the similar rules we shall adopt for other kinds of sentences. [As in the case of the replacement rule for equations, we shall say that we have used the *left-by-right*, or the *right-by-left*, replacement rule for biconditional sentences depending on whether we have replaced an occurrence of the left "side" of a biconditional sentence by its right side, or an occurrence of its right side by its left.]

There is also a rule analogous to the introduction rule for equations:

The Reflexive Rule for Biconditional Sentences

Any sentence of the form '$p \Longleftrightarrow p$' is valid.

The use of a biconditional sentence in a proof is most often for the purpose of justifying replacing one of its components [''sides''] by the other, somewhere in a second sentence. The justification can sometimes be carried out without much trouble by first inferring either the if-part or the only if-part of the biconditional sentence and using this part in justifying the desired replacement. It is, however, always easier to use the replacement rule on page 99 directly. Since this rule can be justified on the basis of our other rules of logic — those present and those to come — it is only reasonable to use it. [In addition, the rule has a certain intuitive appeal. A biconditional sentence appears to say that its components, themselves, ''say the same thing''. That it actually does say this can be inferred only from the fact that the replacement rule does follow from the basic rules on page 98 which characterize the meaning of '$\Longleftrightarrow$'.]

The assertion made by the introduction rule for biconditional sentences stems from the validity of sentences of the form '$p \Longrightarrow p$' and of inferences of the form:

$$\frac{p \Longrightarrow p \qquad p \Longrightarrow p}{p \Longleftrightarrow p}$$

The replacement rule and introduction rule for biconditional sentences are analogous to the similarly-named rules for equations. But, although the latter tell us all there is to be told about the logical predicate '=', the former do not do so thorough a job for the logical connective '$\Longleftrightarrow$'. This is because a biconditional sentence can be ''split up'' into two conditional sentences, while an equation cannot. [True, when dealing with, say, real numbers, the equation '$a = b$' has the same content as '$a \geq b$ and $b \geq a$'. But, greater than is not a logical relation. Except when the subject matter — in this case, the real numbers — is subject to an order relation like greater than, the identity relation is unanalyzable.]

Exercises

Part A

In each exercise, complete the inference to illustrate the replacement rule for biconditional sentences. Tell whether you used the left-by-right rule or the right-by-left rule.

1. $\underline{2a + 5 = 0 \longleftrightarrow a = {}^5/_3 \quad \text{not } (2a + 5 = 0)}$

2. $\underline{b < 0 \longleftrightarrow -b > 0 \quad -b \ngtr 0}$ [*Hint*: '$-b \ngtr 0$' is short for 'not $(-b > 0)$'.]

3. $\underline{B + \vec{a} = A \longleftrightarrow \vec{a} = A - B \quad B + \vec{a} = A}$

4. $\underline{Q + \vec{t} = P \longleftrightarrow \vec{t} = P - Q \quad \vec{t} \neq P - Q}$

5. $\underline{A + \vec{a} = A + \vec{b} \longleftrightarrow \vec{a} = \vec{b} \quad \vec{a} \neq \vec{b} \text{ and } A = B}$

6. $\underline{P + \vec{p} = P + \vec{q} \longleftrightarrow \vec{p} = \vec{q} \quad (P + \vec{p} = \text{P} + \vec{q} \text{ and } P + \vec{q} = Q) \longrightarrow P + \vec{p} = Q}$

Part B

Show, without using the replacement rule, that inferences of the following forms are valid.

Sample. $\dfrac{p \longleftrightarrow q \quad p}{q}$

Solution. $$\dfrac{p \quad \dfrac{p \longleftrightarrow q}{p \longrightarrow q}\,\text{(Part (b) of "Rules for Biconditional Sentences")}}{q}\,\text{(Modus ponens)}$$

1. $\dfrac{p \longleftrightarrow q \quad p \longrightarrow r}{q \longrightarrow r}$ [*Hint*: See example preceding statement of the replacement rule.]

2. $\dfrac{p \longleftrightarrow q \quad r \longrightarrow p}{r \longrightarrow q}$

Part C

A *conjunction sentence* is a sentence of the form:

p and q

Examples: John is present and Jack is absent.

$\vec{p} = \vec{q}$ and $P + \vec{q} = Q$
$ab = 0$ and $a \neq 0$

What you probably mean when you say 'and' [between two sentences] can be formulated in:

Answers for Part A

1. not (a = 5/3) [The reason for the 'not' rather than a slash through the '=' is that, when beginning to learn the use of the replacement rule, students sometimes have difficulty in seeing that 'a = b' occurs in 'a ≠ b'. Perhaps a projectural with 'a = b' and an overlay with '/' would help here.]

2. $b \nless 0$ [In view of the preceding note, it will probably bear remarking that, although '$b \nless 0$' is a sentence and is part of '$-b \nless 0$', the former does not occur as a sentence in the latter. The difference is that, while '/' "operates" on sentences to give new sentences, '−' "operates" on terms.]

3. $\vec{a} = A - B$

4. $Q + \vec{t} \neq P$

5. $A + \vec{a} \neq A + \vec{b}$ and $A = B$

6. $(\vec{p} = \vec{q}$ and $P + \vec{q} = Q) \Longrightarrow P + \vec{p} = Q$

Answers for Part B

[Establishing the validity of inferences like those of the sample, Exercises 1 and 2, and Exercises 3 and 4 of Part C, is a first step toward justifying the replacement rule. Our purpose in giving these exercises is, however, to give students opportunities to practice writing derivation schemes. You might ask students to recall the justification they gave, in Exercise 7 on page 90, for the rule (Syll).]

1. $$\dfrac{\dfrac{p \Leftrightarrow q}{q \Rightarrow p}\,\text{(a)} \qquad p \Rightarrow r}{q \Rightarrow r}\,\text{(Syll)}$$

2. $$\dfrac{r \Rightarrow p \qquad \dfrac{p \Leftrightarrow q}{p \Rightarrow q}\,\text{(b)}}{r \Rightarrow q}\,\text{(Syll)}$$

Rules for Conjunction Sentences

Inferences of any of the forms:

(a) $\dfrac{p \text{ and } q}{p}$ (b) $\dfrac{p \text{ and } q}{q}$

(c) $\dfrac{p \qquad q}{p \text{ and } q}$

are valid.

Show, without using the replacement rule for biconditional sentences, that inferences of the following forms are valid.

1. $\dfrac{[q \longrightarrow p] \text{ and } [p \longrightarrow q]}{p \longleftrightarrow q}$

2. $\dfrac{p \longleftrightarrow q}{[q \longrightarrow p] \text{ and } [p \longrightarrow q]}$

3. $\dfrac{p \longleftrightarrow q \quad p \text{ and } r}{q \text{ and } r}$

4. $\dfrac{p \longleftrightarrow q \quad r \text{ and } p}{r \text{ and } q}$

5. $\dfrac{p \longrightarrow [q \longrightarrow r]}{(p \text{ and } q) \longrightarrow r}$

[This rule of logic is known as *importation.*]

6. $\dfrac{(p \text{ and } q) \longrightarrow r}{p \longrightarrow [q \longrightarrow r]}$

[This rule of logic is known as *exportation.*]

[*Hint for Exercises 5 and 6*: (1) A good technique to use when you are trying to derive a conditional sentence is to adopt the antecedent of this sentence as a premiss which you plan to discharge later by applying the deduction rule. (2) If the *consequent* of the conditional sentence is also a conditional sentence then you will be able to discharge two assumptions by two applications of the deduction rule.]

2.10 Some Theorems

In Section 2.02 we proved two theorems.

(a) $\vec{a} = B - A \Longrightarrow A + \vec{a} = B$

(b) $A + \vec{a} = B \Longrightarrow \vec{a} = B - A$

On page 85 we analyzed the proof of (a) by exhibiting a tree-form derivation which shows that (a) is a consequence of Postulate 2(a). In answering Exercise 1 of Part A on page 86 you constructed a similar derivation which shows that (b) is a consequence of Postulate 2(b). Now that we have adopted appropriate rules for biconditional sentences we can derive from (a) and (b) a biconditional sentence. Since (a) and (b) are theorems, so is the biconditional sentence.

Theorem 2-1 $A + \vec{a} = B \longleftrightarrow \vec{a} = B - A$

A translation maps A on B if and only if it is the translation from A to B.

Answers for Part C

[Note that there are two conjunctions each of which is spelled 'and'. One serves to connect nouns, the other, sentences. It is the latter we are concerned with here.]

1.
$$\dfrac{\dfrac{[q \Rightarrow p] \text{ and } [p \Rightarrow q]}{q \Rightarrow p} \qquad \dfrac{[q \Rightarrow p] \text{ and } [p \Rightarrow q]}{p \Rightarrow q}}{p \Leftrightarrow q}$$

2.
$$\dfrac{\dfrac{p \Leftrightarrow q}{q \Rightarrow p} \qquad \dfrac{p \Leftrightarrow q}{p \Rightarrow q}}{[q \Rightarrow p] \text{ and } [p \Rightarrow q]}$$

[Exercises 1 and 2 show that, by virtue of the rules we have adopted as basic for biconditional sentences and for conjunction sentences, corresponding sentences of the forms '$p \Leftrightarrow q$' and '$[q \Rightarrow p]$ and $[p \Rightarrow q]$' do "say the same thing".]

3.
$$\dfrac{\dfrac{p \Leftrightarrow q \qquad \dfrac{p \text{ and } r}{p}}{q} \qquad \dfrac{p \text{ and } r}{r}}{q \text{ and } r}$$

[The inference having '$p \Leftrightarrow q$' as a premiss has been shown to be valid in the sample for Part B.]

4.
$$\dfrac{\dfrac{r \text{ and } p}{r} \qquad \dfrac{p \Leftrightarrow q \qquad \dfrac{r \text{ and } p}{p}}{q}}{r \text{ and } q}$$

5.
$$\dfrac{\dfrac{\dfrac{p \overset{*}{\text{and}} q}{q} \qquad \dfrac{\dfrac{p \overset{*}{\text{and}} q}{p} \qquad p \Rightarrow [q \Rightarrow r]}{q \Rightarrow r}}{r}}{(p \text{ and } q) \Rightarrow r}\;*$$

6.
$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\dagger}{p} \qquad \overset{*}{q}}{p \text{ and } q} \qquad (p \text{ and } q) \Rightarrow r}{r}}{q \Rightarrow r}\;*}{p \Rightarrow [q \Rightarrow r]}\;\dagger$$

If students wish more practice like that given by Exercises 5 and 6, they may justify inferences of the forms:

$$\frac{r \Rightarrow (p \text{ and } q)}{[r \Rightarrow p] \text{ and } [r \Rightarrow q]} \qquad \frac{[r \Rightarrow p] \text{ and } [r \Rightarrow q]}{r \Rightarrow (p \text{ and } q)}$$

Here's how:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{*}{r} \qquad r \Rightarrow (p \text{ and } q)}{p \text{ and } q}}{p}}{r \Rightarrow p}{}^{*} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{*}{r} \qquad r \Rightarrow (p \text{ and } q)}{p \text{ and } q}}{q}}{r \Rightarrow q}{}^{*}}{[r \Rightarrow p] \text{ and } [r \Rightarrow q]}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{*}{r} \qquad \dfrac{[r \Rightarrow p] \text{ and } [r \Rightarrow q]}{r \Rightarrow p}}{p} \qquad \dfrac{\overset{*}{r} \qquad \dfrac{[r \Rightarrow p] \text{ and } [r \Rightarrow q]}{r \Rightarrow q}}{q}}{p \text{ and } q}}{r \Rightarrow (p \text{ and } q)}{}^{*}$$

This last derivation scheme makes the point that the same premiss may be used more than once in a derivation and all its occurrences be discharged by one application of the deduction rule. The previous scheme shows that this does not always happen.

Sample Quiz

Match each of the logical patterns given in the left column with the most appropriate name for the pattern from the right column. You may select the same name more than once.

___ 1. If p then q

___ 2. p and q

___ 3. p only if q

___ 4. p if and only if q

___ 5. $\frac{p \text{ and } q}{q}$

___ 6. $\frac{\text{If } p \text{ then } q}{\text{If } q \text{ then } p}$

___ 7. $\frac{p}{p \text{ and } q}$

___ 8. $\frac{p \text{ if and only if } q}{\text{If } q \text{ then } p}$

(a) biconditional sentence

(b) definition

(c) conditional sentence

(d) conjunction sentence

(e) antecedent

(f) valid inference

(g) invalid inference

(h) consequent

(i) modus ponens

Answers: 1. c 2. d 3. c 4. a 5. f 6. g 7. g 8. f

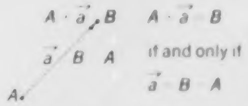


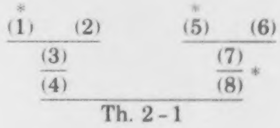
Fig. 2–5

[Sometimes the conditional sentence (a) is called *the if-part* of Theorem 2 – 1, and (b) is called *the only if-part* of Theorem 2 – 1. Do you see why?]

Exercises

Part A

A tree-form proof of Theorem 2 – 1 looks like:



Construct such a proof [by writing the appropriate sentences in the indicated positions] and, for each step, tell the rule of logic which justifies it.

Part B

1. Write a tree-form proof of the sentence:

(*) $$\vec{a} = \vec{b} \longrightarrow A + \vec{a} = A + \vec{b}$$

[*Hint*: Since you are attempting to prove a conditional sentence, you may take its antecedent as a premiss which you plan to discharge later. As other premisses you may take valid sentences, postulates, or previously proved theorems. (As a last resort, study the proof of '$a = b \longrightarrow a + c = b + c$' on page 84.)]

2. **(a)** Write the converse of the sentence (*) in Exercise 1.
   (b) If you think that the converse of (*) is a theorem, try to prove it. If not, look for a counter-example.
3. If you have proved the converse of (*) then, since (*) is a theorem, you have shown that the sentence:

(**) $$A + \vec{a} = A + \vec{b} \longleftrightarrow \vec{a} = \vec{b}$$

is a theorem. On the other hand, if you could prove (**) then you would know that the converse of (*) is a theorem. [Explain why.] You can prove (**) by deriving it from Theorem 2 – 1 and a postulate. Do so.

4. Show that Theorem 2 – 1 is a consequence of (**) and a postulate.

Part A of the exercises is — you will be glad to know — almost the last time we shall have occasion to revert to the proofs of (a), (b), and Theorem 2-1. We shall, however, have many occasions to use Theorem 2-1, via the replacement rule for biconditional sentences.

Answer for Part A

[To save space we list the sentences which should occur in the tree-form proof. The result is a column-proof of the theorem. The first column-proof in the text occurs on page 107. Now, however, is a good time for you to steal a march on the text. We urge that you do so.]

(1)	$\vec{a} = B - A$	[assumption]*
(2)	$A + (B - A) = B$	[Postulate 2(a)]
(3)	$A + \vec{a} = B$	[(1), (2)]
(4)	$\vec{a} = B - A \Longrightarrow A + \vec{a} = B$	[(3), *(1)]
(5)	$A + \vec{a} = B$	[assumption]†
(6)	$\vec{a} = (A + \vec{a}) - A$	[Postulate 2(b)]
(7)	$\vec{a} = B - A$	[(5), (6)]
(8)	$A + \vec{a} = B \Longrightarrow \vec{a} = B - A$	[(7), †(5)]
(9)	$A + \vec{a} = B \Longleftrightarrow \vec{a} = B - A$	[(4), (8)]

This sequence of nine sentences is an example of a column-proof. The numerals are for reference purposes, and are used in the column of bracketed remarks to indicate the source of some of the lines of the proof. What makes the column of sentences a proof of the sentence which is its last line is this: Each line either is a theorem or is an assumption which is discharged at some later line, or is a consequence of preceding lines. [Recall that postulates, definitions, and valid sentences are all included under the heading 'theorem'.] Since, as indicated by the remarks, the only undischarged premisses are Postulates 2(a) and 2(b) this proof shows that its final line, Theorem 2-1, is a consequence of Postulate 2.

The remark for line (3) merely shows that this line is a consequence of lines (1) and (2). The remark might be expanded to 'from (1) and (2) by the replacement rule for equations'. [In another context, the same remark might refer to some other 2-premiss kind of inference — for example, modus ponens.] The remark for line (4) might be expanded to 'from (3) by conditionalizing, thus discharging (1)'. The remark for line (9) might be expanded to 'from (4) and (8) by part (c) of the rules for biconditional sentences' — or, 'by the introduction rule for '⟺''.

While the structure of a proof is most clearly shown when the proof is exposed as a tree, trees take up a good deal of space. A proof [or, more generally, a derivation] in the form of a column occupies space more efficiently. If it is supplemented by a tree-diagram like that given in the statement of Part A, all the advantages of tree-proofs are preserved.

As proofs become more complex, column-proofs become excessively long. To some extent this can be counter-acted by adopting conventions for abbreviating such proofs by omitting easily supplied lines. It must again be emphasized, however, that students should not be required to give even abbreviated column-proofs of any but a few theorems. What they should learn is to give relatively short paragraph-proofs which show the main line of the argument and, so, could be expanded to complete proofs in column-form. Proofs of the former kind are just as "rigorous" as those of the latter. Our attention to logical detail in this and the next few chapters is not so much to teach students what they must put into a proof as to show them what they may leave out. Roughly, they may leave out what anyone who is familiar with logical reasoning could, at need, supply. Aside from this, columns and trees will continue to serve a useful purpose when one wishes to analyze a small portion of a proof concerning whose validity there may be doubts.

Answers for Part B

[To give you examples of column-proofs, we give, as answers, column-proofs and tree-diagrams.]

1. (1) $\vec{a} = \vec{b}$ [assumption]*
(2) $A + \vec{a} = A + \vec{a}$ [valid]
(3) $A + \vec{a} = A + \vec{b}$ [(1), (2)]
(4) $\vec{a} = \vec{b} \Longrightarrow A + \vec{a} = A + \vec{b}$ [(3), *(1)]

$$\cfrac{\cfrac{\overset{*}{(1)} \qquad (2)}{(3)}}{(4)}\,*$$

[In writing line (2), we assume that this sentence is easily recognized as valid. Formally, of course, it results from 'A = A' by a substitution whose legitimacy is guaranteed by Postulate 1(b). The upper right-hand corners of your student's tree-proofs may look like this:

$$\dfrac{A = A \qquad A + \vec{a} \in \mathcal{E}}{[(2)]}$$

In this connection, see the remarks following the answer, below, for Exercise 3.]

2. (a) $A + \vec{a} = A + \vec{b} \Longrightarrow \vec{a} = \vec{b}$

(b) (1) $A + \vec{a} = A + \vec{b}$ [assumption]*
(2) $\vec{a} = (A + \vec{a}) - A$ [Post. 2(b)]
(3) $\vec{a} = (A + \vec{b}) - A$ [(1), (2)]
(4) $\vec{b} = (A + \vec{b}) - A$ [(2)]
(5) $\vec{a} = \vec{b}$ [(4), (3)]
(6) $A + \vec{a} = A + \vec{b} \Longrightarrow \vec{a} = \vec{b}$ [(5), *(1)]

$$\cfrac{\cfrac{\dfrac{(2)}{(4)} \qquad \dfrac{\overset{*}{(1)} \qquad (2)}{(3)}}{(5)}}{(6)}\,*$$

[Note how, in the column-proof, line (2) does double duty. In the tree-proof this sentence must be repeated. The remark for line (4) might be expanded to 'from (2) by (Subst)'. Note that this use of (Subst) does not violate the proviso to the deduction rule because, although '$\vec{a}$' occurs in the assumption (1), this assumption is not used in deriving (4).]

3. The converse of (☆) is the only if-part of (☆☆) and, so, is a consequence of the latter. Hence, a proof of (☆☆) could easily be extended by one step to give a proof of (☆).

(1) $A + \vec{a} = B \Longleftrightarrow \vec{a} = B - A$ [Th. 2-1]
(2) $A + \vec{a} = A + \vec{b} \Longleftrightarrow \vec{a} = (A + \vec{b}) - A$ [(1) (Post. 1(b))]
(3) $\vec{a} = (A + \vec{a}) - A$ [Post. 2(b)]
(4) $\vec{b} = (A + \vec{b}) - A$ [(3)]
(5) $A + \vec{a} = A + \vec{b} \Longleftrightarrow \vec{a} = \vec{b}$ [(4), (2)]

$$\cfrac{\dfrac{(3)}{(4)} \qquad \dfrac{(1)}{(2)}}{(5)}$$

[As remarked, we shall eventually not bother to refer to Postulate 1 for the purpose of showing that (Subst)-inferences are what they claim to be. We begin this down-grading of Postulate 1 here by placing such a reference in the remark for line (2) — thus getting it out of the proof proper. The tree-diagram we give takes account of this. Your students will, however — if they follow previous instructions — present tree-diagrams whose upper right corners look like this:

$$\dfrac{A + \vec{a} = B \overset{[(1)]}{\Longleftrightarrow} \vec{a} = B - A \qquad \dfrac{A + \vec{a} \in \mathcal{E}}{A + \vec{b} \in \mathcal{E}}\,\text{(Subst)}}{[(2)]}$$

Whether they do or not, point out that the temporary instructions to include the indicated (Subst)-inferences are still in effect, but that failure to observe them is not criminal — and is perhaps even commendable!

One of the conventions which might be adopted for abbreviating column-proofs would allow the omission of line (3), and giving 'Post. 2(b)' as the comment for line (4).]

4. (1) $A + \vec{a} = A + \vec{b} \Longleftrightarrow \vec{a} = \vec{b}$ [(☆☆)]
(2) $A + \vec{a} = A + (B - A) \Longleftrightarrow \vec{a} = B - A$ [(1) (Post. 1(a))]
(3) $A + (B - A) = B$ [Post. 2(a)]
(4) $A + \vec{a} = B \Longleftrightarrow \vec{a} = B - A$ [(3), (2)]

$$\cfrac{(3) \qquad \dfrac{(1)}{(2)}}{(4)}$$

*

Do not fall into the trap of interpreting Theorem 2-2 as though it were the weaker statement:

$$\forall_X\ X + \vec{a} = X + \vec{b} \Longleftrightarrow \vec{a} = \vec{b}$$

The if-parts of this statement and of Theorem 2-2 do "say the same thing". The only if-part of this statement is true merely because functions — in this case, translations — which have the same arguments and have the same values for the same arguments are the same function. The only if-part of Theorem 2-2 asserts that translations which have the same value for a single argument are the same translation. This matter need not be brought up at this time; it is discussed in a later chapter.

*

Since your students' aim should be that of learning to write paragraph proofs, the exercises of Part B may well be discussed to show how columns aid in writing paragraphs. Comparing the answers given above for Exercises 1 - 4 with the following paragraphs may be helpful.

Exercise 1:

Suppose that $\vec{a} = \vec{b}$. Since $A + \vec{a} = A + \vec{a}$ it follows that $A + \vec{a} = A + \vec{b}$. Hence, if $\vec{a} = \vec{b}$ then $A + \vec{a} = A + \vec{b}$.

*

Theorem 2-2 $A + \vec{a} = A + \vec{b} \Longleftrightarrow \vec{a} = \vec{b}$

A translation determines the images, under it, of all points and is determined by the image of any given point.

Part C

1. Prove:

Theorem 2-3 $A - C = B - C \Longleftrightarrow A = B$

[*Hint*: First, prove '$A = B \Longrightarrow A - C = B - C$', then, prove its converse.]

2. If you have given a tree-form proof of Theorem 2-3, translate your proof into words. [*Suggestion*: Write three short paragraphs:

Suppose that $A = B$. Since . . .

. . . Hence, if $A = B$ then $A - C = B - C$.

Suppose that $A - C = B - C$. Since . . .

. . . Hence, if $A - C = B - C$ then $A = B$.

Since if $A = B$ then $A - C = B - C$ and since . . . it follows that . . .

Part D

1. Earlier we showed that Postulate 2(a) is a consequence of the if-part of Theorem 2-1—that is, of sentence (a) on page 101. Write a paragraph to show that this is the case. [*Suggestion*. Start out: Since [(a)] if $\vec{a} = B - A$ then $A + \vec{a} = B$ it follows [using Postulate 1(a)] that if . . .]
2. Write a paragraph to show that another of our postulates is a consequence of the only if-part of Theorem 2-1.
3. What single sentence might we have used as a postulate in place of Postulate 2(a) and Postulate 2(b)?

Part E

1. Write a paragraph-proof of the sentence:

$$A = B \Longrightarrow A + \vec{a} = B + \vec{a}$$

2. (a) Do you think that the sentence:

$$A + \vec{a} = B + \vec{a} \Longrightarrow A = B$$

is true?

(b) What does the sentence of part (a) say about any translation?

(c) Do you think that the sentence of part (a) is [now] a theorem—that is, that it is a consequence of Postulates 1 and 2?

Exercise 2(b):

Suppose that $A + \vec{a} = A + \vec{b}$. Since, by Postulate 2(b), $\vec{a} = (A + \vec{a}) - A$ it follows that $\vec{a} = (A + \vec{b}) - A$. Since, again by Postulate 2(b), $\vec{b} = (A + \vec{b}) - A$ it follows that $\vec{a} = \vec{b}$. Hence, if $A + \vec{a} = A + \vec{b}$ then $\vec{a} = \vec{b}$.

Exercise 3:

Since [Th. 2-1] $A + \vec{a} = B$ if and only if $\vec{a} = B - A$ it follows that $A + \vec{a} = A + \vec{b}$ if and only if $\vec{a} = (A + \vec{b}) - A$. Since, by Postulate 2(b), $\vec{b} = (A + \vec{b}) - A$ it follows that $A + \vec{a} = A + \vec{b}$ if and only if $\vec{a} = \vec{b}$.

Exercise 4:

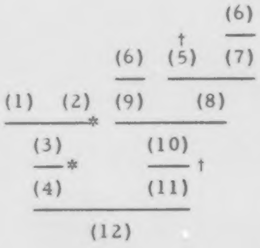
Since [(☆☆)] $A + \vec{a} = A + \vec{b}$ if and only if $\vec{a} = \vec{b}$ it follows that $A + \vec{a} = A + (B - A)$ if and only if $\vec{a} = B - A$. Since, by Postulate 2(a), $A + (B - A) = B$ it follows that $A + \vec{a} = B$ if and only if $\vec{a} = B - A$.

TC 103 (1)

Answers for Part C

1.

(1)	$A = B$	[assumption]*
(2)	$A - C = A - C$	[valid]
(3)	$A - C = B - C$	[(1), (2)]
(4)	$A = B \Longrightarrow A - C = B - C$	[(3), *(1)]
(5)	$A - C = B - C$	[assumption]†
(6)	$A + (B - A) = B$	[Post. 2(a)]
(7)	$C + (B - C) = B$	[(6)]
(8)	$C + (A - C) = B$	[(5), (7)]
(9)	$C + (A - C) = A$	[(6)]
(10)	$A = B$	[(9), (8)]
(11)	$A - C = B - C \Longrightarrow A = B$	[(10, †(5)]
(12)	$A - C = B - C \Longleftrightarrow A = B$	[(4), (11)]



2. Suppose that $A = B$. Since $A - C = A - C$ it follows that $A - C = B - C$. Hence, if $A = B$ then $A - C = B - C$.

Suppose that $A - C = B - C$. Since, by Postulate 2(a), $C + (B - C) = B$ it follows that $C + (A - C) = B$. Since, again by Postulate 2(a), $C + (A - C) = A$ it follows that $A = B$. Hence, if $A - C = B - C$ then $A = B$.

Since if $A = B$ then $A - C = B - C$ and since if $A - C = B - C$ then $A = B$ it follows that $A - C = B - C$ if and only if $A = B$.

[Note the use of three paragraphs to separate out clearly the parts of the argument.]

Answers for Part D

1. Since [(a)] if $\vec{a} = B - A$ then $A + \vec{a} = B$ it follows [using Post. 1(a)] that if $B - A = B - A$ then $A + (B - A) = B$. Hence, since $B - A = B - A$, $A + (B - A) = B$.

[Obviously, the wording of your students' paragraph proofs is likely to differ from that given in the answers. What they should all say is that an instance of (a) has a valid antecedent and has Postulate 2(a) as its consequent. So [by modus ponens], Postulate 2(a) is a consequence of (a). How they choose to say this is a matter for individual initiative. Style and grammar will follow from practice and from the observation of suitable examples.

In discussing the answers you may find it worthwhile to translate them into a column-proof. This is easy. The tree-diagram for such a proof is that of Exercise 4 of Part B. If it includes deriving '$B - A = B - A$' from '$\vec{a} = \vec{a}$', the diagram will be that of Exercise 3.]

2. Since if $A + \vec{a} = B$ then $\vec{a} = B - A$ it follows that if $A + \vec{a} = A + \vec{a}$ then $\vec{a} = (A + \vec{a}) - A$. Hence, since $A + \vec{a} = A + \vec{a}$, $\vec{a} = (A + \vec{a}) - A$.

3. Theorem 2-1.

[Theorem 2-1 might be used in place of both parts of Postulate 2. Another possibility is to use Theorem 2-2 — or merely its only if-part as a replacement for Postulate 2(b). Challenge students to show that the latter is a consequence of Postulate 2(a) and the only if-part of Theorem 2-2:

(1) $A + \vec{a} = A + \vec{b} \Longrightarrow \vec{a} = \vec{b}$ [Th. 2-2]

(2) $A + \vec{a} = A + [(A + \vec{a}) - A] \Longrightarrow \vec{a} = (A + \vec{a}) - A$ [(1)]

(3) $A + (B - A) = B$ [Post. 2(a)]

(4) $A + [(A + \vec{a}) - A] = A + \vec{a}$ [(3)]

(5) $A + \vec{a} = A + \vec{a} \Longrightarrow \vec{a} = (A + \vec{a}) - A$ [(4), (2)]

(6) $A + \vec{a} = A + \vec{a}$ [valid]

(7) $\vec{a} = (A + \vec{a}) - A$ [(6), (5)]

Similarly, the only if-part of Theorem 2-3 might be used as a replacement for Postulate 2(a). However, Theorems 2-2 and 2-3 may not be used together to replace both parts of Postulate 2. With each theorem one needs one part of the postulate if one is to infer the other part.]

Answers for Part E

1. Suppose that $A = B$. Since $A + \vec{a} = A + \vec{a}$ it follows that $A + \vec{a} = B + \vec{a}$. Hence, if $A = B$ then $A + \vec{a} = B + \vec{a}$.

2. (a) Yes. [This sentence has been discussed earlier. See the discussion of Exercise 1, Part B on TC 94(1) - (2).]

   (b) Any translation is one-to-one [or: has an inverse].

   (c) No. [See discussion referred to for part (a).]

   [The sentence '$C - A = C - B \Longrightarrow A = B$' is another example of a true sentence which is not yet a theorem.]

2.11 The Bypass Postulate

Because we decided to think of the image of a point under a translation as the result of "adding" the translation to the point, Postulates 1 and 2 tell us that a translation is a mapping of $\mathcal{E}$ into itself and that, given a point A and a point B, there is one and only one translation – the translation $B - A$ from A to B – which maps A on B. In the preceding sections you have found different ways of saying some of these things which Postulates 1 and 2 tell us. [This is not very important. The important things you have learned are various rules of logic and how to use them in deducing the consequences of given premisses.]

In Chapter 1 you learned much more about translations than is formulated in our two postulates. [Part of what you learned is summarized on pages 47 and 48.] For example, you learned that a translation is a one-to-one mapping. As you probably decided when working Exercise 2 of Part E on page 103, this fact about translations is not part of the content of Postulates 1 and 2. Since we wish all true statements about translations, and about how they operate on points, to be consequences of our postulates, we might adopt the sentence:

$$(1) \qquad A + \vec{a} = B + \vec{a} \Longrightarrow A = B$$

as a third postulate. Before doing so, however, let's recall some of the other facts we know about translations. In addition to knowing that each translation has an inverse, we know that the inverse of any translation is a translation. More specifically, we know that the inverse of the translation from A to B is the translation from B to A. Recalling a theorem on inverses from Section 1.04, this amounts to saying that

(i) $(A - B) \circ (B - A) = i_{\mathcal{E}}$ and (ii) $(B - A) \circ (A - B) = i_{\mathcal{E}}$.
[See page 26.]

Since either of (i) and (ii) implies the other, we might adopt (i), say, as our third postulate rather than (1). Since we decided earlier to write, for example, '$\vec{a} + \vec{b}$' instead of '$\vec{b} \circ \vec{a}$', and '$\vec{0}$' instead of '$i_{\mathcal{E}}$', our third postulate would be:

$$(2) \qquad (B - A) + (A - B) = \vec{0}$$

There is still more basic fact about translations than that which is expressed by (2). It is that the set $\mathcal{T}$ of translations is closed under function composition. The resultant of a translation $\vec{a}$ followed by a translation $\vec{b}$ is a translation.

The text of section 2.11 is an attempt to illustrate the sort of considerations which guide one in choosing postulates. Something like this has been done previously in section 1.07; but, there, the emphasis was mostly on choosing a potentially helpful notation. This earlier work might, nevertheless, be recalled here to point out that, while Postulates 1 and 2 may be thought of as explaining the use of '-' and one of the two uses of '+' which were decided on in section 1.07, we have no postulate which so explains the second usage of '+'. We are certainly in need of a postulate which, in some way, formulates our decision to use '+' in such a way that $\vec{a} + \vec{b}$ is to the resultant of $\vec{a}$ followed by $\vec{b}$. This would suggest, directly, the adoption of Theorem 2-5 on page 109 as our third postulate. The sentence which we do [on page 105] adopt as our third postulate is another way of saying what Theorem 2-5 does.

Students will probably recognize Postulate 3 as the sentence used as a premiss in Parts C and D on pages 91 - 92.

In particular, for any points A, B, and C, $(B - A) + (C - B)$ is the resultant of the translation $B - A$ followed by the translation $C - B$.

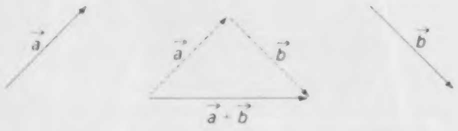

$\vec{a}$
$\vec{a}$
$\vec{b}$
$\vec{a} + \vec{b}$
$\vec{b}$

Fig. 2–6

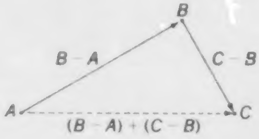

B
B − A
C − B
A
C
(B − A) + (C − B)

Fig. 2–7

Since this resultant is a translation and since it maps A on C it follows that it is the translation $C - A$.

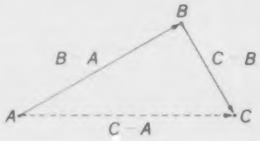


Fig. 2–8

So, another possible choice for our third postulate is:

$$(B - A) + (C - B) = C - A \tag{3}$$

This turns out to be the most satisfactory choice, and it is the one we shall adopt.

> **Postulate 3** [The Bypass Postulate.]
>
> $$(B - A) + (C - B) = C - A$$
>
> The translation which maps A on B followed by the translation which maps B on C is the translation which maps A on C.

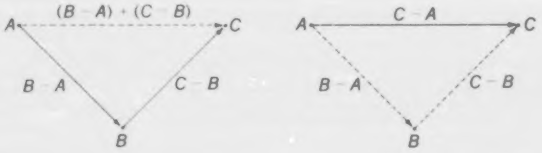

(B − A) + (C − B)
A
C
B − A
C − B
B
C − A
A
C
B − A
C − B
B

Fig. 2–9

Exercises

Part A

Complete each of the following sentences and draw a figure which illustrates it.

1. $(A - P) + (B - A) =$ ________
2. $[(A - P) + (B - A)] + (P - B) =$ ________
3. $[(B - A) + (D - B)] + (C - D) =$ ________

Part B

1. (a) Use variables 'a', 'b', and 'c' to write a sentence about real numbers which is analogous to Postulate 3 on page 105.
   (b) Is the sentence you wrote in part (a) true?
2. Here is a different sentence about real numbers:

$$(b + c) - (a + b) = c - a$$

What do you notice when you try to write an analogous sentence about points? [Exercises 1 and 2 illustrate an important fact. A true sentence about points often "translates" into a true sentence about real numbers when point-variables are replaced by real number-variables; but a true sentence about addition and subtraction of real numbers may become nonsense when real number-variables are replaced by point-variables.]

Part C

1. (a) Mark three noncollinear points P, Q, and R. Draw arrows to describe the translations $P - Q$ and $R - P$. On the same picture, draw an arrow to describe $R - Q$. Your picture illustrates an instance of Postulate 3. What instance?
   (b) Repeat part (a) for three collinear points P, Q, and R.
2. Mark three points P, Q, and R. Draw arrows to describe $P - R$ and $Q - R$. Complete the following:
   (a) In order to illustrate the instance:

$$(Q - R) + (\_\_\_) = P - R$$

of Postulate 3 we must draw an arrow to describe the translation ____.

   (b) In order to illustrate the instance:

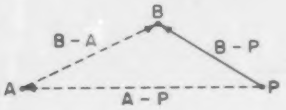
$$(P - R) + (\_\_\_\_\_) = Q - R$$

we must draw an arrow to describe the translation ____.

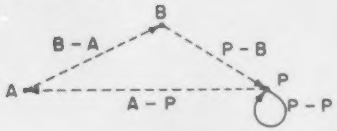
3. Complete each of the following to obtain an instance of Postulate 3:
   (a) $(R - P) + (\_\_\_ - \_\_\_) =$ ________
   (b) $(\_\_\_ - \_\_\_) + (R - P) =$ ________
   (c) $(\_\_\_ - \_\_\_) + (\_\_\_ - \_\_\_) = R - P$
   (d) $(\_\_\_ - \_\_\_) + (\_\_\_ - \_\_\_) = (R + \vec{r}) - (P + \vec{p})$

Answers for Part A

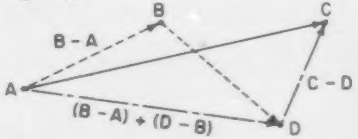
1. B - P



2. P - P [or: $\vec{0}$]



3. C - A

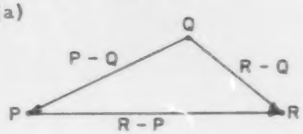


Answers for Part B

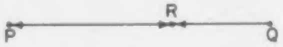
1. (a) $(b - a) + (c - b) = c - a$ (b) Yes
2. Substituting the corresponding capital letters for lower case letters yields nonsense. [The point remarked on in the text is made very precise in section 3.07.]

Answers for Part C

1. (a)



(b)



$(P - Q) + (R - P) = R - Q$

2. [Figure like that in Exercise 1.]
   (a) P - Q [twice] (b) Q - P [twice]
3. (a) A, R, A - P [Any capital letter — for that matter, any point-term — will do as well as 'A'. For example, '$(R - P) + ((P + \vec{a}) - R) = (P + \vec{a}) - P$' is one of the instances of (3) which students might suggest. If you promote a bit of competition in giving answers, your students will gain practice in constructing point-terms — and in making sure that they are such — as well as becoming acquainted with the pattern of instances of (3).]
   (b) P, A, R - A [Again, any point-term may be used in place of 'A'.]
   (c) A, P, R, A (d) A, $(P + \vec{p})$, $(R + \vec{r})$, A

Part D

1. Using the notation of our algebra of points and translations, write a sentence which says that, for any translations $\vec{a}$ and $\vec{b}$, the image of a point A under $\vec{a} + \vec{b}$ is the same as its image under $\vec{a}$ followed by $\vec{b}$.
2. Write a sentence which says that $\mathcal{T}$ is closed under function composition.
3. Do you think that, since we have adopted Postulate 3, your answers for Exercises 1 and 2 are theorems?

2.12 More Theorems

Your answer for Exercise 3 of Part D should have been 'Yes.', for you have already shown while doing Parts C and D on pages 91–92 that Postulates 1–3 imply the two sentences in question. We shall show this again – mostly to illustrate a third way of writing proofs. We begin by guessing at another theorem.

Consider the two following figures:

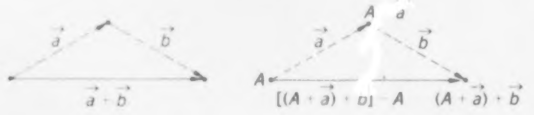


Fig. 2–10

Notice that the '+' in the left-hand figure has a different meaning than do the '+'s in the right-hand figure. The left-hand figure illustrates our convention of using '+' to refer to function composition. The right-hand figure illustrates our other convention of using '+' to refer to function application. The two figures suggest that the sentence:

$$(*) \qquad \vec{a} + \vec{b} = [(A + \vec{a}) + \vec{b}] - A$$

is true. As a matter of fact, this equation, with its '+' of function composition on the left side and its '–' on the right side, looks very much like an instance of Postulate 3. So, to show that (*) is [not only true, but] a theorem, let's try to obtain it by making substitutions in this postulate.

(1) $(B - A) + (C - B) = C - A$ [Postulate 3]
(2) $(B - A) + [(B + \vec{b}) - B]$ [from (1); '$B + \vec{b}$' for 'C']
$= (B + \vec{b}) - A$
(3) $\vec{b} = (B + \vec{b}) - B$ [from Postulate 2(b)]
(4) $(B - A) + \vec{b} = (B + \vec{b}) - A$ [from (3) and (2)]
(5) $[(A + \vec{a}) - A] + \vec{b}$ [from (4); '$A + \vec{a}$' for 'B']
$= [(A + \vec{a}) + \vec{b}] - A$

Answers for Part D

1. $A + (\vec{a} + \vec{b}) = (A + \vec{a}) + \vec{b}$ [Compare with: $[\vec{b} \circ \vec{a}](A) = \vec{b}(\vec{a}(A))$.]
2. $\vec{a} + \vec{b} \in \mathcal{T}$
3. Yes. [Recall Parts C and D on pages 117 - 119.]

[In discussing Exercise 1 make certain that students realize that the second '+' on the left side of the equation has an entirely different interpretation than do the other three '+'s. In particular, although our notation has been chosen as it has in order to bring out formal analogies with the algebra of real numbers, the equation in question does not assert the associativity of any operation.]

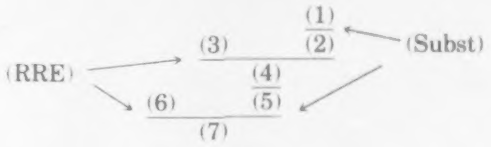
The "third way of writing proofs" refers to column-proofs. Your students have, we hope, made acquaintance with these in your discussion of the exercises in section 2.10.

(6) $\overrightarrow{a} = (A + \overrightarrow{a}) - A$ [Postulate 2(b)]
(7) $\overrightarrow{a} + \overrightarrow{b} = [(A + \overrightarrow{a}) + \overrightarrow{b}] - A$ [from (6) and (5)]

The preceding is an example of what we shall call a *column-proof.* Such a proof is a sequence of sentences some of which are postulates, previously proved theorems, or valid sentences [or assumptions, if we plan to use the deduction rule]. Each of the other sentences must be a consequence of sentences which precede it. [Actually, the sequence (1)–(7) does not quite satisfy this requirement unless we agree that sentence (3) is a "previously proved theorem". We could fix this up by writing line (6) between lines (2) and (3), but we shall often allow column-proofs to contain sentences which like (3) are obviously substitution-instances of postulates, previously proved theorems, or valid sentences.]

The numerals to the left of the sentences of a column-proof are merely for reference purposes. The bracketed comments to the right are intended to explain what is going on—they are not, strictly speaking, part of the proof.

Column-proofs are somewhat more explicit than are paragraph-proofs, and take up less space than tree-proofs. Sometimes it is helpful to supplement a column-proof by giving a *tree-diagram.* This is like a tree-proof, with the reference numerals in place of the corresponding sentences.



[If, as suggested two paragraphs back, we wished to show that (3) is a consequence of (6), we would show another substitution-inference by writing '(6)' above the '(3)' in the tree-diagram.]

Theorem 2–4 $\overrightarrow{a} + \overrightarrow{b} = [(A + \overrightarrow{a}) + \overrightarrow{b}] - A$

A paragraph-proof of Theorem 2–4 might go as follows:

By Postulate 3, $(B - A) + [(B + \overrightarrow{b}) - B] = (B + \overrightarrow{b}) - A$ and so, by Postulate 2(b), $(B - A) + \overrightarrow{b} = (B + \overrightarrow{b}) - A$. From this it follows that $[(A + \overrightarrow{a}) - A] + \overrightarrow{b} = [(A + \overrightarrow{a}) + \overrightarrow{b}] - A$. So, by Postulate 2(b), $\overrightarrow{a} + \overrightarrow{b} = [(A + \overrightarrow{a}) + \overrightarrow{b}] - A$.

Had we chosen Theorem 2-4, rather than the bypass postulate as our third postulate, this choice would have led to precisely the same theorems. This fact is likely to suggest that we might merely have defined addition of translations by:

(☆) $\vec{a} + \vec{b} = [(A + \vec{a}) + \vec{b}] - A$

and that, as a consequence, our third postulate might be dispensed with. This is incorrect and, for a proper understanding of definitions, it is important to see why. To this end, suppose that (☆) were accepted as a definition. What this means is that, whatever instance of (☆) we may choose, the left side of this instance is to be used, at will, as an abbreviation for the right side. So, for example, both of the sentences:

(1) $[(A + \vec{a}) + \vec{b}] - A = [(A + \vec{a}) + \vec{b}] - A$

and:

(2) $[(B + \vec{a}) + \vec{b}] - B = [(A + \vec{a}) + \vec{b}] - A$

might be abbreviated to:

(3) $\vec{a} + \vec{b} = [(A + \vec{a}) + \vec{b}] - A$

This being the case, one who is presented with sentence (3) has no way of knowing whether he should consider it as an abbreviation for the valid sentence (1) or for the true but nonvalid sentence (2). An abbreviation — such as we are assuming here that '$\vec{a} + \vec{b}$' is intended to be — must be such that there is never any doubt as to what it is that it abbreviates. In particular, an equation which, like (☆), has a variable in its right side which does not occur in its left side may not be used as a definition.

The preceding objection to using (☆) as a definition may also be put in another way. Although the adoption of a definition does make it possible to prove additional theorems — namely, theorems which contain the defined expression — it should not make it possible to prove any "really new theorems". Any theorems obtainable by using the definition should be merely abbreviations of theorems which could be proved without it. Now, (☆), an instance of (☆) ['B' for 'A'], and the valid sentence (1) together imply (2). So, if (☆) is adopted — either as a definition or as a less specific kind of postulate — (3) becomes a theorem. With it, by Postulate 2(a), the sentence:

$$(B + \vec{a}) + \vec{b} = B + ([(A + \vec{a}) + \vec{b}] - A)$$

becomes a theorem. This sentence tells us that the image of any point B under the mapping which is the resultant of $\vec{a}$ followed by $\vec{b}$ is the same as the image of B under a certain translation. That is, it tells us that the resultant of $\vec{a}$ followed by $\vec{b}$ is a translation. Now, while this is true enough, it does not follow from our Postulates 1 and 2. So, the adoption of (☆) has made it possible to prove something new. Hence, (☆) accomplishes more than a definition is allowed to.

Exercises

Part A

1. Write a column to show that Postulate 3 is a consequence of Theorem 2-4 and part of Postulate 2.
2. Make a tree-diagram of the argument you gave in answer to Exercise 1.
3. Write a paragraph which serves the same purpose as your answer for Exercise 1.

Part B

1. Prove:

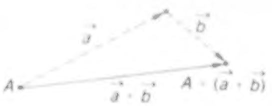
> Theorem 2-5
>
> (a) $\vec{a} + \vec{b} \in \mathcal{T}$
> (b) $A + (\vec{a} + \vec{b}) = (A + \vec{a}) + \vec{b}$

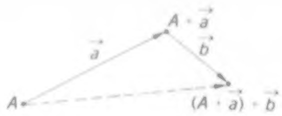
[*Hint*: You may use Theorem 2-4, as well as still earlier theorems. If you prove Theorem 2-5(a) first, you may use it in proving Theorem 2-5(b).]

2. By using Theorem 2-1, show that if, in place of Postulate 3, we had adopted both parts of Theorem 2-5 as postulates then Postulate 3 would still have been a theorem.

Part C

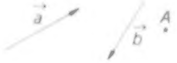
Theorem 2-5 might have been suggested by these figures:





1. Which '+'s refer to function composition? Which to function application?
2. In each part of this exercise you are given two arrows and a dot. Copy these and make a drawing like the left-hand figure above.

(a)



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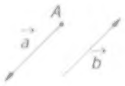
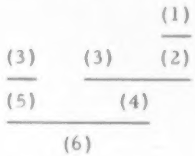


The purpose of the exercises in Part A is to show an alternate way of organizing the properties we have about points and translations. Again, the exercises should be used for practice on derivation. It is not crucial to the course to be able to derive, say, Postulate 3 from Theorem 2-4 and Postulate 2. This is only an interesting observation to most students at this point. If your students are progressing well with derivation, you may wish to omit Part A. Exercise 2 of Part B is in this category also.

Answers for Part A

1. (1) $\vec{a} + \vec{b} = [(A + \vec{a}) + \vec{b}] - A$ [Th. 2-4]
(2) $(B - A) + (C - B) = [(A + (B - A)) + (C - B)] - A$ [(1)]
(3) $A + (B - A) = B$ [Post. 2(a)]
(4) $(B - A) + (C - B) = [B + (C - B)] - A$ [(3), (2)]
(5) $B + (C - B) = C$ [(3)]
(6) $(B - A) + (C - B) = C - A$ [(5), (4)]

2.



3. Since $\vec{a} + \vec{b} = [(A + \vec{a}) + \vec{b}] - A$ it follows that $(B - A) + (C - B) = [(A + (B - A)) + (C - B)] - A$. Since, by Postulate 2(a), $A + (B - A) = B$ it follows that $(B - A) + (C - B) = [B + (C - B)] - A$. Since, again by Postulate 2(a), $B + (C - B) = C$ it follows that $(B - A) + (C - B) = C - A$.

Answers for Part B

1. [Proof of part (a)] By Postulate 1(b), $(A + \vec{a}) + \vec{b} \in \mathcal{E}$. So, by Postulate 1(a), $[(A + \vec{a}) + \vec{b}] - A \in \mathcal{T}$. Hence, by Theorem 2-4, $\vec{a} + \vec{b} \in \mathcal{T}$. [The 'Hence', refers to an application of the replacement rule for equations.]

[Proof of part (b)] Since $\vec{a} + \vec{b} \in \mathcal{T}$, $A + (\vec{a} + \vec{b}) = A + (\vec{a} + \vec{b})$. It follows by Theorem 2-4 that $A + (\vec{a} + \vec{b}) = A + ([(A + \vec{a}) + \vec{b}] - A)$. So, by Postulate 2(a), $A + (\vec{a} + \vec{b}) = (A + \vec{a}) + \vec{b}$.

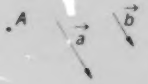
[Alternatively, Theorem 2-5(b) can be derived by modus ponens from Theorem 2-4 and an instance of the if-part of Theorem 2-1. The instance required is:

$$\vec{a} + \vec{b} = [(A + \vec{a}) + \vec{b}] - A \Longrightarrow A + (\vec{a} + \vec{b}) = (A + \vec{a}) + \vec{b}$$

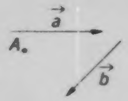
The writing of sentences as long as this is sometimes tiresome, and the multiplicity of grouping symbols which may occur furnishes opportunities to blunder. These difficulties may be reduced by a procedure which amounts to introducing abbreviations for complex terms. We illustrate the procedure and then discuss it.

> Suppose that $A + (\vec{a} + \vec{b}) = B$. Since $\vec{a} + \vec{b} \in \mathcal{T}$ it follows by Theorem 2-1 that $\vec{a} + \vec{b} = B - A$. So, by Theorem 2-4, $B - A = [(A + \vec{a}) + \vec{b}] - A$ and, by Theorem 2-3, $B = (A + \vec{a}) + \vec{b}$. Since, by assumption, $A + (\vec{a} + \vec{b}) = B$ it follows that $A + (\vec{a} + \vec{b}) = (A + \vec{a}) + \vec{b}$.

(c)



(d)



☆ Part D

On page 104 we considered the sentence:

$$(1) \qquad A + \vec{a} = B + \vec{a} \longrightarrow A = B$$

This sentence is true because it says that any translation is one-to-one. Although it is true, sentence (1) is not yet a theorem. [It will become a theorem when, in the next chapter, we adopt another postulate.] We can show that (1) is not a consequence of our three postulates by giving a different interpretation to our language. With this new interpretation our postulates will again be true but (1) will be false. Since consequences of true sentences are true, this will show that (1) is not a consequence of our postulates.

For this new interpretation we shall continue to think of $\mathscr{E}$ as the set of all points and of $\mathscr{T}$ as a set of mappings of $\mathscr{E}$ into itself. Instead, however, of the members of $\mathscr{T}$ being translations, now the members of $\mathscr{T}$ are to be *constant mappings*. [A constant mapping is one which maps each point of $\mathscr{E}$ on some one point.] You may think of a constant mapping as one which "shrinks" all of $\mathscr{E}$ to a single point. Evidently, a constant mapping is as far as you can get from being one-to-one.

The only other change we shall make in our interpretation is to take $B - A$ to be the constant mapping which maps A on B. $A + \vec{a}$ is still the image of the point A under the mapping $\vec{a}$, and $\vec{a} + \vec{b}$ is the resultant of the mapping $\vec{a}$ followed by the mapping $\vec{b}$. So, as before, $B - A \in \mathscr{T}$ and $A + \vec{a} \in \mathscr{E}$.

With this *new interpretation* in mind, do the following exercises.

1. $C + (B - A) = ?$ [*Hint*: What is the image of C under the *constant* mapping which maps A on B?]
2. $\vec{a} + \vec{b} = ?$ [*Hint:* Suppose that $\vec{a}$ is the constant mapping which maps each point of $\mathscr{E}$ on the point A and that $\vec{b}$ maps each point of $\mathscr{E}$ on the point B. What is the image of a point C under the resultant of $\vec{a}$ followed by $\vec{b}$?]
3. Are both parts of Postulate 2 true?
4. Is Postulate 3 true? [*Hint*: See Exercise 2.]
5. Is the sentence (1) true?

(Instead of appealing to Theorem 2-3 we might, of course, have made use of Postulate 2(a).) The preceding argument shows that Theorem 2-5(b) is a consequence of several theorems together with an assumption. One's intuitive feelings are that, as far as the conclusion is concerned, the assumption may be discharged. This feeling changes to absolute certainty on noting that substituting '$A + (\vec{a} + \vec{b})$' for 'B' throughout the argument leaves the conclusion unaffected and turns the assumption into a valid sentence. Of course, actually carrying out the substitution would lose us the gains obtained by using 'B' as an abbreviation for '$A + (\vec{a} + \vec{b})$'. What we may do, however, is to continue the argument by discharging the assumption, making the substitution in the resulting conclusion, and applying modus ponens:

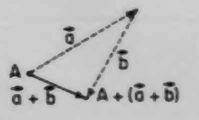
> Hence, if $A + (\vec{a} + \vec{b}) = B$ then $A + (\vec{a} + \vec{b}) = (A + \vec{a}) + \vec{b}$. In particular, if $A + (\vec{a} + \vec{b}) = A + (\vec{a} + \vec{b})$ then $A + (\vec{a} + \vec{b}) = (A + \vec{a}) + \vec{b}$. So, since $A + (\vec{a} + \vec{b}) = A + (\vec{a} + \vec{b})$, it follows that $A + (\vec{a} + \vec{b}) = (A + \vec{a}) + \vec{b}$.

This concluding paragraph is, essentially, a rubber stamp affair. So, there is really no need to write it down. Abbreviating assumptions like our '$A + (\vec{a} + \vec{b}) = B$' may be ignored once one has arrived at a conclusion in which the variable introduced as an abbreviation does not occur. For, then, the rubber stamp can be used to "complete" the proof.]

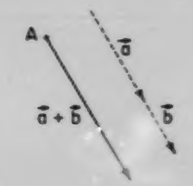
2. By Theorem 2-1 it follows from Theorem 2-5(a) that $A + (\vec{a} + \vec{b}) = (A + \vec{a}) + \vec{b}$ if and only if $\vec{a} + \vec{b} = [(A + \vec{a}) + \vec{b}] - A$. So, by Theorem 2-5(b), $\vec{a} + \vec{b} = [(A + \vec{a}) + \vec{b}] - A$. Hence, as in Exercise 1 of Part A, $(B - A) + (C - B) = C - A$.

Answers for Part C

1. The first and the last of the three '+'s in the left-hand figure refer to function composition; all others refer to function application.
2. (a)



(b)



2.13 Chapter Summary

Vocabulary Summary

- postulate
- translation
- valid sentence
- conditional sentence
- consequent
- converse of a conditional
- biconditional sentence
- constant mapping
- theorem
- mapping
- equation
- antecedent
- valid derivation
- counter-example
- conjunction sentence
- linear function

Postulates

1. (a) $B - A \in \mathscr{T}$ **(b)** $A + \vec{a} \in \mathscr{E}$

2. (a) $A + (B - A) = B$ **(b)** $\vec{a} = (A + \vec{a}) - A$

3. $(B - A) + (C - B) = C - A$

Other Theorems

2–1. $A + \vec{a} = B \Longleftrightarrow \vec{a} = B - A$

2–2. $A + \vec{a} = A + \vec{b} \Longleftrightarrow \vec{a} = \vec{b}$

2–3. $A - C = B - C \Longleftrightarrow A = B$

2–4. $\vec{a} + \vec{b} = [(A + \vec{a}) + \vec{b}] - A$

2–5. (a) $\vec{a} + \vec{b} \in \mathscr{T}$ **(b)** $A + (\vec{a} + \vec{b}) = (A + \vec{a}) + \vec{b}$

Basic Rules of Logic

Dealing with variables

Substitution Rule [See page 69 and page 87, following (*).]
Any sentence which is used to make an assertion about all values of some variable implies each of its substitution-instances with respect to this variable.

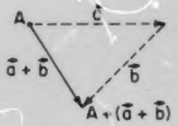
Dealing with equations

Replacement Rule for Equations [See page 74.]
Given an equation and a second sentence, if either side of the equation is replaced, somewhere in the second sentence, by the other side, the resulting sentence is a consequence of the given equation and sentence.

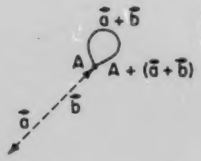
Introduction Rule for Equations [See page 75.]
The equations '$A = A$', '$\vec{a} = \vec{a}$', and '$a = a$' are valid sentences.

Answers for Part C [cont.]

2. (c)



(d)



Answers for Part D

[These optional exercises illustrate a procedure which can sometimes be used to show that a given sentence is not a theorem. The procedure is based on the fact that, since consequences of true sentences are true, a false sentence cannot be a theorem in a system based on true postulates. The power of the procedure depends on the fact that the rules of logic — including the rules of sentence structure — are purely formal, so that whether or not a sentence is true depends on what meanings are assigned to the mathematical symbols; but whether or not a sentence is a consequence of others is completely independent of these meanings. Since we know that our postulates are true when our symbols have the meanings we have chosen to express by them, we know that no sentence which is false "under this interpretation of our symbolism" can be a theorem. The test is to find a new interpretation of the symbolism under which the postulates are, again, true. If the sentence in question turns out to be false under this interpretation then it cannot be a theorem. If it turns out to be true, we have learned nothing.

Whether or not the procedure just described works (in the case of a sentence which is actually not a theorem) depends on how clever — or lucky — we are in finding new interpretations. The more postulates there are whose truth has to be maintained, the less likely we are to succeed.]

1. Since B - A is the constant mapping which maps A on B, B - A maps each point on B. So, C + (B - A) = B.
2. $\vec{a} + \vec{b} = \vec{b}$. [It is the nature of a constant mapping to "absorb" any other.]
3. Yes. [Postulate 2(a) follows from the result obtained in Exercise 1. As to Postulate 2(b), $(A + \vec{a}) - A$ is the constant mapping which maps each point on $A + \vec{a}$. Since this is what $\vec{a}$ does, $\vec{a} = (A + \vec{a}) - A$.]
4. Yes. [By Exercise 2, (B - A) + (C - B) = C - A. But, C - B and C - A are, both of them, the constant mapping which maps each point on C.]
5. No. [For any constant mapping $\vec{a}$, $A + \vec{a} = B + \vec{a}$ no matter what points A and B are. Any two points, then, give a counter-example for (1).]

Dealing with conditional sentences

Modus Ponens [See page 78.]
Any inference of the form:

$$\frac{p \quad p \longrightarrow q}{q}$$

is valid.

Deduction Rule [See pages 87 and 88.]
Any inference of the form:

$$\frac{\begin{matrix}[p]\\ q\end{matrix}}{p \longrightarrow q}$$

is valid.

Dealing with biconditional sentences

Elimination Rule [See page 98.]
Any inference of either of the forms:

$$\frac{p \longleftrightarrow q}{q \longrightarrow p} \qquad \frac{p \longleftrightarrow q}{p \longrightarrow q}$$

is valid.

Introduction Rule [See page 98.]
Any inference of the form:

$$\frac{q \longrightarrow p \quad p \longrightarrow q}{p \longleftrightarrow q}$$

is valid.

Dealing with conjunction sentences

Elimination Rule [See page 101.]
Any inference of either of the forms:

$$\frac{p \text{ and } q}{p} \qquad \frac{p \text{ and } q}{q}$$

is valid.

Introduction Rule [See page 101.]
Any inference of the form:

$$\frac{p \quad q}{p \text{ and } q}$$

is valid.

Other Rules of Logic

Hypothetical Syllogism [See page 90.]
Any inference of the form: $\frac{p \longrightarrow q \quad q \longrightarrow r}{p \longrightarrow r}$ is valid.

Replacement Rule for Biconditional Sentences [See page 99.]
Given a biconditional sentence and a second sentence, if either side of the biconditional sentence is replaced, somewhere in the second sentence, by the other side, the resulting sentence is a consequence of the given sentences.

Reflexive Rule for Biconditional Sentences [See page 99.]
Any sentence of the form '$p \longleftrightarrow p$' is valid.

Importation and Exportation [See page 101.]
Any inference of either of the forms:

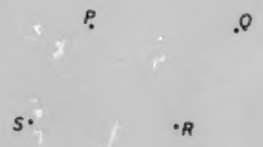
$$\frac{p \longrightarrow [q \longrightarrow r]}{(p \text{ and } q) \longrightarrow r} \qquad \frac{(p \text{ and } q) \longrightarrow r}{p \longrightarrow [q \longrightarrow r]}$$

is valid.

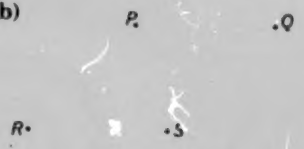
Chapter Test

1. Complete the following so that when the results are treated as universal generalizations, they are true ones. [If it is not possible to do so, say so.]
   - **(a)** $(R - P) + ((P + \overrightarrow{q}) - R) =$ ________
   - **(b)** $(S - (R + \overrightarrow{q})) + (R - S) =$ ________
   - **(c)** $(B - C) + C \in$ ________
   - **(d)** $B + (C - D) \in$ ________
   - **(e)** $(A +$ ________ $) - A = A + \overrightarrow{a}$
   - **(f)** $R + ($________ $- R) = R + \overrightarrow{r}$
2. Suppose that P, Q, R, and S are four points such that $P - Q = R - S$. Which of the following diagrams *do not* illustrate this assumption?

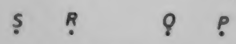
(a)



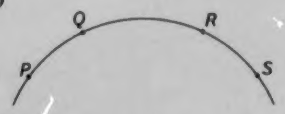
(b)



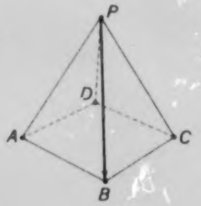
(c)



(d)



3. Consider the diagram at the right. Using 'P', 'A', 'B', 'C', 'D', give three different pairs of translations (differences of points) which have as their resultant the translation $B - P$.



4. Which of the following expressions is *not* a substitution-instance of '$\vec{a} = (A + \vec{a}) - A$'?

(a) $\vec{p} = ((Q + \vec{q}) + \vec{p}) - (Q + \vec{q})$ **(b)** $P - Q = (Q + (P - Q)) - Q$
(c) $P + \vec{p} = (Q + (P + \vec{p})) - Q$ **(d)** $\vec{p} + \vec{q} = (Q + (\vec{p} + \vec{q})) - Q$

5. True or false?

(a) $P - Q$ is the translation that maps Q on P.
(b) $\vec{q}$ is the translation that maps Q on $P + (Q - P)$.
(c) $(R - S) + (Q - R)$ is the inverse of $Q - S$.
(d) If $\vec{a} = (A + \vec{b}) - B$ then $\vec{b} = (B + \vec{a}) - A$

6. (a) Draw a diagram that illustrates this theorem about points and translations:

$$A - (B + \vec{b}) = \vec{a} \Longrightarrow A - B = \vec{b} + \vec{a}$$

(b) Prove the theorem in part (a).
(c) Now consider this sentence:

$$A - (B + \vec{b}) = \vec{a} \Longrightarrow A - B = \vec{a} + \vec{b}$$

Is this sentence true? Is it a theorem? Explain your answers.

Key to Chapter Test

1. (a) $\vec{q}$ (b) $-\vec{q}$ [or: $R - (R + \vec{q})$]
(c) Not possible (d) $\mathcal{E}$
(e) Not possible (f) $R + \vec{r}$

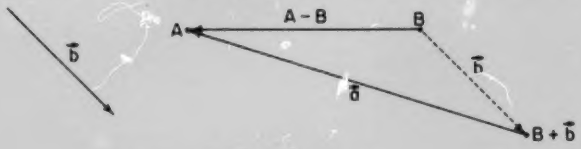
2. (a) and (d) do not illustrate the given points.

3. Each of the answers is of the form '$X - P$, $B - X$' where the possible values for 'X' are A, C, D, and B [or, P, but not both]. Some students may have difficulty seeing that '$(D - P)$, $(B - D)$' is one of the answers because there is no "dotted line" from D to B to cue this choice.

4. (c), because '$P + \vec{p}$' is not a translation-term

5. (a) True.
(b) False.
(c) False.
(d) True.

6. (a) Here is an appropriate diagram to illustrate the theorem:



(b) Proof. Suppose that $A - (B + \vec{b}) = \vec{a}$. Then, [by Theorem 2-1] $A = (B + \vec{b}) + \vec{a}$. Since $(B + \vec{b}) + \vec{a} = B + (\vec{b} + \vec{a})$, it follows that $A = B + (\vec{b} + \vec{a})$. So, $A - B = \vec{b} + \vec{a}$. Hence, if $A - (B + \vec{b}) = \vec{a}$ then $A - B = \vec{b} + \vec{a}$.

(c) Yes. By our work in Chapter 1, composition of translations is commutative.

No. We need to have '$\vec{a} + \vec{b} = \vec{b} + \vec{a}$' as a theorem (postulate) in order to be able to derive the given sentence. Since we cannot derive it from the present postulates, the sentence is not a theorem.

Chapter Three

The Algebra of Points and Translations

3.01 Some Properties of Translations

We have already discovered that, on the basis of Postulates 1–3, it is a theorem that the set $\mathcal{T}$ of all translations is closed under function composition. We stated this fact in Theorem 2–5(a):

$$\vec{a} + \vec{b} \in \mathcal{T} \tag{1}$$

The other part of Theorem 2–5:

$$A + (\vec{a} + \vec{b}) = (A + \vec{a}) + \vec{b}$$

is just a way of saying that addition of translations *is* function composition. Since, as you learned in Chapter 1, composition of functions is associative, it should be possible to use Theorem 2–5(b) in proving that addition of translations is associative:

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \tag{2}$$

As a matter of fact, doing this is not only possible, but quite easy. [You will soon have a chance to do it as an exercise.]

Another thing you learned in Chapter 1 is that each translation has a translation as its inverse. In fact the inverse of the translation from, say, P to Q is the translation from Q to P. As we pointed out in Section 2.11, this would follow from two instances of the sentence:

$$(B - A) + (A - B) = i_{\mathcal{E}} \tag{3}$$

[Explain.] [*Hint*: What do we need to know about the resultants $(P - Q) \circ (Q - P)$ and $(Q - P) \circ (P - Q)$ in order to conclude that $P - Q$ is the inverse of $Q - P$?]

TC 115

The text of this section, like that of section 2.11, is meant to initiate students into the secrets of where postulates come from.

In order to conclude that P - Q is the inverse of Q - P, we need to know that $(P - Q) \circ (Q - P)$ is the identity mapping on the domain of Q - P, namely $i_{\mathcal{E}}$, and that $(Q - P) \circ (P - Q)$ is the identity mapping on the range of Q - P.

Note that (4) could not be adopted as a definition of '$\vec{0}$' — there is a variable on the right which does not occur in what would be the defined expression.

Note, also, that (4) and (5) are related to one another in just the same way as are Theorem 2-4 and Theorem 2-5. In either case, the second sentence follows from the first and Postulate 2(a), while the first follows from the second and Postulate 2(b).

The results obtained in the exercises of this section serve as good illustrations of how similar the algebra of points and translations is to the algebra of real numbers. As a matter of fact, we are beginning to lay the foundation for the "convenient rule" stated in section 3.07 which says, essentially, that a sentence about points and translations is a theorem if and only if the corresponding real number sentence is a theorem. [In addition, these exercises give the students some good (and probably needed) practice in employing the logical principles developed so far.] Do not hesitate to stress the apparent similarities with real number theorems. Also, try to stress the value of drawing pictures to illustrate what the theorems and other sentences "say".

TC 116 (1)

The exercises of Parts A - F are an attempt to show some of the alternate ways we could select postulates. However, there is a great deal of work involved if each student does each derivation for himself. In fact, done individually these exercises would probably require more than one homework assignment. By this time the student will have lost sight of the alternate organizations we intended to point out.

In order to confine the duration of these exercises we suggest assigning each exercise to a team of students. The team is to solve the exercise, and either write their solution on an overhead transparency, or prepare their solution in dark pencil so that a Thermo-fax transparency may be made. Then about three-fourths of the class period should be spent discussing the individual derivations.

Following this the teacher should summarize the lesson by outlining some possible organizations. Here is a chart which, if constructed in stages on the chalkboard, should aid your summary.

Now that we have Postulate 3 we see that sentence (3) is just a complicated way of saying that, for any point A, $A - A = i_{\mathscr{E}}$. ["A translation which leaves any point fixed leaves each point fixed."] If we recall our intention to use '$\vec{0}$' instead of '$i_{\mathscr{E}}$', it appears that we should be able to deal with inverses of translations if we were to adopt as a new postulate the sentence:

$$(4) \qquad \vec{0} = A - A$$

We would wish, of course, to be able to prove that $\vec{0}$ is a translation and that $\vec{0}$ is—as we intend it to be—the identity mapping of $\mathscr{E}$ onto itself. Since this last means that any point is its own image under the mapping $\vec{0}$, the two theorems we would wish to be able to prove are:

$$(5)\ (a)\ \vec{0} \in \mathscr{T} \qquad (b)\ A + \vec{0} = A$$

Neatly enough, both parts of (5) follow from (4) together with earlier postulates. And, (4) follows from an earlier postulate and the two parts of (5).

Since either (4) or (5) tells us that translations have inverses, we should be able to derive from either (4) or (5) and Postulates 1–3 the sentence:

$$A + \vec{a} = B + \vec{a} \Longrightarrow A = B$$

This sentence tells us that any translation is a one-to-one mapping.

In the following exercises, you will check up on some of the notions which have been outlined above as well as some others. After having done so, we shall be in a better position to decide on what our fourth postulate should be.

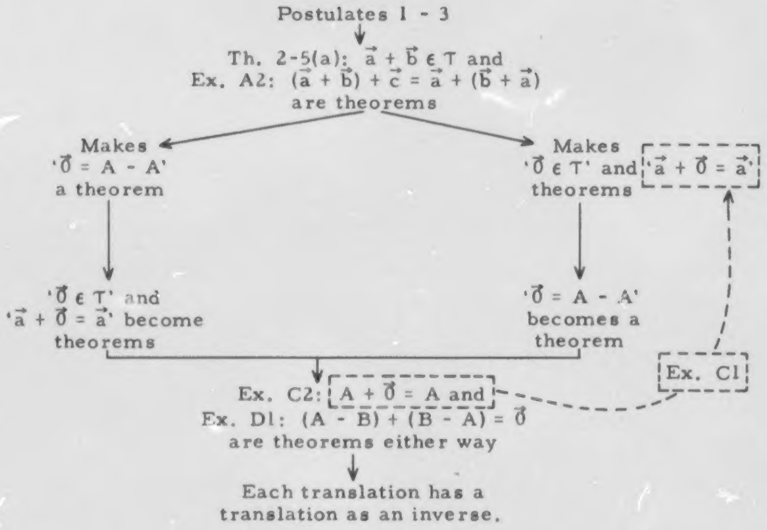
Exercises

Part A

1. Mark a point A and draw arrows to describe three translations, $\vec{a}$, $\vec{b}$, and $\vec{c}$ which have different directions.
   - **(a)** Draw an arrow from A to describe the translation $\vec{a} + \vec{b}$ and then draw another such arrow to describe $(\vec{a} + \vec{b}) + \vec{c}$.
   - **(b)** Draw an arrow from the point $A + \vec{a}$ to describe the translation $\vec{b} + \vec{c}$, and draw an arrow from A to describe $\vec{a} + (\vec{b} + \vec{c})$.
2. Prove sentence (2) on page 115. [*Hint:* If you can prove:

$$A + [(\vec{a} + \vec{b}) + \vec{c}] = A + [\vec{a} + (\vec{b} + \vec{c})]$$

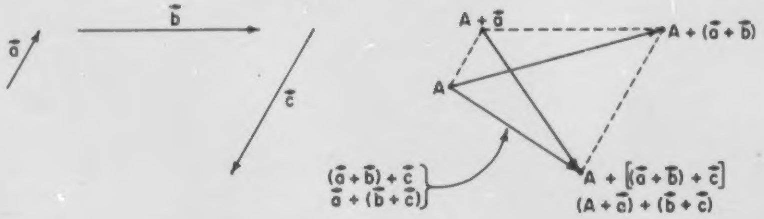
then [using (1) on page 115] an instance of an earlier theorem will yield (2).]


Postulates 1 - 3
Th. 2-5(a): $\vec{a} + \vec{b} \in T$ and
Ex. A2: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{a})$
are theorems
Makes '$\vec{0} = A - A$' a theorem
Makes '$\vec{0} \in T$' and '$\vec{a} + \vec{0} = \vec{a}$' theorems
'$\vec{0} \in T$' and '$\vec{a} + \vec{0} = \vec{a}$' become theorems
'$\vec{0} = A - A$' becomes a theorem
Ex. C1
Ex. C2: $A + \vec{0} = A$ and
Ex. D1: $(A - B) + (B - A) = \vec{0}$
are theorems either way
Each translation has a translation as an inverse.

We urge you to not put too much emphasis on the individual exercises of Parts A - F. Nc one derivation is crucial to the students success in this course. We want the student to gain som insights and appreciation for alternate selections of postulates. Such activities are not appropriate for measuring pupil progress and, if used for this purpose, will probably produce negative rather than positive attitudes.

Answers for Part A

1. The students should have diagrams something like this:


$\vec{a}$
$\vec{b}$
$\vec{c}$
$A + \vec{a}$
$A + (\vec{a} + \vec{b})$
A
$(\vec{a} + \vec{b}) + \vec{c}$
$\vec{a} + (\vec{b} + \vec{c})$
$A + [(\vec{a} + \vec{b}) + \vec{c}]$
$(A + \vec{a}) + (\vec{b} + \vec{c})$

2. [We give an abbreviated proof.]

$$A + [(\vec{a} + \vec{b}) + \vec{c}] = [A + (\vec{a} + \vec{b})] + \vec{c} = [(A + \vec{a}) + \vec{b}] + \vec{c}$$

Th. 2-5

$$A + [\vec{a} + (\vec{b} + \vec{c})] = (A + \vec{a}) + (\vec{b} + \vec{c}) = [(A + \vec{a}) + \vec{b}] + \vec{c}$$

Hence, $A + [(\vec{a} + \vec{b}) + \vec{c}] = A + [\vec{a} + (\vec{b} + \vec{c})]$. Since, by Theorem 2-5(b), each of $(\vec{a} + \vec{b}) + \vec{c}$ and $\vec{a} + (\vec{b} + \vec{c})$ is a translation it follows from Theorem 2-2 that $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

Part B

1. Using (4) on page 116 as a premiss, derive both parts of (5).
2. Using the two parts of (5) as premisses, derive (4).

[In both Exercises 1 and 2, your other premisses should be postulates or previously proved theorems.]

Part C

1. One way of saying that $\vec{0}$ is the identity mapping of $\mathscr{E}$ onto itself is to assert:

(5) (b) $$A + \vec{0} = A$$

So, from this it should follow that the resultant of any translation $\vec{a}$ followed by $\vec{0}$ is $\vec{a}$. Show that it does. [*Hint*: Use the same technique as suggested for Exercise 2 of Part A.]

2. Conversely, it follows from our three postulates and:

(*) $$\vec{a} + \vec{0} = \vec{a}$$

that $\vec{0}$ is the identity mapping of $\mathscr{E}$ onto itself. Here is an argument to this effect. Tell what should be written in the blanks.

$A + \vec{0} = [A + (A - A)] + \vec{0}$ [Postulate 2(a) (and Post. 1(b) and '$\vec{0} \in \mathscr{T}$')]
$= A + [(A - A) + \vec{0}]$ [Theorem ______ (and ______ and ______)]
$= A +$ ________ [(*) (and ______)]
$=$ ________ [Postulate ______]

Hence, $A + \vec{0} = A$.

Part D

As indicated in the text, we should be able, using (5) on page 116, to derive:

(**) $$A + \vec{a} = B + \vec{a} \Longrightarrow A = B$$

If you recall how you might prove an analogous real number theorem:

$$a + c = b + c \Longrightarrow a = b$$

you should see that we might hope to prove (**) by "adding the opposite of $\vec{a}$" on both sides of '$A + \vec{a} = B + \vec{a}$'. Since, by Postulate 2(b), $\vec{a} = (C + \vec{a}) - C$, for any point C, we know—whether we can yet prove it or not—that the inverse [or: opposite] of $\vec{a}$ is $C - (C + \vec{a})$.

1. As pointed out in the text, to show that, for any points P and Q, $P - Q$ is the inverse of $Q - P$ it is sufficient to derive:

$$(B - A) + (A - B) = \vec{0}$$

Show that this sentence is a consequence of '$\vec{0} = A - A$' and our postulates. (Note that when we do this, we will know that '$(B - A)$

Answers for Part B

1. [5(a)] Since, by (4), $\vec{0} = A - A$ and, by Postulate 1(a), $A - A \in T$ it follows that $\vec{0} \in T$.

   [5(b)] Since, by (4), $\vec{0} = A - A$ it follows that $A + \vec{0} = A + (A - A)$ and, by Postulate 2(a), that $A + \vec{0} = A$. [Alternatively, 5(b) is equivalent to (4) by Theorem 2-1.]

2. By (5), $(A + \vec{0}) - A = A - A$ and, so, by Postulate 2(b), $\vec{0} = A - A$ [since $\vec{0} \in T$]. [For an alternative answer, see remark, above, concerning second answer for Exercise 1.]

Answers for Part C

1. $A + (\vec{a} + \vec{0}) = (A + \vec{a}) + \vec{0} = A + \vec{a}$, by Theorem 2-5(b) and (5)(b) on page 116. Hence, by Theorem 2-2, $\vec{a} + \vec{0} = \vec{a}$. [If required to show that the first equation is an instance of Theorem 2-5(b), cite '$\vec{0} \in T$'; this membership sentence plays a similar role in connection with the application of Theorem 2-2.]

2. 2-5(b), Postulate 1(a), '$\vec{0} \in T$'; (A - A), Postulate 1(a); A, 2(a)

[It follows from these two exercises that, as additional postulates (5)(a) and (5)(b) or 5(a) and (☆) will serve the same purposes.]

Answers for Part D

1. By Postulate 3, $(B - A) + (A - B) = A - A$. So, assuming that $\vec{0} = A - A$ it follows that $(B - A) + (A - B) = \vec{0}$.

$+ (A - B) = \vec{0})$ will become a theorem once we enlarge our set of postulates in such a way that '$\vec{0} = A - A$' is a theorem.)

2. Use the result of Exercise 1 to show that if '$\vec{0} = A - A$' were a theorem then the sentences:
   (a) $\vec{a} + [C - (C + \vec{a})] = \vec{0}$ **(b)** $(A + \vec{a}) + [C - (C + \vec{a})] = A$
   would be theorems.
3. Derive (**). [*Hint*: Suppose that $A + \vec{a} = B + \vec{a}$. It follows, by the result in Exercise 2 that ____________ $= A$. But, by Exercise 2 itself, $(B + \vec{a}) + [C - (C + \vec{a})] =$ ________. So, $A = B$. Hence, if ________]
4. In deriving (b) of Exercise 2 you probably used (a) together with Theorem 2-5(b) and '$A + \vec{0} = A$' [which, by Exercise 1 of Part B, is a theorem if '$\vec{0} = A - A$' is]. If, in place of these last two premisses you use (2) on page 115 and '$\vec{a} + \vec{0} = \vec{a}$', you can derive:

   (c) $(\vec{b} + \vec{c}) + [C - (C + \vec{a})] = \vec{b}$

   Having done so, you can then derive:

   (***) $\vec{b} + \vec{a} = \vec{c} + \vec{a} \longrightarrow \vec{b} = \vec{c}$

   Derive (c) and (***).

Part E

As Part D illustrates, if '$\vec{0} = A - A$' is a theorem then whenever we wish to refer to the inverse of a translation $B - A$ we can write '$A - B$', and whenever we wish to refer to the inverse of a translation $\vec{a}$ we can write '$C - (C + \vec{a})$'. It is much simpler, however, to have a notation for the inverse of $\vec{a}$ and, as we decided in Chapter 1, the most convenient notation is '$-\vec{a}$'. [Read '$-\vec{a}$' as 'the inverse of $\vec{a}$' or as '$\vec{a}$ inverse' or as 'the *opposite* of $\vec{a}$'.] If we wish to include this notation in our algebra we need a postulate:

$$-\vec{a} \in \mathscr{T}$$

to serve the same purposes as do the parts of Postulate 1, and we need a postulate:

$$\vec{a} + -\vec{a} = \vec{0}$$

to say what '$-$' means.

1. Mark a point A and draw arrows to describe a translation $\vec{a}$ and the translation $-\vec{a}$. Draw a figure to show that

$$(A + \vec{a}) + -\vec{a} = A.$$

2. Make drawings to illustrate each of the following:

   (a) $-\vec{b} + -\vec{a} = -(\vec{a} + \vec{b})$ **(b)** $(\vec{a} + \vec{b}) + -\vec{a} = \vec{b}$

Answers for Part D [cont.]

2. (a) By Postulate 2(b), $\vec{a} = (C + \vec{a}) - C$ and so

$$\vec{a} + [C - (C + \vec{a})] = [(C + \vec{a}) - C] + [C - (C + \vec{a})].$$

Using an instance of the sentence displayed in Exercise 1 we could infer '$\vec{a} + [C - (C + \vec{a})] = \vec{0}$'.

(b) By (a) and Theorem 2-5(b) it follows that $(A + \vec{a}) + [C - (C + \vec{a})] = A + \vec{0}$. Since '$A + \vec{0} = A$' would be a theorem if '$\vec{0} = A - A$' were, it follows that in this latter case (b) would be a theorem.

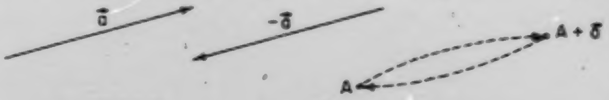
3. $(B + \vec{a}) + [C - (C + \vec{a})] = A$; B; $A + \vec{a} = B + \vec{a}$ then $A = B$.

4. (c) By (a) and (2) it follows that $(\vec{b} + \vec{a}) + [C - (C + \vec{a})] = \vec{b} + \vec{0}$. So, assuming that $\vec{b} + \vec{0} = \vec{b}$ it follows that $(\vec{b} + \vec{a}) + [C - (C + \vec{a})] = \vec{b}$.

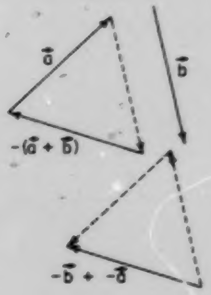
   (☆☆☆) Suppose that $\vec{b} + \vec{a} = \vec{c} + \vec{a}$. It follows by (c) that $(\vec{c} + \vec{a}) + [C - (C + \vec{a})] = \vec{b}$. But, from (c) itself, $(\vec{c} + \vec{a}) + [C - (C + \vec{a})] = \vec{c}$. So, $\vec{b} = \vec{c}$. Hence, if $\vec{b} + \vec{a} = \vec{c} + \vec{a}$ then $\vec{b} = \vec{c}$.

Answers for Part E

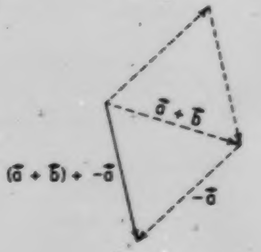
1. [The students should have diagrams something like these.]



2. (a)



(b)



Part F

1. Mark a point A and draw two arrows describing translations $\overrightarrow{a}$ and $\overrightarrow{b}$ in different directions.
   (a) Draw an arrow from A to describe the translation $\overrightarrow{a} + \overrightarrow{b}$.
   (b) Draw an arrow from A to describe the translation $\overrightarrow{b} + \overrightarrow{a}$.
   (c) Use our notation to write a sentence saying what your results show.
2. Show that if '$\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{b} + \overrightarrow{a}$' were adopted as a postulate then '$A - A = B - B$' would be a theorem. [*Hint*: Consider an appropriate instance of Postulate 3.]

3.02 A Fourth Postulate

We have seen that, on the basis of Postulates 1–3, the sentences:

$$\overrightarrow{a} + \overrightarrow{b} \in \mathscr{T} \quad \text{and:} \quad (\overrightarrow{a} + \overrightarrow{b}) + \overrightarrow{c} = \overrightarrow{a} + (\overrightarrow{b} + \overrightarrow{c})$$

are theorems. We have also seen that if we enlarge our set of postulates so that '$\overrightarrow{0} = A - A$' is a theorem then the sentences:

$$\overrightarrow{0} \in \mathscr{T} \quad \text{and:} \quad \overrightarrow{a} + \overrightarrow{0} = \overrightarrow{a}$$

are theorems, and *vice versa*. In either case the sentences:

$$A + \overrightarrow{0} = A \quad \text{and:} \quad (A - B) + (B - A) = \overrightarrow{0}$$

will be theorems. From these last it follows that—as you learned in Chapter 1—each translation has a translation for its inverse. As was pointed out in Part E, it will be convenient to express this fact about inverses explicitly in postulates:

$$-\overrightarrow{a} \in \mathscr{T} \quad \text{and:} \quad \overrightarrow{a} + -\overrightarrow{a} = \overrightarrow{0}$$

Finally, as Part F reminded you, composition of translations is commutative. We shall wish to express this important fact in a postulate:

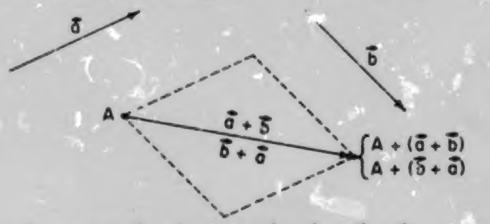
$$\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{b} + \overrightarrow{a}$$

Evidently, we need to adopt several new postulates. For reasons which will become apparent later in this chapter, we shall group them together into a single postulate. For the same reasons we shall include as parts of this postulate two sentences which are already theorems. [There is certainly no harm in "postulating more than is necessary".] Our fourth postulate turns out, then, to have five main parts. We state them at once, then discuss them separately.

Answers for Part F

1. [The students should have diagrams something like this.]

   (a), (b)



   (c) $A + (\vec{a} + \vec{b}) = A + (\vec{b} + \vec{a})$ [or: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$]

2. By Postulate 3, $(B - A) + (A - B) = A - A$ and $(A - B) + (B - A) = B - B$. So, if $(B - A) + (A - B) = (A - B) + (B - A)$ then $A - A = B - B$.

Postulate 4_0 (a) $\vec{a} + \vec{b} \in \mathscr{T}$ (b) $\vec{0} \in \mathscr{T}$ (c) $-\vec{a} \in \mathscr{T}$

Postulate 4_1 $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

Postulate 4_2 $\vec{a} + \vec{0} = \vec{a}$

Postulate 4_3 $\vec{a} + -\vec{a} = \vec{0}$

Postulate 4_4 $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Some parts of this postulate are theorems before its adoption. These parts are Postulate 4_0(a) and Postulate 4_1. [Also, we could derive Postulate 4_0(b) from Postulate 4_0(a) and Postulate 4_3 by using Postulate 4_0(c).] The principle use of the parts of Postulate 4_0 is—like that of the parts of Postulate 1—to make sure that substitution-instances of sentences are what we claim them to be. Since we shall seldom go to the trouble of doing this [after this chapter] we shall not often have occasion to refer to Postulate 4_0.

As you have seen in Exercise 2 of Part C, Postulate 4_0(b) and Postulate 4_2 have as a consequence sentence '$A + \vec{0} = A$'. So, this sentence is now a theorem. Hence, by Exercise 2 of Part B, '$\vec{0} = A - A$' is also a theorem. We take note of this:

Theorem 3-1 (a) $A + \vec{0} = A$ (b) $A - A = \vec{0}$

Another pair of theorems which are easily proved now are:

Theorem 3-2 (a) $A + \vec{a} = A \Longleftrightarrow \vec{a} = \vec{0}$
(b) $B = A \Longleftrightarrow B - A = \vec{0}$

[Do you see how to derive Theorem 3-2(a) from Theorem 2-2 and Theorem 3-1(a)?] You will investigate these theorems in the exercises which follow.

The reasons for adopting Postulate 4_0(c) and Postulate 4_3 have already been given. Using them you can give a shorter proof of the cancellation principle:

$$(**) \qquad A + \vec{a} = B + \vec{a} \Longrightarrow A = B$$

than you gave in Part D. [You can use Theorem 2-5(b) and Theorem 3-1(a). Do you see how?] From your work in Chapter 2, you are probably well aware that the converse of (**) is a theorem. [Explain.] So, we have:

Theorem 3-3 $A + \vec{a} = B + \vec{a} \Longleftrightarrow A = B$

Just as we might have chosen Theorem 2-4:

$$\vec{a} + \vec{b} = [(A + \vec{a}) + \vec{b}] - A$$

as a postulate, in place of Postulate 3, so we might choose:

$$\text{(i)} \qquad \vec{0} = A - A$$

and:

$$\text{(ii)} \qquad -\vec{a} = A - (A + \vec{a})$$

as postulates to introduce '$\vec{0}$' and '−'. Adopting (i) as a postulate would insure that '$\vec{0} \in T$' and '$\vec{a} + \vec{0} = \vec{a}$' are theorems; adopting (ii) would insure that '$-\vec{a} \in T$' is a theorem; with both (i) and (ii) as postulates [using, also, Postulates 2(b) and (3)] '$\vec{a} + -\vec{a} = \vec{0}$' would be a theorem.

Because of the occurrences of the variable 'A', neither (i) nor (ii) should be called a definition; for, as pointed out earlier, if (i) were used as a definition it would always be uncertain what term had been abbreviated to '$\vec{0}$', — and a similar remark can be made concerning (ii). It has also been pointed out that such "quasi-definitions" often differ from honest definitions through being "creative" — the adoption of (i), for example, makes it possible to prove a theorem:

$$\text{(iii)} \qquad A - A = B - B$$

which is not derivable from Postulates 1 - 3 alone. Similarly, the adoption of (ii) makes it possible to prove:

$$\text{(iv)} \qquad A - (A + \vec{a}) = B - (B + \vec{a})$$

[That neither (iii) nor (iv) follows from Postulates 1 - 3 can be shown by using the constant function interpretation of Part D on page 110.] Note that by (iii) and Postulate 2(a) $B + (A - A) = B + (B - B) = B$ which tells us that the translation $A - A$ maps any point B on itself — in particular, that the identity mapping of $\mathcal{E}$ onto itself is a translation. Similarly, by (iv) and Postulate 2(a), the translation $A - (A + \vec{a})$ maps the image $B + \vec{a}$, of any point B under $\vec{a}$ on B — in particular, that any translation has a translation as its inverse. So, even without Postulate 3, we could postulate that the identity mapping is a translation by adopting (iii) as a postulate, and that any translation has a translation as its inverse by adopting (iv). All that (i) adds to (iii) is that '$\vec{0}$' is to be used as a name for the identity mapping; all that (ii) adds to (iv) is that '−' is to be used as an inversing operator.

In view of what has been said in the preceding paragraph it is of interest to note that on the basis of Postulates 1 - 3, each of (iii) and (iv) can be derived from the other. In other words, having adopted only our first three postulates, it follows that all translations have translations as inverses if and only if the identity mapping is a translation. [The only if-part is not very surprising, in view of the fact that Postulates 1 - 3 imply the closure of T under composition of functions. The if-part, on the other hand, is a rather strong result.] To show that this is the case we first derive (iv) by using (iii) and then derive (iii) by using (iv).

Derivation of (iv):

$$\begin{aligned}(B+\vec{a})+[A-(A+\vec{a})] &= B+(\vec{a}+[A-(A+\vec{a})]) && [\text{Theorem 2-5(b)}]\\ &= B+([(A+\vec{a})-A]+[A-(A+\vec{a})]) && [\text{Postulate 2(b)}]\\ &= B+(A-A) && [\text{Postulate 3}]\\ &= B+(B-B) && [\text{(iii)}]\\ &= B && [\text{Postulate 2(a)}]\end{aligned}$$

So, by Theorem 2-1, $A - (A + \vec{a}) = B - (B + \vec{a})$.

Derivation of (iii):

$$\begin{aligned}B+(A-A) &= B+([(A+\vec{a})-A]+[A-(A+\vec{a})]) && [\text{Postulate 3}]\\ &= B+([(B+\vec{a})-B]+[B-(B+\vec{a})]) && [\text{Postulate 2(b), (iv)}]\\ &= B+(B-B) && [\text{Postulate 3}]\\ &= B && [\text{Postulate 2(a)}]\end{aligned}$$

So, by Theorem 2-1, $A - A = B - B$.

Now, an even more surprising result becomes apparent if we recall Exercise 2 of the preceding Part F. As shown there, (iii) is a consequence of Postulate 3 and:

$$\text{(v)} \qquad \vec{a} + \vec{b} = \vec{b} + \vec{a}$$

In other words, from the closure of $\mathcal{T}$ under composition and the commutativity of composition when restricted to members of $\mathcal{T}$ it follows that the identity mapping is a translation and [using also Postulate 1 and 2] that translations have translations as inverses.

Put in another way, if (v) is adopted as a postulate then (i) and (ii) are not "creative" — the only purpose they serve is to introduce the symbols '$\vec{0}$' and '$-$'. In view of this loss of creativity which (i) and (ii) suffer when (v) is adopted as a postulate, some writers would, in this context, style (i) and (ii) definitions. Nevertheless, the original objection to calling them so still stands.

In view of the strength of (v) as a postulate, it is worthwhile to discover its geometrical significance. To do so, note that by Theorem 2-2, (v) is equivalent to:

$$A + (\vec{a} + \vec{b}) = A + (\vec{b} + \vec{a})$$

and that this, by Theorem 2-5(b) is equivalent to:

$$(A + \vec{a}) + \vec{b} = (A + \vec{b}) + \vec{a}$$

As a figure will show, what this says — in geometrical terms which have not as yet been defined — is that, given three points, $A + \vec{a}$, A, and $A + \vec{b}$, there is a fourth point, D, such that the four points, in the order listed, are vertices of a quadrilateral whose opposite sides are parallel and have the same length. A familiar theorem about parallelograms now suggests that, as a potential postulate, (v) is equivalent to:

$$\text{(vi)} \qquad (B + \vec{a}) - (A + \vec{a}) = B - A \qquad [\text{Compare this with (iv)}]$$

In the next section we shall adopt (v) as a postulate and, later, it will turn out that (vi) is a theorem. So, to check the suggestion it will be sufficient, here, to show that if (vi) were adopted as a postulate, (v) would be a theorem. We suggest a proof [using Postulate 2(b), (vi), Postulate 3 (twice), and Postulate 2(b)]:

$$\begin{aligned}(B-A)+(C-D) &= ([D+(B-A)]-D)+([C+(B-A)]-[D+(B-A)])\\ &= [C+(B-A)]-D\\ &= (C-D)+([C+(B-A)]-C)\\ &= (C-D)+(B-A)\end{aligned}$$

As pointed out in the preceding commentary there are many equivalent choices for our fourth postulate. We choose the particular form given in the text because it may be restated in the form given on page 130. This restatement permits of very simple modifications to include all of the additional postulates which we shall adopt in later portions of the course.

The derivation of 4_0(b) which is mentioned in the paragraph which follows the statement of the postulate is:

$$\dfrac{\vec{a} + -\vec{a} = \vec{0} \qquad \dfrac{\vec{a} + \vec{b} \in \mathcal{T} \qquad -\vec{a} \in \mathcal{T}}{\vec{a} + -\vec{a} \in \mathcal{T}}\,(\text{Subst})}{\vec{0} \in \mathcal{T}}\,(\text{RRE})$$

The two proofs mentioned in the paragraph which follows the one just referred to are given below in the answers for Exercise 1 of Part A and Exercise 1 of Part B, respectively. The explanation concerning the converse of (☆☆) which is asked for amounts to recognizing that all equality principles are valid sentences and, so, are theorems.

Comparing Theorem 3-3 with Theorem 2-2, and recalling Theorem 2-3, suggests that there is another theorem matching up with the last of these. There is.

Theorem 3-4 $C - A = C - B \longleftrightarrow A = B$

Now that we have Theorem 3-3, it is not difficult to prove Theorem 3-4. A little later we shall find lots of use for Postulate 4_1.

Exercises

Part A

1. There is a very short proof of Theorem 3-2(a):
   By Theorem 3-1(a), $A + \vec{a} = A$ if and only if $A + \vec{a} = A + \vec{0}$. By Theorem 2-2 [and Postulate 4_0(b)] $A + \vec{a} = A + \vec{0}$ if and only if $\vec{a} = \vec{0}$. So, $A + \vec{a} = A$ if and only if $\vec{a} = \vec{0}$.

   Putting this argument in tree-form you will get something like this:

$$\dfrac{\dfrac{[\text{Th. 3-1(a)}] \quad [\text{valid sentence}]}{p \longleftrightarrow q} \qquad \dfrac{[\text{Th. 2-2}]}{q \longleftrightarrow r}}{[\text{Th. 3-2(a)}]}$$

   (a) Write out the tree-proof.
   (b) Explain why, on the basis of the rules summarized at the end of Chapter 2, inferences of the form:

$$\dfrac{p \longleftrightarrow q \quad q \longleftrightarrow r}{p \longleftrightarrow r}$$

   are valid.

2. A longer proof of Theorem 3-2(a) involves proving its if-part:

$$\vec{a} = \vec{0} \longrightarrow A + \vec{a} = A$$

   and its only if-part:

$$A + \vec{a} = A \longrightarrow \vec{a} = \vec{0}$$

   separately.
   (a) Prove the if-part by an argument which begins:
   Suppose that $\vec{a} = \vec{0}$. Since, by Theorem 3-1(a), ...
   (b) Prove the only if-part by an argument which begins:
   Suppose that $A + \vec{a} = A$. It follows by Theorem 2-1 that . . .

Answers for Part A

We recommend that one of Exercises 1 and 2 be used as a class exercise rather than for homework. The reason for this is that students are sometimes annoyed by a requirement to produce several proofs for one theorem. Such activities are best treated by the class as a whole. Similar comments apply to Exercise B1.

1. (a) [In the diagram, students should replace the bracketed theorem-names by the theorems themselves, replace '[valid sentence]' by '$A + \vec{a} = A \Longleftrightarrow A + \vec{a} = A$', and replace 'p', 'q', and 'r' by '$A + \vec{a} = A$', '$A + \vec{a} = A + \vec{0}$', and '$\vec{a} = \vec{0}$', respectively. The validity of the final inference is the subject of part (b).]

   (b) Inferences of the kind in question are valid by virtue of the replacement rule for biconditional sentences. Such inferences can also be justified, more laboriously, by using the elimination and introduction rules for $\Longleftrightarrow$ and the rule (Syll).

2. (a) Suppose that $\vec{a} = \vec{0}$. Since, by Theorem 3-1(a), $A + \vec{0} = A$ it follows that $A + \vec{a} = A$. Hence, if $\vec{a} = \vec{0}$ then $A + \vec{a} = A$.

   (b) Suppose that $A + \vec{a} = A$. It follows by Theorem 2-1 that $\vec{a} = A - A$ and, so, by Theorem 3-1(b), that $\vec{a} = \vec{0}$. Hence, if $A + \vec{a} = A$ then $\vec{a} = \vec{0}$.

(c) The only if-part of Theorem 3-2(a) tells you that any translation which leaves some point fixed is $\vec{0}$. The if-part of this theorem tells you—just as Postulate 4_2 does—that $\vec{0}$ leaves each point fixed. So, the whole of Theorem 3-2(a) formulates one of the facts about translations which are summarized on pages 47 and 48. Which one?

3. Prove Theorem 3-2(b). [*Suggestions*: One way depends on noticing that Theorem 3-2(b) is almost an instance of Theorem 3-2(a). Another is to note that, since equality is symmetric, you may as well prove '$A = B \longleftrightarrow \vec{0} = B - A$'. What early theorem does this remind you of?]

Part B

1. Give the "shorter proof" of:

$$(**) \qquad A + \vec{a} = B + \vec{a} \longrightarrow A = B$$

which is suggested on page 120.

2. Prove the converse of (**).
3. Prove Theorem 3-4. [*Hint*: By Theorem 2-1, $C - A = C - B$ if and only if $B + (C - A) = C$. (Now, prepare to use Theorem 3-3, by using Postulate 2(a).)]

Part C

Prove:

1. $\vec{a} + \vec{c} = \vec{b} + \vec{c} \longrightarrow \vec{a} = \vec{b}$ [*Hint*: Compare with Exercise 1 of Part B.]
2. $\vec{c} + \vec{a} = \vec{c} + \vec{b} \longrightarrow \vec{a} = \vec{b}$ [*Hint*: Use Postulate 4_4.]
3. $\vec{a} + \vec{b} = \vec{0} \longrightarrow -\vec{a} = \vec{b}$ [*Hint*: Use Exercise 2.]
4.

Theorem 3-5 (a) $-(B - A) = A - B$
(b) $-\vec{a} = A - (A + \vec{a})$

5. $-\vec{0} = \vec{0}$
6. (a) $-\vec{a} + \vec{a} = \vec{0}$ (b) $--\vec{a} = \vec{a}$
7. (a) $\vec{a} = \vec{b} \longrightarrow -\vec{a} = -\vec{b}$ (b) $-\vec{a} = -\vec{b} \longrightarrow \vec{a} = \vec{b}$
8. $-(\vec{a} + \vec{b}) = -\vec{b} + -\vec{a}$ [*Hint*: Use Exercise 3.]
9. $(A + \vec{a}) + -\vec{a} = A$ [*Hint*: There is a proof which uses Postulate 4_3, and there is a proof which uses Theorem 3-5(b). Try for both.]
10. $(A + \vec{a}) + \vec{b} = A \longrightarrow \vec{b} = -\vec{a}$ [*Hint*: Suppose that $(A + \vec{a}) + \vec{b} = A$. It follows by Theorems _?_ and _?_ that $A + (\vec{a} + \vec{b}) = A + \vec{0}$. So, by Theorem _?_, $\vec{a} + \vec{b} = \vec{0}$. It follows by Exercise _?_ that _?_. So, since equality is symmetric, $\vec{b} = -\vec{a}$. Hence, if . . .]

2. (c) A translation which leaves any point fixed leaves each point fixed. [This is (5) on page 47. Another formulation of (5) is '$A + \vec{a} = A \Longrightarrow B + \vec{a} = B$'. This can be derived from the only if-part of Theorem 3-2(a) and an instance of the if-part by using (Syll).]

3. Following the first suggestion:

By Theorem 3-2(a), $A + (B - A) = A$ if and only if $B - A = \vec{0}$. So, by Postulate 2(a), $B = A$ if and only if $B - A = \vec{0}$.

Following the second suggestion:

By Theorem 2-1, $A + \vec{0} = B$ if and only if $\vec{0} = B - A$. So, by Theorem 3-1(a), $A = B$ if and only if $\vec{0} = B - A$. [Since $A = B$ if and only if $B = A$, and $\vec{0} = B - A$ if and only if $B - A = \vec{0}$, it follows [using the replacement rule for biconditional sentences] that $B = A$ if and only if $B - A = \vec{0}$.]

Answers for Part B

1. Suppose that $A + \vec{a} = B + \vec{a}$. It follows [since $(A + \vec{a}) + -\vec{a} = (A + \vec{a}) + -\vec{a}$] that $(A + \vec{a}) + -\vec{a} = (B + \vec{a}) + -\vec{a}$ and so, by Theorem 2-5(b), that $A + (\vec{a} + -\vec{a}) = B + (\vec{a} + -\vec{a})$. So, by 4_3, $A + \vec{0} = B + \vec{0}$ and, by Theorem 3-1(a), $A = B$. Hence, if $A + \vec{a} = B + \vec{a}$ then $A = B$.
2. Suppose that $A = B$. Since $A + \vec{a} = A + \vec{a}$ it follows that $A + \vec{a} = B + \vec{a}$. Hence, if $A = B$ then $A + \vec{a} = B + \vec{a}$.
3. By Theorem 2-1, $C - A = C - B$ if and only if $B + (C - A) = C$. Since, by Postulate 2(a), $C = A + (C - A)$ it follows that $C - A = C - B$ if and only if $B + (C - A) = A + (C - A)$. By Theorem 3-3, $B + (C - A) = A + (C - A)$ if and only if $B = A$ — that is, if and only if $A = B$. So, $C - A = C - B$ if and only if $A = B$.

Answers for Part C

[The theorems of Exercises 1, 2, 3, 5, 6, 7, 8 are obviously analogues of theorems concerning real numbers and, in view of $4_0 - 4_4$, can be proved just as the latter are proved in algebra courses. Although they are important and useful, we shall not assign numbers to such theorems. (See page 126.) It is not necessary that each student derive each exercise in Part C. The more important derivations are for Exercises 4, 9, 10, and 11.]

1. Suppose that $\vec{a} + \vec{c} = \vec{b} + \vec{c}$. It follows [since $(\vec{a} + \vec{c}) + -\vec{c} = (\vec{a} + \vec{c}) + -\vec{c}$] that $(\vec{a} + \vec{c}) + -\vec{c} = (\vec{b} + \vec{c}) + -\vec{c}$ and so, by 4_1, that $\vec{a} + (\vec{c} + -\vec{c}) = \vec{b} + (\vec{c} + -\vec{c})$. So, by 4_3, $\vec{a} + \vec{0} = \vec{b} + \vec{0}$ and, by 4_2, $\vec{a} = \vec{b}$. Hence, if $\vec{a} + \vec{c} = \vec{b} + \vec{c}$ then $\vec{a} = \vec{b}$.
2. By 4_4, $\vec{c} + \vec{a} = \vec{a} + \vec{c}$ and $\vec{c} + \vec{b} = \vec{b} + \vec{c}$. So, by the result of Exercise 1, if $\vec{c} + \vec{a} = \vec{c} + \vec{b}$ then $\vec{a} = \vec{b}$.

[Although this need not concern your students, it is interesting to note that $4_0 - 4_3$, alone, imply '$\vec{0} + \vec{a} = \vec{a}$' and '$-\vec{a} + \vec{a} = \vec{0}$'. So, 4_4 is not needed as a basis for the result of Exercise 2. For an expansion of this remark see the commentary for page 293 of High School Mathematics, Course 3.]

3. By Exercise 2, if $\vec{a} + \vec{b} = \vec{a} + -\vec{a}$ then $\vec{b} = -\vec{a}$. So, by 4_3, if $\vec{a} + \vec{b} = \vec{0}$ then $\vec{b} = -\vec{a}$ — that is, $-\vec{a} = \vec{b}$.

[In the proofs given in answer to Exercises 2 and 3 the desired results have been derived directly from instances of those of Exercises 1 and 2, respectively, by using the replacement rule for equations. For example, the latter proof in tree-form is:

$$\dfrac{\vec{a} + -\vec{a} = \vec{0} \qquad \dfrac{\vec{c} + \vec{a} = \vec{c} + \vec{b} \Longrightarrow \vec{a} = \vec{b}}{\vec{a} + \vec{b} = \vec{a} + -\vec{a} \Longrightarrow \vec{b} = -\vec{a}}\,(\text{Subst})}{\vec{a} + \vec{b} = \vec{0} \Longrightarrow \vec{b} = -\vec{a}}\,(\text{RRE})$$

Here is a less efficient method of proof which students may suggest:

> Suppose that $\vec{a} + \vec{b} = \vec{0}$. Since $\vec{a} + -\vec{a} = \vec{0}$ it follows that $\vec{a} + \vec{b} = \vec{a} + -\vec{a}$. Since, by Exercise 2, if $\vec{a} + \vec{b} = \vec{a} + -\vec{a}$ then $\vec{b} = -\vec{a}$, it follows that $\vec{b} = -\vec{a}$. Hence, if $\vec{a} + \vec{b} = \vec{0}$ then $\vec{b} = -\vec{a}$.

There is nothing wrong with this proof other than its inefficiency. But, it is well to learn to use both replacement rules as effectively as possible. The lesson to be learned is that, since replacements may be made in arbitrary sentences, it is not necessary to break a sentence up into smaller sentences, then make the replacement, then put the sentence back together again. This is rather like taking off one's shoes, then washing one's face, then putting the shoes back on — all just to get one's face washed.]

4. (a) By Postulate 3, $(B - A) + (A - B) = A - A$ which, by Theorem 3-1(b), $= \vec{0}$. So, by [an instance of] the theorem of Exercise 3, $-(B - A) = A - B$.

   (b) By Postulate 2(b), $\vec{a} + [A - (A + \vec{a})] = [(A + \vec{a}) - A] + [A - (A + \vec{a})]$ which, by Postulate 3, $= A - A$ and so, by Theorem 3-1(b), $= \vec{0}$. So, by [an instance of] the theorem of Exercise 3, $-\vec{a} = A - (A + \vec{a})$.

   [Here is an alternative proof: By Theorem 2-1, $-\vec{a} = A - (A + \vec{a})$ if and only if $(A + \vec{a}) + -\vec{a} = A$. But, by Theorem 2-5(b), 4_3, and Theorem 3-1(a), $(A + \vec{a}) + -\vec{a} = A + (\vec{a} + -\vec{a}) = A + \vec{0} = A$. Hence, $-\vec{a} = A - (A + \vec{a})$.]

5. By 4_2, $\vec{0} + \vec{0} = \vec{0}$. By Exercise 3, if $\vec{0} + \vec{0} = \vec{0}$ then $-\vec{0} = \vec{0}$. Hence, $-\vec{0} = \vec{0}$.

6. (a) [By 4_3 and an instance of 4_4.]

   (b) Since, by part (a), $-\vec{a} + \vec{a} = \vec{0}$ it follows, by an instance of the result in Exercise 3 that $--\vec{a} = \vec{a}$.

7. (a) [The simplest proof is like that of any equality principle. Use the valid sentence '$-\vec{a} = -\vec{a}$' as a premiss. For kicks, the result may also be derived by using the instance '$\vec{a} + -\vec{b} = \vec{0} \Longrightarrow -\vec{a} = -\vec{b}$' of the result of Exercise 3. From this and 4_3 it follows that if $\vec{a} + -\vec{b} = \vec{b} + -\vec{b}$ then $-\vec{a} = -\vec{b}$. But, if $\vec{a} = \vec{b}$ then $\vec{a} + -\vec{b} = \vec{b} + -\vec{b}$. Hence, if $\vec{a} = \vec{b}$ then $-\vec{a} = -\vec{b}$.]

   (b) By [an instance of] (a), if $-\vec{a} = -\vec{b}$ then $--\vec{a} = --\vec{b}$. So, by part (b) of Exercise 6, if $-\vec{a} = -\vec{b}$ then $\vec{a} = \vec{b}$.

8. $(\vec{a} + \vec{b}) + (-\vec{b} + -\vec{a}) = [(\vec{a} + \vec{b}) + -\vec{b}] + -\vec{a} = [\vec{a} + (\vec{b} + -\vec{b})] + -\vec{a} = (\vec{a} + \vec{0}) + -\vec{a} = \vec{a} + -\vec{a} = \vec{0}$ [by 4_1 (twice), 4_3, 4_2, and 4_3]. So, by the result of Exercise 3, $-(\vec{a} + \vec{b}) = -\vec{b} + -\vec{a}$.

   [Notice that this result does not depend on 4_4. The more familiar theorem '$-(\vec{a} + \vec{b}) = -\vec{a} + -\vec{b}$' does depend on commutativity.]

9. [The proof using 4_3 is given in the alternative answer for Exercise 4(b). Using Theorem 3-5(b) one could argue as follows:

   > It follows by Theorem 3-5(b) that $(A + \vec{a}) + -\vec{a} = (A + \vec{a}) + [A - (A + \vec{a})]$ which, by Postulate 2(a), $= A$.

   There is, of course, nothing odd in the fact that either of two theorems — in this case, the result in the present exercise and Theorem 3-5(b) — can be used in "proving" the other. Which of the two arguments may be accepted as a proof of its conclusion depends on which result happens to be proved first. For only theorems which have been "previously proved" may be used as premisses of proofs.]

10. [The required theorems and exercise are 2-5(b), 3-1(a), 2-2, and 3. By the same techniques references to points can be introduced into other of the preceding unnumbered theorems. For example, using Exercise 7(b) it is easy to derive '$A + -\vec{a} = A + -\vec{b} \Longrightarrow \vec{a} = \vec{b}$' and, once subtraction is introduced in the next section, to derive '$A - \vec{a} = A - \vec{b} \Longrightarrow \vec{a} = \vec{b}$'.]

11.

Theorem 3-6 $(A + \vec{a}) + \vec{b} = A \Longleftrightarrow \vec{b} = -\vec{a}$

3.03 Subtraction

It is convenient to define

subtraction of translations from points,
and subtraction of translations from translations,

in ways that are analogous to the way that subtraction of real numbers is defined.

Definition 3-1 (a) $A - \vec{a} = A + -\vec{a}$
(b) $\vec{a} - \vec{b} = \vec{a} + -\vec{b}$

(a) $A - \vec{a}$ is the image of A under the inverse of $\vec{a}$.
(b) $\vec{a} - \vec{b}$ is the resultant of $\vec{a}$ followed by the inverse of $\vec{b}$.

Let us summarize the various ways that the symbols '+' and '−' are being used in this algebra of points and translations:

(1) Subtracting a point from a point gives a __________: $B - A \in$ ______

(2) Adding a translation to a point gives a __________: $A + \vec{a} \in$ ______

(3) Adding a translation to a translation gives a __________: $\vec{a} + \vec{b} \in$ ______

(4) The inverse of a translation is a __________: $-\vec{a} \in$ ______

(5) Subtracting a translation from a point [or from a translation] is the same as adding the inverse of the translation to the point [or to the translation]:

$A - \vec{a} = A +$ ______; $\vec{a} - \vec{b} = \vec{a} +$ ______

It may also be worthwhile to note some of the ways that the symbols '+', '—', and '−' are *not* being used in this algebra:

Meaningless expressions:

$\cancel{A + B}$
$\cancel{\vec{a} + A}$
$\cancel{-A}$
$\cancel{\vec{a} - A}$

11. After Exercise 10, all that remains is to prove the converse of the result in that exercise. Suppose that $\vec{b} = -\vec{a}$. It follows that $(A + \vec{a}) + \vec{b} = (A + \vec{a}) + -\vec{a} = A + (\vec{a} + -\vec{a}) = A + \vec{0} = A$ [by Theorem 2-5(b), 4_a, and Theorem 3-1(a)]. Hence, if $\vec{b} = -\vec{a}$ then $(A + \vec{a}) + \vec{b} = A$.

*

As remarked earlier, our emphasis on the rules of logic which constitute the formal basis for proofs is for the purpose of giving students confidence in the validity of informal arguments. So, for example, the answers we have given for exercises have become less formal. There is another kind of formality which, for brevity, we shall adhere to but which need not be imposed on students. This is the use of arbitrary and nondescriptive names — such as 'Theorem 3-1(a)' — for theorems. When such references are scattered through a proof they can be an annoying distraction to a reader who sees what is going on. Even when grouped together, as in the answer given above for Exercise 11, they are an unnecessary annoyance to one who is not quite sure what statement was labelled, say, 'Theorem 3-1(a)' — and is quite sure that it's an imposition to be required to remember this. On the other hand, students do need to have some check imposed on them to prevent irresponsible activity directed merely toward reaching the correct answer. One solution to this problem is to write the theorems used in a proof as footnotes.

*

The blanks should be filled in as follows:

(1) translation; $\mathcal{T}$
(2) point; $\mathcal{E}$
(3) translation; $\mathcal{T}$
(4) translation; $\mathcal{T}$
(5) $-\vec{a}$; $-\vec{b}$

Sample Quiz

1. Write a paragraph-proof of the following:

(✻) $A - \vec{b} = A - \vec{c} \Longrightarrow \vec{b} = \vec{c}$

2. Write the converse of (✻).

3. Draw a picture of a counter-example, or write a paragraph-proof of the converse of (✻).

4. Prove that $-(A - B) = B - A$.

Answers for Sample Quiz

1. Here is a sample paragraph-proof of (✻):

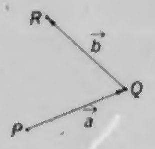
Suppose that $A - \vec{b} = A - \vec{c}$. It follows that $A + -\vec{b} = A + -\vec{c}$ so that $(A + -\vec{b}) - A = (A + -\vec{c}) - A$. Thus, $-\vec{b} = -\vec{c}$ so that $\vec{b} = \vec{c}$. Hence, if $A - \vec{b} = A - \vec{c}$ then $\vec{b} = \vec{c}$.

2. The converse of (✻) is:

$$\vec{b} = \vec{c} \Longrightarrow A - \vec{b} = A - \vec{c}$$

Exercises

Part A



Let $Q = P + \vec{a}$ and $R = Q + \vec{b}$ as in the figure at the right. Complete each of the following by filling the blanks with 'P', 'Q', 'R', or '$\vec{0}$'. [The first exercise has been completed as a sample.]

1. $\vec{a} + \vec{b} = \underline{R} - \underline{P}$
2. $-\vec{b} - \vec{a} = \_\_\_\_ - \_\_\_\_$
3. $-(\vec{a} + \vec{b}) = \_\_\_\_ - \_\_\_\_$
4. $-(R - P) = \_\_\_\_ - \_\_\_\_$
5. $(P + \vec{a}) + \vec{b} = \_\_\_\_$
6. $P + (\vec{a} + \vec{b}) = \_\_\_\_$
7. $(P + \vec{a}) - \vec{a} = \_\_\_\_$
8. $P + (\vec{a} - \vec{a}) = \_\_\_\_$
9. $\vec{a} - \vec{a} = \_\_\_\_$
10. $Q - Q = \_\_\_\_$
11. $\vec{b} = \_\_\_\_ - \_\_\_\_$
12. $\_\_\_\_ + \vec{b} = R$
13. $Q = R - \_\_\_\_$
14. $Q = R - (\_\_\_\_ - \_\_\_\_)$
15. $(Q - P) + (R - Q)$ $= \_\_\_\_ - \_\_\_\_$
16. $(P - Q) + (Q - R)$ $= \_\_\_\_ - \_\_\_\_$
17. $(Q + \vec{a}) + (\vec{b} - \vec{a}) = \_\_\_\_$
18. $[Q + (\vec{a} + \vec{b})] - \vec{a} = \_\_\_\_$
19. $[Q - (\vec{a} - \vec{b})] + \vec{a} = \_\_\_\_$
20. $Q + \vec{b} = \_\_\_\_$

Part B

1. Mark three noncollinear points A, B, and C and draw arrows from C to describe the translations $A - C$ and $B - C$.
2. Draw an arrow from B to describe the inverse of $A - C$ [that is, to describe $-(A - C)$].
3. Using your ruler only as a straightedge, draw arrows to describe **(a)** $(B - C) - (A - C)$ **(b)** $B - A$
4. Prove:

‖ Theorem 3-7 $(B - C) - (A - C) = B - A$

[*Hint*: By Definition 3(_?_), $(B - C) - (A - C) = ?$ So, by Theorem 3-5(a), $(B - C) - (A - C) = ?$ So, by Postulate 4_4, ...]

3.04 Postulate 4 and Definition 3-1(b)

At this stage in developing our algebra of points and translations we have some postulates which refer explicitly to points [Which are they?] and some which deal only with translations. These last are the parts of Postulate 4. We also have a definition which deals only with translations.

4_0. (a) $\vec{a} + \vec{b} \in \mathcal{T}$ (b) $\vec{0} \in \mathcal{T}$ (c) $-\vec{a} \in \mathcal{T}$
4_1. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
4_2. $\vec{a} + \vec{0} = \vec{a}$
4_3. $\vec{a} + -\vec{a} = \vec{0}$
4_4. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
Def. 3-1(b). $\vec{a} - \vec{b} = \vec{a} + -\vec{b}$

3. Here is a sample paragraph-proof of the converse of (✻):

Suppose that $\vec{b} = \vec{c}$. Since $A - \vec{b} = A - \vec{b}$, it follows that $A - \vec{b} = A - \vec{c}$. Hence, if $\vec{b} = \vec{c}$ then $A - \vec{b} = A - \vec{c}$.

4. Here is a sample paragraph-proof:

We know that if $(A - B) + (B - A) = \vec{0}$ then $-(A - B) = B - A$. Since $(A - B) + (B - A) = B - B = \vec{0}$, it follows that $-(A - B) = B - A$.

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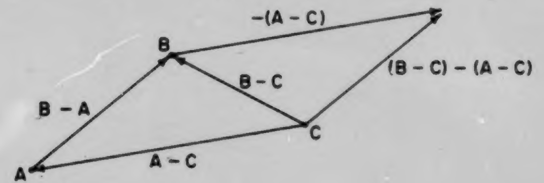
Answers for Part A

1. R, P 2. P, R 3. P, R 4. P, R 5. R
6. R 7. P 8. P 9. $\vec{0}$ 10. $\vec{0}$
11. R, Q 12. Q 13. $\vec{b}$ 14. R, Q [or: $\vec{b}$, $\vec{0}$]
15. R, P 16. P, R 17. R 18. R 19. R 20. R

Answers for Part B

1, 2, 3. The students should have a diagram something like this:



4. By Definition 3-1(b), $(B - C) - (A - C) = (B - C) + -(A - C)$. So, by Theorem 3-5(a), $(B - C) - (A - C) = (B - C) + (C - A)$. So, by Postulate 4_4, $(B - C) - (A - C) = (C - A) + (B - C)$. Hence, by Postulate 3, $(B - C) - (A - C) = B - A$.

The postulates which refer explicitly to points are Postulates 1, 2, and 3. These postulates show how translations act on points. All our remaining postulates will, like the parts of Postulate 4, make no reference to points. [Exceptions to this statement are various definitions of geometric figures.] This dichotomy is one of the important characteristics of this development of geometry. Geometry is seen as the theory of how a set $\mathcal{T}$ of mappings which itself has a certain structure operates on a set $\mathcal{E}$ of points. How the mappings operate on points is described in our first three postulates; the later postulates describe the structure of the set of mappings; the definitions referred to above impose a structure on $\mathcal{E}$ analogous to the structure of $\mathcal{T}$.

The purpose of the present section is to firm-up the analogy between addition of translations and addition of real numbers which students will have discovered in doing the exercises of Part C on page 122. This analogy is generalized in section 3.05.

In your previous study of the algebra of real numbers you have probably come across similar sentences:

(i) $(a + b) + c = a + (b + c)$
(ii) $a + 0 = a$
(iii) $a + -a = 0$
(iv) $a + b = b + a$
(v) $a - b = a + -b$

Depending on how algebra was developed in your course, you may or may not have seen sentences analogous to those in Postulate 4_0:

(o) (a) $a + b \in \mathcal{R}$ (b) $0 \in \mathcal{R}$ (c) $-a \in \mathcal{R}$

Also, instead of using variables, as we are doing, to express generalities, you may have used quantifiers. [For example, instead of using '$a + b = b + a$' to express the commutativity of addition of real numbers, you may have written '$\forall_x \forall_y\, x + y = y + x$'.]

You will also have learned that the properties of addition, 0, oppositing, and subtraction which are stated in (i)–(v) are in some sense basic ones. From these sentences it is possible to derive a great variety of others. In fact, "most" true sentences about the real number 0 and about addition, oppositing, and subtraction of real numbers are consequences of sentences (i)–(v). [One which isn't is '$a + a = 0 \Longrightarrow a = 0$'. Others which you wouldn't expect to be are sentences like '$2 + 2 = 4$' which deal with special properties of particular real numbers.] As an example, you might derive:

$$(a - b) + b = a$$

from (i)–(v) as follows:

$(a - b) + b = (a + -b) + b$ [by (v)]
$(a + -b) + b = a + (-b + b)$ [by (i)]
$a + (-b + b) = a + (b + -b)$ [by (iv)]
$a + (b + -b) = a + 0$ [by (iii)]
$a + 0 = a$ [by (ii)]
So, $(a - b) + b = a$.

[The comment '[by (v)]', for example, refers to the fact that the inference:

$$\frac{a - b = a + -b \quad (a - b) + b = (a - b) + b}{(a - b) + b = (a + -b) + b}$$

is valid [By what rule?], and that its first premiss is an instance of (v), its second premiss is a valid sentence, and its conclusion is the sentence in question. Whether or not you require principles like those in (o) depends on whether or not you insist on being able to prove that sentences like '$(a - b) + b = (a - b) + b$' are valid sentences.]

You can, if you wish, show that '$a + a = 0 \Longrightarrow a = 0$' is not deducible from (o)-(v) by giving a different interpretation to these sentences — one in which (o)-(v) are still true, but the sentence in question is false. One such is to take for $\mathcal{R}$ the set whose members are the two properties odd[ness] and even[ness] of real integers, and take '0' as a name for the latter property. Then, recalling that a sum of two odd numbers is even, etc., it is natural to define addition in the set {odd, even} by:

even + even = even, even + odd = odd,
odd + even = odd, odd + odd = even

Define oppositing by:

−even = odd, −odd = even

and define subtraction so that (v) is satisfied. It is now easy to show that (o)-(iv) are also true but, since odd + odd = even = 0 and odd ≠ 0, '$a + a = 0 \Longrightarrow a = 0$' is not. [You might note, in contrast to this last, that '$a + a = a \Longrightarrow a = 0$' is true. In fact, it has to be because it is a consequence of (o)-(v).]

In order for sentences like '$2 + 2 = 4$' to be theorems we need to add some additional postulates. For example, we might add a new part to (o):

$$1 \in \mathcal{R}$$

and we might then add definitions, '$2 = 1 + 1$', '$3 = 2 + 1$', '$4 = 3 + 1$', etc.

The rule of logic asked for in connection with the displayed inference is the replacement rule for equations.

The quick way to obtain a proof for '$(\vec{a} - \vec{b}) + \vec{b} = \vec{a}$' is, of course, to write arrows over the letters in the given proof of the analogous theorem for real numbers. [The comments would be not quite right — for example, '(v)' should be changed to 'Definition 3(b)' — but, these are comments on the proof rather than a part of it.]

It is probably obvious that there is now a very easy way of showing that the following sentence about translations:

$$(\vec{a} - \vec{b}) + \vec{b} = \vec{a}$$

is a consequence of Postulate 4 and Definition 3-1(b). If you were allowed to write in this book, how could you show this?

On the basis of this example it is easy to generalize as follows:

> Given any sentence about real numbers which can be derived from the sentences (o)-(v), the corresponding sentence about translations can be derived [in exactly the same way] from 4_0-4_4 and Definition 3-1(b).

In your study of the algebra of real numbers you may have called (v) a postulate, or you may have called it a definition of subtraction. As the sample proof we have just given shows, it doesn't matter at all which you did. Definitions and postulates are used in exactly the same ways. In fact, a definition is just a special sort of postulate. What is special about a definition is that it merely introduces a new way of saying things which you can say already. In theory, you can get along without any definitions. In practice, it doesn't work out that way. Think, for example, of the inconvenience of writing '+ –' every time you would, now, write '-'; or of the inconvenience of writing:

> a four-sided figure whose angles are right angles and whose sides are congruent

instead of 'a square'.

Since definitions are, really, postulates, and since a theorem is a sentence which is a consequence of our postulates, any consequence of our postulates and definitions is a theorem. [The phrase 'postulates and definitions' is just a redundant way of saying 'postulates'.] Consequently,

> given any sentence about real numbers which can be derived from the sentences (o)-(v), the corresponding sentence about translations is a theorem of our algebra of points and translations.

All at once, we have an abundance of theorems. [You have already proved some of them in Part C on page 122. Of course, not all the theorems you proved there are of this special kind.] Since, from your previous study of algebra you probably have a good idea of what can be deduced from (o)-(v), we shall not list, or assign numbers to, any of the theorems which are consequences of Postulate 4 and Definition

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If you have time, you should get students to recall the "translations of $\mathcal{R}$" which were discussed in section 1.02. These are the prototypes of the translations we have been studying and, with '$\mathcal{R}$' for '$\mathcal{E}$', 4_0 - 4_4 and Definition 3-1(b) are obviously satisfied when the operators are interpreted as we do interpret them for translations. On the other hand, there is an obvious one-to-one correspondence between translations of $\mathcal{R}$ and real numbers, under which

> the translation $x \to x + a$ corresponds with the real number a.

In this correspondence, the translation $\vec{0}$ corresponds with the real number 0, the inverse of a translation corresponds with the opposite of the number which corresponds to the translation itself, and the number corresponding with a resultant of translations is the sum of the numbers corresponding with the translations themselves. [We say, then, that the correspondence is an isomorphism between the addition-oppositing structure on the set of real numbers and the composition-inversing structure on the set of translations of $\mathcal{R}$.] Because of this correspondence, there is a complete analogy between the addition-oppositing properties of real numbers and the composition-inversing properties of translations of $\mathcal{R}$. More precisely, to each true statement concerning addition and oppositing of real numbers there corresponds [in the obvious way] a true statement about composition and inversion of translations of $\mathcal{R}$, and vice versa.

As the algebra of translations is further developed in later chapters, we shall see further analogies between this algebra and that of the real numbers.

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Answers for Part A

1, 2, 3, 4, 7, 8, 10, 11, 12, 14, and 15 are consequences of Postulate 4 and Definition 3-1(b).

6, 9, 13, and 16 are false, and so cannot be consequences of our postulates. 5, although true, is a consequence of Postulates 1, 2, and 4 and Definition 3-1(b), and therefore is not a consequence of Postulate 4 and Definition 3-1(b) alone.

Here are proofs for the true sentences in Part A. Again, it is not necessary for each student to do all of these derivations.

Proof of 1. $\vec{0} + \vec{a} = \vec{a} + \vec{0} = \vec{a}$. [by Postulates 4_0(a), 4_0(b), 4_4, and 4_2]

Proof of 2. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) = \vec{a} + (\vec{c} + \vec{b}) = (\vec{a} + \vec{c}) + \vec{b}$.

Proof of 3. Suppose that $\vec{c} + \vec{b} = \vec{a}$. Since $\vec{a} + -\vec{b} = \vec{a} + -\vec{b}$, $(\vec{c} + \vec{b}) + -\vec{b} = \vec{a} + -\vec{b}$. Since $(\vec{c} + \vec{b}) + -\vec{b} = \vec{c} + (\vec{b} + -\vec{b}) = \vec{c} + \vec{0} = \vec{c}$ and $\vec{a} + -\vec{b} = \vec{a} - \vec{b}$, it follows that $\vec{c} = \vec{a} - \vec{b}$. Hence, if $\vec{c} + \vec{b} = \vec{a}$ then $\vec{c} = \vec{a} - \vec{b}$.

Proof of 4. $(\vec{a} + \vec{b}) + (\vec{c} + \vec{d}) = [(\vec{a} + \vec{b}) + \vec{c}] + \vec{d} = [\vec{a} + (\vec{b} + \vec{c})] + \vec{d} = [\vec{a} + (\vec{c} + \vec{b})] + \vec{d} = [(\vec{a} + \vec{c}) + \vec{b}] + \vec{d} = (\vec{a} + \vec{c}) + (\vec{b} + \vec{d})$.

Proof of 5. $(A + \vec{b}) + \vec{c} = A + (\vec{b} + \vec{c}) = A + (\vec{c} + \vec{b}) = (A + \vec{c}) + \vec{b}$.

3-1(b) alone. Except when you are asked to prove such a theorem you may merely write down any one you happen to need and comment '[Postulate 4]'. [Doing this does, however, commit you to proving the theorem in case you are asked to.]

Exercises

Part A

Which of the following sentences are consequences of Postulate 4 and Definition 3-1(b)?

1. $\vec{0} + \vec{a} = \vec{a}$
2. $(\vec{a} + \vec{b}) + \vec{c} = (\vec{a} + \vec{c}) + \vec{b}$
3. $\vec{c} + \vec{b} = \vec{a} \longrightarrow \vec{c} = \vec{a} - \vec{b}$
4. $(\vec{a} + \vec{b}) + (\vec{c} + \vec{d}) = (\vec{a} + \vec{c}) + (\vec{b} + \vec{d})$
5. $(A + \vec{b}) + \vec{c} = (A + \vec{c}) + \vec{b}$
6. $\vec{a} - (\vec{b} - \vec{c}) = \vec{a} - \vec{b} - \vec{c}$
7. $\vec{a} + \vec{a} = \vec{a} \longrightarrow \vec{a} = \vec{0}$
8. $(\vec{a} - \vec{b}) - \vec{c} = (\vec{a} - \vec{c}) - \vec{b}$
9. $\vec{a} + \vec{b} = \vec{0} \longrightarrow \vec{a} = \vec{b}$
10. $(\vec{a} - \vec{b}) - (\vec{c} - \vec{d}) = (\vec{a} - \vec{c}) - (\vec{b} - \vec{d})$
11. $-\vec{0} = \vec{0}$
12. $(\vec{a} - \vec{b}) + (\vec{c} - \vec{d}) = (\vec{a} + \vec{c}) - (\vec{b} + \vec{d})$
13. $\vec{a} + \vec{b} = \vec{0} \longrightarrow (\vec{a} = \vec{0} \text{ and } \vec{b} = \vec{0})$
14. $(\vec{a} - \vec{b}) - (\vec{c} - \vec{d}) = (\vec{a} + \vec{d}) - (\vec{c} + \vec{b})$
15. $(\vec{a} - \vec{c}) - (\vec{b} - \vec{c}) = \vec{a} - \vec{b}$
16. $(\vec{a} - \vec{c}) + (\vec{b} - \vec{c}) = \vec{a} + \vec{b}$

Part B

By Postulate 1(a), the sentence:

(*) $\qquad (C - A) + (B - C) = (B - C) + (C - A)$

is a substitution-instance of Postulate 4_4. So, (*) is a consequence of Postulate 4_4. With this example in mind, tell which of the following sentences are consequences of Postulate 4 and Definition 3-1(b).

1. $[\vec{a} + (B - C)] - (B - C) = \vec{a}$
2. $-[(P - Q) - \vec{a}] = \vec{a} - (P - Q)$
3. $(A - B) + \vec{a} = (A + \vec{a}) - B$
4. $[S + (R - S)] + (S - R) = R$
5. $A + [(B - A) - \vec{a}] = B + \vec{a}$
6. $\vec{a} + [(B - A) - \vec{a}] = B - A$

Proof of 7. Suppose that $\vec{a} + \vec{a} = \vec{a}$. Since $\vec{a} + -\vec{a} = \vec{0}$, it follows that $(\vec{a} + \vec{a}) + -\vec{a} = \vec{0}$. Now, $(\vec{a} + \vec{a}) + -\vec{a} = \vec{a} + (\vec{a} + -\vec{a}) = \vec{a} + \vec{0} = \vec{a}$, so that $\vec{a} = \vec{0}$. Hence, if $\vec{a} + \vec{a} = \vec{a}$ then $\vec{a} = \vec{0}$.

Proof of 8. $(\vec{a} - \vec{b}) - \vec{c} = (\vec{a} + -\vec{b}) + -\vec{c} = \vec{a} + (-\vec{b} + -\vec{c}) = \vec{a} + (-\vec{c} + -\vec{b}) = (\vec{a} + -\vec{c}) + -\vec{b} = (\vec{a} - \vec{c}) - \vec{b}$.

Proof of 10. $(\vec{a} - \vec{b}) - (\vec{c} - \vec{d}) = (\vec{a} + -\vec{b}) + -(\vec{c} + -\vec{d}) = (\vec{a} + -\vec{b}) + (-\vec{c} + --\vec{d}) = [(\vec{a} + -\vec{b}) + -\vec{c}] + --\vec{d} = [\vec{a} + (-\vec{b} + -\vec{c})] + --\vec{d} = [\vec{a} + (-\vec{c} + -\vec{b})] + --\vec{d} = [(\vec{a} + -\vec{c}) + -\vec{b}] + --\vec{d} = (\vec{a} + -\vec{c}) + (-\vec{b} + --\vec{d}) = (\vec{a} + -\vec{c}) + -(\vec{b} + -\vec{d}) = (\vec{a} - \vec{c}) - (\vec{b} - \vec{d})$.

Proof of 11. We know that if $\vec{a} + \vec{b} = \vec{0}$ then $-\vec{a} = \vec{b}$ and that $\vec{0} + \vec{0} = \vec{0}$. So, $-\vec{0} = \vec{0}$.

Proof of 12. $(\vec{a} - \vec{b}) + (\vec{c} - \vec{d}) = (\vec{a} + -\vec{b}) + (\vec{c} + -\vec{d}) = (\vec{a} + \vec{c}) + (-\vec{b} + -\vec{d}) = (\vec{a} + \vec{c}) + -(\vec{b} + \vec{d}) = (\vec{a} + \vec{c}) - (\vec{b} + \vec{d})$.

Proof of 14. $(\vec{a} - \vec{b}) - (\vec{c} - \vec{d}) = (\vec{a} + -\vec{b}) + -(\vec{c} + -\vec{d}) = (\vec{a} + -\vec{b}) + (-\vec{c} + \vec{d}) = (\vec{a} + -\vec{b}) + (\vec{d} + -\vec{c}) = (\vec{a} + \vec{d}) + (-\vec{b} + -\vec{c}) = (\vec{a} + \vec{d}) + -(\vec{b} + \vec{c}) = (\vec{a} + \vec{d}) + -(\vec{c} + \vec{b}) = (\vec{a} + \vec{d}) - (\vec{c} + \vec{b})$.

Proof of 15. $(\vec{a} - \vec{c}) - (\vec{b} - \vec{c}) = (\vec{a} + -\vec{c}) + -(\vec{b} + -\vec{c}) = (\vec{a} + -\vec{c}) + (-\vec{b} + --\vec{c}) = (\vec{a} + -\vec{b}) + (-\vec{c} + --\vec{c}) = (\vec{a} + -\vec{b}) + \vec{0} = (\vec{a} + -\vec{b}) = \vec{a} - \vec{b}$.

*

Answers for Part B

1, 2, and 6 are consequences of Postulate 4 and Definition 3-1(b).

3 is true but involves knowing more than the "fact" that differences of points are translations. 4 and 5 are false and, so, cannot be consequences of our postulates.

*

Here are proofs for 1, 2, and 6.

Proof of 1. $[\vec{a} + (B - C)] - (B - C) = [\vec{a} + (B - C)] + -(B - C) = \vec{a} + [(B - C) + -(B - C)] = \vec{a} + \vec{0} = \vec{a}$. [Notice that the only "fact" we need to know about 'B - C' is that its values are translations.]

Proof of 2. We know that if $\vec{a} + \vec{b} = \vec{0}$ then $-\vec{a} = \vec{b}$. In particular, we know that if $[(P - Q) - \vec{a}] + [\vec{a} - (P - Q)] = \vec{0}$ then $-[(P - Q) - \vec{a}] = \vec{a} - (P - Q)$. Now, $[(P - Q) - \vec{a}] + [\vec{a} - (P - Q)] = [(P - Q) + -\vec{a}] + [\vec{a} + -(P - Q)] = [(P - Q) + -(P - Q)] + (\vec{a} + -\vec{a}) = \vec{0} + \vec{0} = \vec{0}$. Hence, $-[(P - Q) - \vec{a}] = \vec{a} - (P - Q)$.

Proof of 6. $\vec{a} + [(B - A) - \vec{a}] = \vec{a} + [(B - A) + -\vec{a}] = (B - A) + (\vec{a} + -\vec{a}) = (B - A) + \vec{0} = B - A$.

Here is a proof of 3. By Theorem 2-1, we know that if $B + [(A - B) + \vec{a}] = A + \vec{a}$ then $(A - B) + \vec{a} = (A + \vec{a}) - B$. Now, $B + [(A - B) + \vec{a}] = [B + (A - B)] + \vec{a} = A + \vec{a}$. Hence, $(A - B) + \vec{a} = (A + \vec{a}) - B$.

Part C

1. The sentence in Exercise 5 of Part A is not a consequence of Postulate 4 and Definition 3-1(b). It does, however, follow from '$\vec{a} + \vec{b} = \vec{b} + \vec{a}$', which is such a consequence, together with Theorem 2-5(b). Show that it does.
2. Write some theorems like that of Exercise 5 which are suggested by Exercises 2, 4, and 8 of Part A.

3.05 Groups

There is more to be said concerning postulates (o) - (iv) for real numbers and Postulate 4 for translations. [Since (v) and Definition 3-1(b) are merely definitions, we shall ignore them for now.] The sentence (o)(a) says that the result of "adding" a first real number and a second real number is, itself, a real number. More briefly,

(o)(a) tells us that '+' refers to a *binary operation* on the set $\mathscr{R}$.
(o)(b) tells us that '0' is a name for some *special member* of $\mathscr{R}$.
(o)(c) tells us that '−' refers to a *singulary operation* on $\mathscr{R}$.
(i)-(iv) say that this binary operation, this special member, and this singulary operation have certain properties. [For example, (i) says that the binary operation is associative.]

On the other hand:

4_0(a) tells us that [in this context] '+' refers to a *binary operation* on $\mathscr{T}$.
4_0(b) tells us that '$\vec{0}$' is a name for some *special member of* $\mathscr{T}$.
4_0(c) tells us that '−' refers to a *singulary operation* on $\mathscr{T}$.
4_1 − 4_4 say that *this* binary operation, *this* special member, and *this* singulary operation have certain properties—

and these are the same properties [associativity, etc.] as are referred to in (i) - (iv).

In both cases we are dealing with

a set [$\mathscr{R}$ or $\mathscr{T}$] on which there is an associative binary operation
[(o)(a) and (i) or 4_0(a) and 4_1] for which there is [in the set] an
(*) identity element [(o)(b) and (ii) or 4_0(b) and 4_2] and [on the set] a
singulary inversing operation [(o)(c) and (iii) or 4_0(c) and 4_3].
Also, the binary operation is commutative [(iv) or 4_4].

The kind of situation described in (*) crops up in very many places in mathematics: One has a set on which there is an [interesting] associative binary operation; the set contains an "identity element" with respect to this binary operation; and each member of the set has an "inverse" with respect to this binary operation and identity element. In many cases—but not in all—the binary operation is commutative. You have already seen one advantage of recognizing this kind of situation—having seen that translations furnish an example, much of

Answers for Part C

1. $(A + \vec{b}) + \vec{c} = A + (\vec{b} + \vec{c}) = A + (\vec{c} + \vec{b}) = (A + \vec{c}) + \vec{b}$ [by Theorem 2-5(b), 4_4, Theorem 2-5(b)]. [In the middle step one uses, besides an instance of 4_4, the valid sentence '$A + (\vec{b} + \vec{c}) = A + (\vec{b} + \vec{c})$' as premisses of a replacement inference.]
2. [Various answers are possible.]

The various ways of referring to a group which are illustrated in the paragraph preceding the exercises are all in common use. According to the first, a group is an ordered quadruple — for example, the additive group of the real numbers is the quadruple $(\mathcal{R}, +, 0, -)$. It is, however, adequate to say that $\mathcal{R}$ is a group with respect to addition — without specifying the appropriate identity element and oppositing operation. For, in saying that addition of real numbers is a group operation one implies that there is a corresponding identity element and a corresponding oppositing relation; and it is not difficult to prove that this element and operation are uniquely determined. [See the previously mentioned commentary for page 293 of High School Mathematics, Course 3.] Using commutativity, it is very easy to show this. To begin with, suppose that 0 and 0′ are both identity elements with respect to a commutative binary operation. Since they are identity elements, $0' + 0 = 0'$ and $0 + 0' = 0$. Since the operation is commutative, $0' = 0$. Hence, a commutative binary operation has at most one identity element. Next, suppose that − and −′ are both inversing operations with respect to a commutative and associative binary operation and an identity element 0. Since − is such an inversing operation it follows [as in Exercise 3 of Part C on page 122] that, for any a and b, if $a + b = 0$ then $-a = b$. Since −′ is such an operation, $a + -'a = 0$. Hence, $-a = -'a$. Consequently, there is at most one identity element and at most one inversing operation associated with a given commutative and associative binary operation.

You may find it worthwhile to illustrate the notion of closure in Part A of the following exercises by a fanciful example. Suppose that a given field is enclosed by a brick wall six inches high. Such a field is not closed to walking since, from any point in it one can by walking, get outside it. It is, however, closed to shuffling. One cannot get outside by shuffling along with his feet remaining in contact with the ground. If it does nothing else, this should point out that it is sets which are closed under operations and not, as some current textbook authors have written, operations which are closed on sets!

what you know about another example [real numbers] can be carried over as knowledge about translations.

Whenever the situation described in (*) occurs one says that the set, together with the binary operation, the identity element, and the inversing operation, is *a commutative group*. More briefly, one may say that the set is a commutative group with respect to the binary operation, and that the binary operation is a commutative *group operation*. For example, $\mathscr{R}$ is a commutative group with respect to addition of real numbers, and $\mathscr{T}$ is a commutative group with respect to composition [or: addition] of translations. Each of these binary operations is a commutative group operation.

Exercises

Part A

We know that addition of real numbers is a binary operation on the set $\mathscr{R}$ of all real numbers. We also know that a sum of positive numbers is a positive number—for short, that the set P of positive numbers is *closed* under addition. Consequently, if we restrict addition to addition of positive numbers, the result is a binary operation on P. Somewhat imprecisely, we can say that addition [of real numbers] is a binary operation on P. On the other hand, a sum of odd numbers is not always an odd number [in fact, it never is]. So, the set of odd numbers is not closed under addition, and addition is not an operation on this set.

For each exercise, tell whether the given set is closed under the given operation. Also, tell whether the operation is binary or singulary.

1. $\mathscr{R}$; subtraction
2. integers; oppositing
3. linear functions; composition
4. rational numbers; multiplication
5. positive rational numbers; square rooting
6. nonzero rational numbers, reciprocating
7. $\{0, 1, -1\}$; multiplication
8. negative numbers; division
9. nonzero real numbers; division
10. negative numbers; addition
11. the set of all functions with domain $\mathscr{R}$ and range contained in $\mathscr{R}$; composition
12. [Same as Exercise 11, but only one-to-one functions whose range is all of $\mathscr{R}$.]

Part B

For some exercises in Part A, the operation is a binary one and the set is closed under it. For each such exercise, tell whether the operation, when restricted to the set,

(a) is associative.

Answers for Part A

1. closed; binary
2. closed; singulary
3. closed; binary
4. closed; binary
5. not closed; singulary
6. closed; singulary
7. closed; binary
8. not closed; binary
9. closed; binary
10. closed; binary
11. closed; binary
12. closed; binary

Answers for Part B

1. (a) No. (b) Yes, 0. (c) Yes, "sameing". (d) No.
2. [not binary]
3. (a) Yes. (b) Yes, $i_{\mathscr{R}}$. (c) Yes, inversing. (d) No.

[So, the set of all linear functions is a noncommutative group with respect to composition. Students may recall that the set of all such functions which leave a given point fixed is a commutative group with respect to composition. This group is, in any case, isomorphic with the multiplicative group of nonzero real numbers — a fact which is very easy to see when the fixed point is 0.]

4. (a) Yes. (b) Yes, 1. (c) No. (d) Yes.

[A number is rational if and only if it is the quotient of an integer by a nonzero integer. Replacing the second 'is', above, by 'can be represented by' makes nonsense. In discussing part (c) it should come out that had the exercise referred to the nonzero rationals then the answer for (c) would have been 'Yes, reciprocating.' and the other answers would have been unchanged. The nonzero rationals form a commutative group with respect to multiplication.]

5. [not binary]
6. [not binary]
7. (a) Yes. (b) Yes, 1. (c) No. (d) Yes.
8. [Not closed.]
9. (a) No. (b) Yes, 1. (c) Yes, "sameing." (d) No.
10. (a) Yes. (b) No. (c) No. (d) Yes.
11. (a) Yes. (b) Yes, $i_{\mathscr{R}}$. (c) No. (d) No.
12. (a) Yes. (b) Yes, $i_{\mathscr{R}}$. (c) Yes, inversing. (d) No.

[So, the set of all one-to-one mappings of $\mathscr{R}$ — or, of any set — onto itself is a noncommutative group with respect to composition.]

(b) has an identity element. [If so, what is it?]
(c) has a corresponding inversing operation.
(d) is commutative.

Part C

Which of the following sets is a commutative group with respect to the given operation?

1. real numbers; subtraction
2. real numbers; addition
3. real numbers; multiplication
4. nonzero real numbers; multiplication
5. integers; addition
6. integers; multiplication
7. nonnegative integers; addition
8. rational numbers; multiplication
9. rational numbers; addition
10. nonzero rational numbers; multiplication

*

In formulating Postulate 4 we included some sentences which we could derive from Postulates 1 – 3, and some sentences which we could derive from other parts of Postulate 4. The principle reason for doing so was in order to be able to restate this postulate as:

> **Postulate $4'''$** $\mathcal{T}$ is a commutative group with respect to composition.

['$4'''$' because we shall add more parts to this postulate, eventually coming out with a 'Postulate 4'.] Since, for any commutative group, we can define subtraction as in Definition 3 – 1(b) or (v), we shall think of Definition 3 – 1(b) as being a part of Postulate $4'''$. So, for example, when we say that a certain theorem is a consequence of Postulate $4'''$, this means that it is a consequence of $4_0 - 4_4$ and Definition 3 – 1(b).

3.06 Other Theorems about Points and Translations

We know that the sentence:

$$(b - c) - (a - c) = b - a \tag{1}$$

is a true sentence about real numbers just because addition of real numbers is a commutative group operation. So, since addition of translations is a commutative group operation, we know that the sentence:

$$(\vec{b} - \vec{c}) - (\vec{a} - \vec{c}) = \vec{b} - \vec{a} \tag{2}$$

Answers for Part C

The given operation is a commutative group operation on the given set in Exercises 2, 4, 5, 7, 9, and 10.

In Chapter 5, Postulate $4'''$ is strengthened to Postulate $4''$ by replacing 'commutative group with respect to composition' by 'vector space over the real numbers'. Postulate $4'$ is obtained by prefixing '3-dimensional' to this last phrase. Postulate 4 is obtained by replacing 'vector' by 'inner product'. Strictly speaking, geometry begins with Postulate $4''$, and Euclidean geometry only with Postulate 4.

The following item was designed to test the student's understanding of the postulates for a commutative group. It was part of a chapter test administered to experimental classes and was handled with apparent ease by the students. You might use it as a quiz or as a take-home exercise.

> Suppose that $(Z = \{a, b\}, *, b, \sim)$ is a commutative group with distinct elements a and b. The following are sentences about this commutative group. If a given sentence is a theorem, write 'T' in the space provided. If it is not a theorem, write 'N' in the space.
>
> ___ 1. $\sim b \in Z$
> ___ 2. $(a * b) * a = (a * a) * b$
> ___ 3. $a * \sim b = b * \sim a$
> ___ 4. $a * \sim a = b$
> ___ 5. $\sim a * a = b * b$
> ___ 6. $a * b \in Z$
> ___ 7. $\sim a * (a * \sim b) = \sim b$
> ___ 8. $a * 0 = a$
> ___ 9. $a * a = a$
> ___ 10. $\sim(a * \sim a) = \sim b * b$
> ___ 11. $b * b = a$
> ___ 12. $\sim(a * a) = \sim a * \sim a$

Answers. 1. T 2. T 3. N 4. T
5. T 6. T 7. T 8. N
9. N 10. T 11. N 12. T

This section 3.06 is preparatory to section 3.07. In particular, pay attention to the manner in which this section leads up to the result which is stated on page 136.

is a theorem about translations. You have also shown that:

$$(3) \qquad (B - C) - (A - C) = B - A$$

is a theorem. [It is Theorem 3–7 on page 124.] Although (3) is a theorem and has the same "shape" as (1) and (2), its proof is quite different from that of (1) and (2). This is to be expected since only one of the '–'s in (3) refers to a group "subtraction operation".

As another example, the sentences:

$$(4) \qquad (a - b) + c = (a + c) - b$$

and:

$$(5) \qquad (\vec{a} - \vec{b}) + \vec{c} = (\vec{a} + \vec{c}) - \vec{b}$$

are both true [and the second is a theorem] just because $\mathscr{R}$ and $\mathscr{T}$ are commutative groups with respect to the addition operations in question, and the subtraction operations are the appropriate ones defined by (v) and Definition 3–1(b). In analogy with (3) we may write:

$$(*) \qquad (A - B) + C = (A + C) - B$$

but (*) is certainly not a theorem. [Why 'certainly'?] There is, however, another analogue of (4) and (5):

$$(6) \qquad (A - B) + \vec{c} = (A + \vec{c}) - B$$

This equation has the advantage, over (*), of at least making sense – both sides refer to translations. Sentence (6) might be true and, if so, it might be a theorem. The following figure shows that (6) is true – you couldn't possibly draw a counter-example:

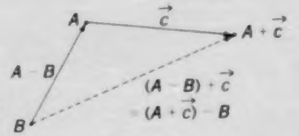


Fig. 3–1

The question as to whether (6) is a theorem is settled almost as easily. [*Hint*: You can show that $(A - B) + \vec{c}$ and $(A + \vec{c}) - B$ are the same translation by finding a point which has the same image under both of them. What is the easiest point to use?] We shall list (6) as 'Theorem 3–8'.

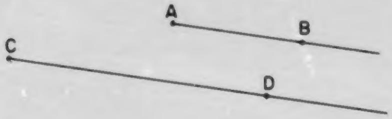
The "easiest point to use" in proving (6) is, fairly obviously, B. By Theorem 2-1, (6) will follow at once from '$B + [(A - B) + \vec{c}] = A + \vec{c}$'; and this is an immediate consequence of Theorem 2-5(b) and Postulate 2(a). Sentence (6) is, for that matter, a consequence of Postulate 2(b) and an instance of Postulate 3. [The instance of Postulate 3 is '$(A - B) + [(A + \vec{c}) - A] = (A + \vec{c}) - B$'.]

*

Quizzes like the following have the dual purposes of reviewing the students over ideas which were developed some weeks past, but which have not been needed during the discussion at hand, and to determine whether students have some feeling for one or more of the topics to be studied in the near future. That this quiz appears on this page should not be interpreted as meaning that it [or one like it] must be administered when students reach page 131 of the text. You will note that the items of the quiz have little local relationship to what is being discussed in the text at this juncture and so the quiz could be given at any time beyond midchapter.

Sample Quiz

Here is a diagram of parallel rays $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ with the same sense.



Suppose that $\overleftrightarrow{CD}$ is twice as long as $\overleftrightarrow{AB}$, that $\vec{b} = B - A$, and that $\vec{c} = C - A$. Which of the following are true (T) and which are false (F)?

___ 1. $\vec{b} + \vec{b} = D - C$
___ 2. D is the image of B under $\vec{c} + \vec{c}$.
___ 3. $D - B = \vec{c} - \vec{b}$
___ 4. $B + \vec{c}$ is a point on the segment $\overleftrightarrow{CD}$.
___ 5. There is a translation which maps $\overleftrightarrow{CD}$ onto $\overleftrightarrow{AB}$.
___ 6. The lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ have a point in common.
___ 7. There is a translation that maps line $\overleftrightarrow{AB}$ onto line $\overleftrightarrow{CD}$.
___ 8. The image of C under $\vec{b}$ is a point on line $\overleftrightarrow{CD}$.
___ 9. The lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ do not have a point in common.
___10. $A - D = -(\vec{b} + \vec{c} + \vec{b})$

Answers for Sample Quiz

1. T 2. F 3. F 4. T 5. F
6. T 7. T 8. T 9. F 10. T

Notice that the sentence:

$$(A - B) + -\vec{c} = (A + -\vec{c}) - B$$

is an instance of (6) [because '$-\vec{a} \in \mathscr{T}$' is a postulate] and that this is equivalent to:

$$(6') \qquad (A - B) - \vec{c} = (A - \vec{c}) - B$$

[Why?] Since (6) is a theorem, so is (6′). [Similarly, starting from (6′) it is easy to derive (6).] We shall not bother to list separately theorems which, like (6) and (6′) differ only in that one involves adding a translation and the other involves subtracting it. If you should need to mention (6′) you may do so by writing 'Theorem 3-8', just as you will for (6).

(3), (*), and (6) are alike in that if the variables in any of them are replaced by real number-variables then the result is a sentence about real numbers which is true just because $\mathscr{R}$ is a group with respect to addition of real numbers.

(3) and (6) differ from (*) in that (3) and (6) "make sense"—that is, they are *sentences* of our algebra of points and translations—but (*) doesn't.

(3) and (6) not only make sense—they are consequences of Postulates 1 through 4‴ and, so, are theorems. [(*), of course, is not a theorem—it's not even a sentence.]

Exercises

Part A

In each of the following exercises you are given an equation of the algebra of points and translations. For each equation:

(a) Check to see that it really is a sentence, tell what kind of thing—points or translations—its sides refer to.

(b) Write out a corresponding sentence about real numbers, and decide whether you could derive *this* sentence from (o) – (v) on page 125.

(c) Draw a figure which illustrates the given sentence, and decide whether you think that this sentence is true.

1. $A + (B - C) = B + (A - C)$ **2.** $(A - B) + \vec{c} = A - (B - \vec{c})$

3. $(A - B) - (C - D) = (A - C) - (B - D)$ [*Hint*: You can draw several quite different looking pictures to illustrate this sentence, depending on how you choose to mark the points *A*, *B*, *C*, and *D*. One easy choice is like this:

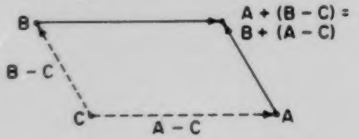
Complete a figure like this for the sentence, and then draw one or two other pictures as different from this one as you can.]

Answer for 'Why?' following (6′): Definition 3-1(b) and Definition 3-1(a).

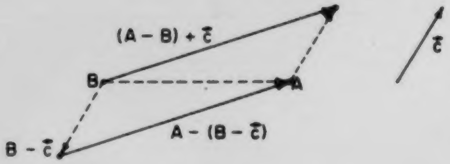
Answers for Part A

In each of the three exercises the expression given is a sentence and the corresponding real number sentences ['$a + (b - c) = b + (a - c)$', '$(a - b) + c = a - (b - c)$', '$(a - b) - (c - d) = (a - c) - (b - d)$'] is derivable from (o)-(v). In Exercise 1, both sides of the equation refer to points; in Exercises 2 and 3, both sides refer to translations. Possible figures illustrating these equations are:

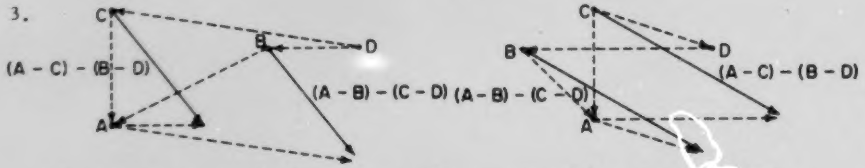
1.



2.



3.



[Use differently colored chalk for the constructions of values of the two sides of the equations. Interchanging the points B and D, or choosing $\overleftrightarrow{BD} \parallel \overleftrightarrow{AC}$ yields rather different figures.]

Part B

Prove:

1. Theorem 3–8. $(A - B) + \vec{c} = (A + \vec{c}) - B$ [*Hint:* See the discussion of (6) on page 131.]
2. Theorem 3–9. $A + (B - C) = B + (A - C)$ [*Hint:* You can show that the point $A + (B - C)$ is the same as the point $B + (A - C)$ by showing that both have the same image under some translation. [What theorem?] Try using the translation $C - B$.]
3. Theorem 3–10. $(A - B) + \vec{c} = A - (B - \vec{c})$ [*Hint:* It's easy to find the image of B under the translation $(A - B) + \vec{c}$. You can find the image of B under $A - (B - \vec{c})$ by using Theorem 3–9, Theorem 3–5(b), and a theorem about translation alone.]
4. Theorem 3–11. $\vec{a} - (B - C) = C - (B - \vec{a})$ [*Hint*: The right side of this equation is rather like that of Theorem 3–10. Can you transform the left side so that you can use Theorem 3–10?]
5. Theorem 3–12. $A - (B - C) = C - (B - A)$ [*Hint:* Use Theorem 3–9.]
6. Theorem 3–13. $(A - B) - (C - D) = (A - C) - (B - D)$ [*Hint:* Use Theorem 3–11 and Theorem 3–12.]
7. Corollary. $A - B = C - D \Longrightarrow A - C = B - D$ [A *corollary* is a a theorem which is suggested by the theorem preceding it and whose proof uses "mostly" the preceding theorem. *Hint*: Note that $\vec{a} = \vec{b} \Longleftrightarrow \vec{a} - \vec{b} = \vec{0}$ is a theorem.]
8. Derive Theorem 3–4 from the theorem of Exercise 7 and Theorems 3–1(b) and 3–2(b).

Part C

In each of the following, fill the blanks so that the resulting statement is a theorem. If it isn't possible to complete a given sentence to make a theorem, explain why.

1. $[A + (\vec{a} + \vec{b})] + \vec{c} = (A + \vec{a}) + (\underline{\qquad\qquad})$
2. $(A - \vec{a}) + \vec{b} = (A + \vec{b}) - \underline{\qquad\qquad}$
3. $\vec{a} = -\vec{b} \Longleftrightarrow -\vec{a} = \underline{\qquad\qquad}$
4. $A - \vec{a} = A + \vec{b} \Longleftrightarrow \vec{a} + \vec{b} = \underline{\qquad\qquad}$
5. $C - A = C - B \Longleftrightarrow \underline{\qquad\qquad}$
6. $C + A = C + B \Longleftrightarrow \underline{\qquad\qquad}$
7. $A - (B + \vec{c}) = (A - \underline{\qquad\qquad}) - \vec{c}$
8. $A + (B + C) = \underline{\qquad\qquad} + (A + C)$
9. $(A - \vec{b}) - (\vec{c} - \vec{d}) = (A - C) - (\underline{\qquad\qquad} - \underline{\qquad\qquad})$
10. $\vec{a} + A = A + \underline{\qquad\qquad}$

Answers for Part B

In order to expedite the discussion of Part B you may wish to assign each exercise to a team of students rather than have each student do each exercise.

1. By Theorem 2-5(b), $B + [(A - B) + \vec{c}] = [B + (A - B)] + \vec{c} = A + \vec{c}$, the last by Postulate 2(a). So, by Theorem 2-1, $(A - B) + \vec{c} = (A + \vec{c}) - B$. [Alternative proof: By Postulate 3, $(A - B) + [(A + \vec{c}) - A] = (A + \vec{c}) - B$. So, by Postulate 2(b), $(A - B) + \vec{c} = (A + \vec{c}) - B$.]

2. By Theorem 2-5(b), $[A + (B - C)] + (C - B) = [A + (B - C)] + -(B - C) = A$, by Theorems 3-5(a) and 3-6. On the other hand, $[B + (A - C)] + (C - B)] = B + [(A - C) + (C - B)] = B + [(C - B) + (A - C)] = B + (A - B) = A$. So, by Theorem 3-3, $A + (B - C) = B + (A - C)$.

3. $B + [(A - B) + \vec{c}] = [B + (A - B)] + \vec{c} = A + \vec{c}$. $B + [A - (B - \vec{c})] = A + [B - (B - \vec{c})]$, by Theorem 3-9 [of Exercise 2]. But, $B - (B - \vec{c}) = B - (B + -\vec{c}) = --\vec{c} = \vec{c}$, by Definition 3-1(a), Theorem 3-5(b), and the theorem of Exercise 6(b) of Part C on page 155. So, $B + [A - (B - \vec{c})] = A + \vec{c}$. So, by Theorem 2-2, $(A - B) + \vec{c} = A - (B - \vec{c})$.

4. By Definition 3-1(b), Theorem 3-5(a), and Postulate 4_4, $\vec{a} - (B - C) = \vec{a} + -(B - C) = \vec{a} + (C - B) = (C - B) + \vec{a}$. By Theorem 3-10, $(C - B) + \vec{a} = C - (B - \vec{a})$. So, $\vec{a} - (B - C) = C - (B - \vec{a})$. [Students should draw figures illustration Theorem 3-11. In fact, it should be standard procedure to illustrate all theorems by figures.]

5. $A - (B - C) = A + -(B - C) = A + (C - B) = C + (A - B) = C + -(B - A) = C - (B - A)$ [Definition 3-1(a), Theorem 3-5(a), Theorem 3-9, Theorem 3-5(a), Definition 3-1(a)]

6. $(A - B) - (C - D) = D - [C - (A - B)] = D - [B - (A - C)] = (A - C) - (D - B)$ [Theorem 3-11, Theorem 3-12, Theorem 3-11]

7. $A - B = C - D$ if and only if $(A - B) - (C - D) = \vec{0}$ [by the theorem quoted in the hint]. By Theorem 3-13, $(A - B) - (C - D) = \vec{0}$ if and only if $(A - C) - (B - D) = \vec{0}$. By the theorem quoted in the hint, $(A - C) - (B - D) = \vec{0}$ if and only if $A - C = B - D$. Hence, $A - B = C - D$ if and only if $A - C = B - D$.

   [Intuitively, the corollary to Theorem 3-13 includes the statement that if two opposite sides of a quadrilateral are parallel and of the same length then so are the other two opposite sides.]

8. By the corollary, $C - A = C - B$ if and only if $C - C = A - B$. So, by Theorem 3-1(b) and 3-2(b), $C - A = C - B$ if and only if $A = B$.

Answers for Part C

1. $\vec{b} + \vec{c}$ [or: $\vec{c} + \vec{b}$]
2. $\vec{a}$
3. $\vec{b}$
4. $\vec{0}$ [or: $A - A$]
5. $A = B$
6. Not possible. [No completions will make this a sentence for the antecedent of this biconditional is not an equation — '$C + A$' and '$C + B$' are not terms.]
7. B
8. Not possible. [The left expression is not a term, and the right side cannot be made into a term by any fill-in.]
9. Not possible. [The left expression has values which are points while the right expression will have values which are translations for any sensible fill-ins.]
10. Not possible. [The left expression is nonsense.]

* * *

There are, of course, infinitely many theorems like Theorems 3-8 through 3-13 and its corollary; and it is not difficult to find among them theorems which — especially when illustrated by figures — suggest interesting properties of geometric figures. The principal purpose of the preceding exercises is, however, to prepare students for the result stated on page 136. This result [which is proved on TC 136(1-4)] gives a general view of all theorems which are consequences of Postulates 1 - 4‴ and renders it unnecessary to prove any additional theorems of this kind. [Of course, there will be ample need for proofs once we have adopted additional postulates.]

Before continuing on to section 3.07, however, it may be worthwhile to add some remarks concerning an alternative set of postulates for commutative groups and a related alternative to Postulates 1 - 4‴. We begin by recalling that, given any commutative group, it is possible to define a second binary operation à la Definition 3-1(a). Taking as our example the additive group of the real numbers, one can, in terms of the group operator '+' and the inversing operator '−', define a subtraction operator '-' by:

$$a - b = a + -b$$

On the other hand, if one were acquainted only with subtraction, we could define both oppositing and addition by:

$$-a = (a - a) - a, \quad a + b = a - -b$$

In some sense, then, subtraction is more basic than is addition. In fact, it can be shown that instead of adopting (o)-(v) of page 158 to describe the additive group structure of the real numbers we might equally well adopt the two definitions for oppositing and addition, a closure postulate '$a - b \in \mathcal{R}$' and the following two postulates concerning subtraction:

$$(\text{☆}) \qquad a - (b - c) = c - (b - a), \quad a - (a - b) = b$$

From these two postulates we can derive '$a - (b - b) = a$' [$a - (b - b) = b - (b - a) = a$] and, using this theorem, '$a - a = b - b$' [$a - a = a - [a - (b - b)] = b - b$]. This last is analogous to (iii) on TC 120(1).

and, as pointed out in the ensuing discussion, it follows that the adoption of '$0 = a - a$' as a postulate does no more than introduce a convenient name for the unique value of the expression '$a - a$'. Adopting this quasi-definition, our first theorem yields '$a - 0 = a$', whence, by the definition of '−', $-0 = (0 - 0) - 0 = 0 - 0 = 0$ and, by the definition of '+', $a + 0 = a - -0 = a - 0 = a$. Also, since $-a = 0 - a$, $--a = 0 - (0 - a) = a - (0 - 0) = a - 0 = a$, whence $a + -a = a - --a = a - a = 0$. Finally, addition, as defined in terms of subtraction, can be shown to be associative and commutative, but we shall not go into the details of this.

As a consequence of the preceding it is fair to say that the fact that the real numbers form a commutative group with respect to addition is expressed by the pair of sentences (☆) [together, if one insists on closure postulates, with '$a - b \in \mathcal{R}$']. It follows then that Postulate 4‴ might be replaced by the corresponding pair of sentences:

$$(\text{I}) \qquad \vec{a} - (\vec{b} - \vec{c}) = \vec{c} - (\vec{b} - \vec{a}), \quad \vec{a} - (\vec{a} - \vec{b}) = \vec{b}$$

Now, the following analogues of the sentences (I) are theorems:

$$(\text{II}) \qquad A - (B - \vec{c}) = \vec{c} - (B - A), \quad A - (A - \vec{b}) = \vec{b}$$

$$(\text{III}) \qquad A - (B - C) = C - (B - A), \quad A - (A - B) = B$$

[The first sentences in these pairs are Theorems 3-11 and 3-12; the second sentences are easily proved.] And, as it turns out, (I) - (III), when augmented by the closure postulates:

$$B - A \in \mathcal{T}, \quad \vec{b} - \vec{a} \in \mathcal{T}, \quad A - \vec{a} \in \mathcal{E}$$

and the definitions:

$$-\vec{a} = (\vec{a} - \vec{a}) - \vec{a}, \quad \vec{a} + \vec{b} = \vec{a} - -\vec{b}, \quad A + \vec{a} = A - -\vec{a}$$

are entirely equivalent to Postulates 1 - 4‴ and Definition 3-1. [''entirely equivalent'' except that the latter basis introduces the name '$\vec{0}$' for the identity mapping of $\mathcal{E}$ onto itself. But, from (II) it follows that $\vec{a} = A - (A - \vec{a}) = \vec{a} - (A - A)$ whence, $\vec{a} - \vec{a} = \vec{a} - [\vec{a} - (A - A)] = A - A$, by the second sentence in (I). Consequently, '$\vec{0}$' may be introduced merely as a convenient name for the common unique value of the expressions '$\vec{a} - \vec{a}$' and '$A - A$'.]

The symmetry of this alternative postulational basis bears testimony to the simplicity of our algebra of points and translations as it has been developed up to now. It should, however, be noted that this symmetry exhibited by (I) - (III) may be misleading. While (I) expresses the fact that the subtraction operation there referred to is the subtraction operation in a commutative group, similar remarks do not apply to (II) and (III). Each of these deals with two subtraction operations. And, neither oppositing nor addition of points can be defined by analogy with the definitions of oppositing and addition of translations.

*

3.07 A Bargain in Theorems

Up to now we have adopted four postulates – Postulates 1, 2, 3, and $4'''$ – and one definition – Definition 3-1 – and have shown that these imply a considerable variety of theorems. A large number of such theorems we obtained very cheaply. These include the sentences about translations which are theorems because we have postulated that $\mathscr{T}$ is a commutative group with respect to addition of translations. How many such theorems you can recognize depends just on how well acquainted you are with the algebra of real numbers. Other ways to obtain theorems cheaply are suggested in Parts B and C on pages 127 and 128 and in the discussion of (6′) on page 132.

Other theorems, like those to which we have chosen to assign numbers have, up to now, required proofs before we could be sure that they are theorems. Also, some of these proofs are not easy to discover. You may be happy to learn that there is a very easy way to tell whether a sentence of our algebra of points and translations is a consequence of our *present* postulates and definitions. [You may already guess what this way is; but let's consider a few more examples. If you have a guess, you can check it against the examples.]

Exercises

In each exercise you are given a sentence (R) about real numbers which may or may not be true. You are also given several analogous "sentences" about points and translations. [As experience may lead you to suspect, not all of the latter are sentences – they just look the part.] List the sentences you think are theorems. For those which are not sentences, tell why they are not. For any sentence which you think is not a theorem, try to draw a counter-example.

1. (R) $a + (b - a) = b$
   (i) $\overrightarrow{a} + (\overrightarrow{b} - \overrightarrow{a}) = \overrightarrow{b}$
   (ii) $A + (\overrightarrow{b} - A) = \overrightarrow{b}$
   (iii) $\overrightarrow{a} + (B - \overrightarrow{a}) = \overrightarrow{b}$
   (iv) $A + (B - A) = B$
2. (R) $(a + b) - a = b$
   (i) $(\overrightarrow{a} + \overrightarrow{b}) - \overrightarrow{a} = \overrightarrow{b}$
   (ii) $(A + \overrightarrow{b}) - A = \overrightarrow{b}$
   (iii) $(\overrightarrow{a} + B) - \overrightarrow{a} = B$
   (iv) $(A + B) - A = B$
3. (R) $(c - b) + (a - c) = a - b$
   (i) $(\overrightarrow{c} - \overrightarrow{b}) + (\overrightarrow{a} - \overrightarrow{c}) = \overrightarrow{a} - \overrightarrow{b}$
   (ii) $(C - \overrightarrow{b}) + (\overrightarrow{a} - C) = \overrightarrow{a} - \overrightarrow{b}$
   (iii) $(\overrightarrow{c} - B) + (\overrightarrow{a} - \overrightarrow{c}) = \overrightarrow{a} - B$

Answers for Exercises

[Proofs of some theorems follow the answers.]

1. (i) and (iv) are theorems; (ii) and (iii) are not sentences — '$\overrightarrow{b} - A$' and '$\overrightarrow{a} + (B - \overrightarrow{a})$' are not terms.
2. (i) and (ii) are theorems; (iii) and (iv) are not sentences — '$\overrightarrow{a} + B$' and '$A + B$' are not terms.
3. (i), (vii), and (viii) are theorems; (ii), (iii), (iv), (v), and (vi) are not sentences — '$\overrightarrow{a} - C$' and '$\overrightarrow{c} - B$' are not terms.

(iv) $(\vec{c} - \vec{b}) + (A - \vec{c}) = A - \vec{b}$
(v) $(C - B) + (\vec{a} - C) = \vec{a} - B$
(vi) $(\vec{c} - B) + (A - \vec{c}) = A - B$
(vii) $(C - \vec{b}) + (A - C) = A - \vec{b}$
(viii) $(C - B) + (A - C) = A - B$

4. (R) $a - (b - c) = (a - b) - c$
(i) $\vec{a} - (\vec{b} - \vec{c}) = (\vec{a} - \vec{b}) - \vec{c}$
(ii) $A - (\vec{b} - \vec{c}) = (A - \vec{b}) - \vec{c}$
(iii) $A - (B - \vec{c}) = (A - B) - \vec{c}$
(iv) $A - (B - C) = (A - B) - C$

5. (R) $a - (b - c) = (a - b) + c$
(i) $\vec{a} - (\vec{b} - \vec{c}) = (\vec{a} - \vec{b}) + \vec{c}$
(ii) $A - (\vec{b} - \vec{c}) = (A - \vec{b}) + \vec{c}$
(iii) $\vec{a} - (B - \vec{c}) = (\vec{a} - B) + \vec{c}$
(iv) $\vec{a} - (\vec{b} - C) = (\vec{a} - \vec{b}) + C$
(v) $A - (B - \vec{c}) = (A - B) + \vec{c}$
(vi) $\vec{a} - (B - C) = (\vec{a} - B) + C$
(vii) $A - (\vec{b} - C) = (A - \vec{b}) + C$
(viii) $A - (B - C) = (A - B) + C$

6. (R) $c - a = c - b \Longleftrightarrow a = b$
(i) $\vec{c} - \vec{a} = \vec{c} - \vec{b} \Longleftrightarrow \vec{a} = \vec{b}$
(ii) $C - \vec{a} = C - \vec{b} \Longleftrightarrow \vec{a} = \vec{b}$
(iii) $C - A = C - \vec{b} \Longleftrightarrow A = \vec{b}$
(iv) $C - A = C - B \Longleftrightarrow A = B$

The most obvious thing about the expressions in the preceding exercises is that many of them are not sentences. For example, Exercise 1(ii) is not a sentence because, although we have three kinds of subtraction, none of them allows for subtracting a point from a translation. Also, Exercise 6(iii) is not a sentence because, since '$C - A$' is a translation-term and '$C - \vec{b}$' is a point-term, '$C - A = C - \vec{b}$' is not an equation.

Next, we notice that, in each exercise, the sentence (R) is a sentence about real numbers which is either true because $\mathscr{R}$ is a group with respect to addition or [in one case] is false. So, we are not surprised to find that, in all but one exercise, the sentence (i) is a theorem.

In Exercise 4, the sentence (R) is false. Also, although (i), (ii), and (iii) of this exercise are sentences, none of them is a theorem. Does this happen just by chance, or is it always the case that when a real number-sentence is false none of the corresponding sentences of our algebra can be derived from Postulates 1 through $4'''$? If this were the case, it would be helpful because it would give us a way of testing suspected theorems—if the real number-sentence corresponding to one of our sentences turned out to be false then we would know, at least, that our sentence couldn't be derived from the postulates we now

4. (R) is false, and none of the sentences is a theorem; however, (i), (ii), and (iii) are sentences — any appropriate figure with $\vec{c} \neq \vec{0}$ will furnish a counter-example; (iv) is not a sentence because '$(A - B) - C$' is not a term.

5. (i), (ii), and (v) are theorems; (iii) is not a sentence since neither of its components is a sentence — for example, '$A = \vec{b}$' is not a sentence because one side is a point-term while the other is a translation-term.

6. (i), (ii), and (iv) are theorems; (iii) is not a sentence since neither of its components is a sentence — for example, '$A = \vec{b}$' is not a sentence because one side is a point-term while the other is a translation-term.

In each exercise for which (i) is a theorem, this theorem is readily proved as in ordinary algebra. As to some of the other claimed theorems,

1. (iv) is Postulate 2(a), 2. (ii) is equivalent to Postulate 2(b),
3. (viii) is Postulate 3, 5. (v) is equivalent to Theorem 3-10,
and 6. (iv) is Theorem 3-4.

As to 3. (vii), $(C - \vec{b}) + (A - C) = A + [(C - \vec{b}) - C] = A + -\vec{b} = A - \vec{b}$, by Theorem 3-9, Definition 3-1(b) and Postulate 2(b), and Definition 3-1(b).

As to 5. (ii), $A - (\vec{b} - \vec{c}) = A + -(\vec{b} + -\vec{c}) = A + (-\vec{b} + \vec{c}) = (A + -\vec{b}) + \vec{c} = (A - \vec{b}) + \vec{c}$, by Definition 3-1, '$-(\vec{b} + -\vec{c}) = -\vec{b} + \vec{c}$', Theorem 2-5(b), and Definition 3-1(b).

As to 6. (ii), this follows from Theorem 2-2, by way of Definition 3-1(b) and the theorem '$-\vec{a} = -\vec{b} \Longleftrightarrow \vec{a} = \vec{b}$'.

have. Luckily, this is the case. The reason is that the real number-sentences which correspond with our postulates are true. So, if, for example, we could derive Exercise 4(iii) from our postulates then we could in the same way derive Exercise 4(R) from true sentences about real numbers. Since Exercise 4(R) is false, we can't do this. So, Exercise 4(iii) cannot be a theorem.

Looking now at the other exercises, we see something even more interesting. In each of these exercises the real number sentence is true [because $\mathcal{R}$ is a group with respect to addition] *and* each of the analogous *sentences* of our algebra is a consequence of Postulates 1 through 4‴. Is *this* chance? If it isn't then we have a lot of low-cost theorems. For, in this case, we shall know that a given sentence of our algebra is a theorem if the corresponding real number-sentence is derivable from (o) – (v) of page 125. This is so, and there is even a way of turning a derivation of a real number-sentence from (o) – (v) into a proof of any of the corresponding sentences of our algebra. We shall not go into the details of this. You have probably examined enough theorems by now to be convinced.

Combining the discoveries of the preceding two paragraphs, we have the following rule:

> A sentence of our algebra is a consequence of Postulates 1 through 4‴ if and only if the corresponding real number sentence is true just because $\mathcal{R}$ is a group with respect to addition.

Once we are convinced of the correctness of this rule, we have no more need to give proofs of theorems of the kind we have been dealing with, and no need to list other theorems of this kind. [We shall still, however, have to give proofs for theorems which depend on postulates we shall introduce later.]

A useful consequence of the rule we have discovered is this:

> We can transform expressions referring to points and translations just as though both kinds of addition and all three kinds of subtraction were addition and subtraction of real numbers *as long as we take care that the expressions we write make sense.*

Exercises

Part A

Consider the following and tell which are theorems and which are not. Explain each of your answers.

1. $(A + \vec{a}) - (B + \vec{a}) = A - B$
2. $A - (B + \vec{a}) = -B + (A - \vec{a})$
3. $A - (\vec{a} + B) = (A - B) - \vec{a}$

For your information we give a proof of the rule stated in the last paragraph on page 136.

* * *

The proof of this important rule requires two steps. We must show that

(☆) if a sentence of our algebra is derivable from Postulates 1 - 4‴ and Definition 3-1 then the corresponding sentence about real numbers is derivable from the group postulates (o)-(v) on page 125.

and we must show that

(☆☆) if the real number-sentence corresponding to a sentence of our algebra is derivable from (o)-(v), then the latter sentence is derivable from Postulates 1 - 4‴ and Definition 3-1.

Now, (☆) is easy to establish. The real number-sentences which correspond with our postulates are all consequences of (o)-(v). [Those corresponding with the parts of Postulate 4‴ and Definition 3-1 are precisely the sentences (o)-(v); those corresponding with Postulate 1 are '$a - a \in \mathcal{R}$' and '$a + a \in \mathcal{R}$'; those corresponding with Postulate 2 are '$a + (b - a) = b$' and '$(a + b) - a = b$', and that corresponding with Postulate 3 is '$(b - a) + (c - b) = c - a$'. Clearly, all are consequences of (o)-(v).] Hence, any proof of a theorem from our postulates can be transformed into a derivation of the corresponding sentence from consequences of (o)-(v) merely by replacing all variables in the proof by real number variables, replacing '$\vec{\mathcal{O}}$' by '0', and replacing '$\mathcal{E}$' and '$\mathcal{T}$' by '$\mathcal{R}$'.

So, (☆) is correct, and we know that any sentence of our algebra whose real number analogue is not derivable from (o)-(v) is, itself, not derivable from our present postulates. [In particular any sentence whose real number analogue is false is not derivable from our present postulates.]

The proof of (☆☆) is scarcely more difficult; but, since it is longer and rather more subtle, we have not given it in the text. To establish (☆☆) we proceed as follows:

> Consider any sentence ϕ of our algebra, let ϕ^* be its real number analogue, and suppose that ϕ^* is a consequence of (o)-(v). Choose a point-variable — say, 'O' — which does not occur in ϕ and let ψ be the sentence obtained from ϕ by inserting '-O' after each occurrence of a point-variable in ϕ. For example,
>
> if ϕ is '$C + (B - A) = B + (C - A)$'
>
> then ϕ^* is '$c + (b - a) = b + (c - a)$'
>
> and ψ is '$(C - O) + [(B - O) - (A - O)] = (B - O) + [(C - O) - (A - O)]$'.

[As indicated, in constructing ψ from ϕ, it is necessary to introduce additional punctuation. The precise procedure is to, first, enclose each occurrence in ϕ of a point-variable in a pair of parentheses, and then insert '- O' between each point-variable and the immediately following right parenthesis.] The proof of (☆☆) is now carried out in two steps by showing, first, that

(☆☆)′ because ϕ^* is a consequence of (o)-(v), ψ is a consequence of Postulate 4‴ and Definition 3-1(a),

and, second, that

(☆☆)″ ϕ is a consequence of ψ, Theorem 2-3, Theorem 3-7, Theorem 3-8, and Definition 3-1(b).

Combining these results we see that ϕ is a theorem if ϕ^* is a consequence of (o)-(v).

It remains to establish (☆☆)′ and (☆☆)″. We begin with the former. We begin by noting that, since ϕ^* is a consequence of (o)-(v), any sentence obtained by substituting different translation-variables for the different real number variables in ϕ^* is a consequence of Postulate 4‴ and Definition 3-1(a). We obtain one such sentence χ as follows. Certain variables in ϕ^* may have gotten there through substitution for translation-variables in ϕ. For these, we substitute these same translation-variables. For each of the other variables in ϕ^* we substitute any translation-variable we please, taking care only not to choose the same translation-variable twice, and not to choose a translation-variable which occurs in ϕ. In the example above,

χ might be '$\vec{c} + (\vec{b} - \vec{a}) = \vec{b} + (\vec{c} - \vec{a})$'.

Since, as should be obvious, ψ is an instance of the sentence χ so obtained, it follows that ψ is a consequence of Postulate 4‴ and Definition 3-1(a).

Having established (☆☆)′, our final task is to establish (☆☆)″. It is this part of the argument which is, perhaps, too subtle for your students. Before entering into it, it will be helpful to illustrate the course the argument will take by applying it to our example. [Recall that, by (☆☆)′, we know that, because '$c + (b - a) = b + (c - a)$' is derivable from (o)-(v), the sentence:

$$(C - O) + [(B - O) - (A - O)] = (B - O) + [(C - O) - (A - O)]$$

is derivable from Postulate 4‴ and Definition 3-1(a). We even know a mechanical way in which to modify a derivation of the first sentence to obtain a derivation of the second. Our task is to continue this derivation in such a way as to emerge with a proof of our original sentence '$C + (B - A) = B + (C - A)$'.] To begin with, it follows from the sentence displayed above and two instances of Theorem 3-7 that

$$(C - O) + (B - A) = (B - O) + (C - A).$$

And, from this and two instances of Theorem 3-8, it follows that

$$[C + (B - A)] - O = [B + (C - A)] - O.$$

Finally, from this and an instance of Theorem 2-3 it follows that

$$C + (B - A) = B + (C - A).$$

In establishing (☆☆)″ we shall at first consider the case in which — as in our example — the sentence ϕ is an equation. We may then suppose that ϕ is $\sigma = \tau$ where σ and τ are both point-terms or are both translation-terms. In this case the sentence ψ is $\sigma(O) = \tau(O)$, where $\sigma(O)$ is obtained from σ — and $\tau(O)$ from τ — in the same way in which ψ is obtained from ϕ, that is, by inserting '$- O$' after each point-variable. We shall show that in case σ is a point-term then $\sigma(O) = \sigma - O$ is a theorem which can be derived from Theorem 3-7, Theorem 3-8, and Definition 3-1(b); while if σ is a translation-term then $\sigma(O) = \sigma$ is such a theorem. We show this by induction on the manner in which terms are constructed. The simplest terms are variables and the one constant '$\vec{0}$'. Now, if σ is a point-variable then, by definition, $\sigma(O)$ is merely $\sigma - O$, and the equation $\sigma(O) = \sigma - O$ is a valid sentence. If σ is a translation-variable or '$\vec{0}$' then, by definition, $\sigma(O)$ is merely σ, and the equation $\sigma(O) = \sigma$ is valid. Thus, we have established the initial step of our induction. The inductive step depends on noting that if σ is a point-term which is not merely a variable then it is $\sigma_1 + \sigma_2$ or $\sigma_1 - \sigma_2$, where σ_1 is a point-term and σ_2 is a translation-term, while if σ is a translation-term which is not a variable or '$\vec{0}$' then it is either $\sigma_1 - \sigma_2$, where σ_1 and σ_2 are point-terms, or it is $\sigma_1 + \sigma_2$, $\sigma_1 - \sigma_2$ or $-\sigma_1$, where σ_1 and σ_2 are translation-terms. Our procedure now is to make the inductive hypothesis that the result we are trying to establish does hold for the terms σ_1 and σ_2 and, from this, argue that it holds for σ.

We consider first the case in which the point-term σ is $\sigma_1 + \sigma_2$ where σ_1 is a point-term and σ_2 a translation-term. By definition, $\sigma(O)$ is $\sigma_1(O) + \sigma_2(O)$ and, by the inductive hypothesis, $\sigma_1(O) = \sigma_1 - O$ and $\sigma_2(O) = \sigma_2$ are consequences of Theorems 3-7 and 3-8, and Definition 3-1(b). It follows that $\sigma(O) = (\sigma_1 - O) + \sigma_2$ is also such a consequence. Hence, $\sigma(O) = (\sigma_1 + \sigma_2) - O$ — that is, $\sigma(O) = \sigma - O$ — is such a consequence since it follows from the preceding one and Theorem 3-8. The case in which σ is $\sigma_1 - \sigma_2$, where σ_1 is a point-term and σ_2 is a translation-term is treated in exactly the same way except that, in the final step we need the theorem '$(A - B) - \vec{c} = (A - \vec{c}) - B$' which is an immediate consequence of Theorem 3-8 and Definition 3-1(b).

We next consider the case in which σ is $\sigma_1 - \sigma_2$, where σ_1 and σ_2 are both point-terms. By definition, $\sigma(O)$ is $\sigma_1(O) - \sigma_2(O)$ and, by the inductive hypothesis $\sigma_1(O) = \sigma_1 - O$ and $\sigma_2(O) = \sigma_2 - O$ are consequences of Theorems 3-7 and 3-8 and Definition 3-1(b). It follows that $\sigma(O) = (\sigma_1 - O) - (\sigma_2 - O)$ is such a consequence. Hence, $\sigma(O) = \sigma_1 - \sigma_2$ — that is, $\sigma(O) = \sigma$ — is such a consequence since it follows from the preceding one and Theorem 3-7.

Finally, we consider the case in which σ is $\sigma_1 + \sigma_2$, $\sigma_1 - \sigma_2$, or $-\sigma_1$, where σ_1 and σ_2 are translation-terms. By definition, $\sigma(O)$ is $\sigma_1(O) + \sigma_2(O)$, $\sigma_1(O) - \sigma_2(O)$, or $-\sigma_1(O)$ and, by the induction hypothesis, $\sigma_1(O) = \sigma_1$ and $\sigma_2(O) = \sigma_2$ are consequences of Theorems 3-7 and 3-8 and Definition 3-1(b). Hence [in any of the three subcases], $\sigma(O) = \sigma$ is also such a consequence.

Having established this result concerning terms, we return to the consideration of the sentence ϕ which we are supposing is an equation, $\sigma = \tau$, where σ and τ are both point-terms or are both translation terms. By definition, the sentence ψ is, then, $\sigma(O) = \tau(O)$. In the first case $\sigma(O) = \sigma - O$ and $\tau(O) = \tau - O$ are both consequences of Theorems 3-7 and 3-8 and Definition 3-1(b). Consequently, the sentence $\sigma(O) = \tau(O) \Longleftrightarrow \sigma - O = \tau - O$ is such a consequence. Hence, the sentence $\sigma(O) = \tau(O) \Longleftrightarrow \sigma = \tau$ is a consequence of Theorems 2-3, 3-7, and 3-8, and Definition 3-1(b). In the second case [that in which σ and τ are translation-terms] the equations $\sigma(O) = \sigma$ and $\tau(O)$ are consequences of Theorems 3-7 and 3-8, and Definition 3-1(b). So, in this case, the sentence $\sigma(O) = \tau(O) \Longleftrightarrow \sigma = \tau$ is also such a consequence.

It follows at once that in case ϕ is an equation then $\psi \Longleftrightarrow \phi$ is a consequence of Theorems 2-3, 3-7, and 3-8, and Definition 3-1(b). So, (☆☆)″ is established in this case. The general case now follows

4. $(A + \vec{a}) - (B + \vec{b}) = (A - B) + (\vec{a} - \vec{b})$
5. $A - (B + \vec{c}) = \vec{c} \Longleftrightarrow B + \vec{c} = A + \vec{c}$
6. $(A + \vec{b}) - (C + \vec{d}) = A - [C - (\vec{b} - \vec{d})]$
7. $(\vec{a} + B) - [A + (B - A)] = \vec{a}$
8. $(P + \vec{p}) - [Q - (Q - R)] = (P - R) + \vec{p}$
9. $(B - \vec{a}) - [(A - \vec{a}) - (A - B)] = \vec{0}$
10. $\vec{b} + [\vec{b} - (\vec{a} - (A - B))] = \vec{b}$

Part B

Simplify the following expressions, and tell whether the given expression refers to points or translations.

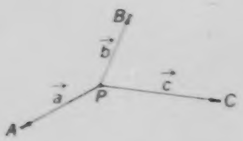
1. $[A - (A - B)] + (A - C)$
2. $[(A + \vec{a}) - B] + (B - A)$
3. $[A - (B - C)] + (B - A)$
4. $(\vec{c} - \vec{b}) + (\vec{a} - \vec{c}) - (\vec{a} - \vec{b})$
5. $[C + (B - C)] + \vec{a}$
6. $[A - (A - B)] + [(C + \vec{c}) - C]$
7. $(A - B) + (C - A)$
8. $A + (A - A)$
9. $[(A - B) + (C - A)] + (B - C)$
10. $\{Q - [R + (P - R)]\} + (P - Q)$

Part C

In each of the following exercises, complete the sentences in terms '$\vec{a}$', '$\vec{b}$', '$\vec{c}$', '$\vec{d}$' or '$\vec{0}$'. Also, on a copy of the given diagram draw dotted arrows to describe the translations listed in the exercises.

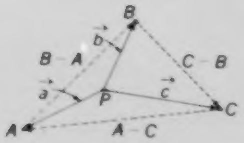
Example.

Given: $A = P + \vec{a}, B = P + \vec{b}, C = P + \vec{c}$



(a) $B - A =$ __________
(b) $C - B =$ __________
(c) $A - C =$ __________
(d) $(B - A) + (C - B) + (A - C) =$ __________

Solution.



(a) $B - A =$ $\vec{b} - \vec{a}$
(b) $C - B =$ $\vec{c} - \vec{b}$
(c) $A - C =$ $\vec{a} - \vec{c}$
(d) $(B - A) + (C - B) + (A - C) =$ $\vec{0}$

at once. For any sentence ϕ is built up out of equations and connectives ['$\Longrightarrow$', '$\Longleftrightarrow$', 'and', 'or', and 'not'], and, by definition, ψ is built up in the same way out of corresponding equations. If one of the building blocks for ϕ is ϕ_i then the corresponding one for ψ is $\phi_i(O)$ and, as has just been shown, $\phi_i(O) \Longleftrightarrow \phi_i$ is a consequence of Theorems 2-3, 3-7, and 3-8, and Definition 3-1(b). So, by the replacement and introduction rules for biconditional sentences it follows that $\psi \Longleftrightarrow \phi$ is such a consequence. In particular, ϕ is a consequence of ψ and these theorems and definition.

This completes the argument for (☆☆)″ and, so, completes the proof of the rule on page 136.

Incidentally, it is by the kind of induction used in establishing (☆☆)″ that one justifies the replacement rule for biconditional sentences. In this case, the induction is with respect to the structure of the sentence in which the replacement is to be made rather than, as in the case of (☆☆)″, on the structure of the term σ for which we wish to establish our result.

Answers for Part A

1. Theorem. ['$(a + c) - (b + c) = a - b$' is algebraically correct and each of '$(A + \vec{a}) - (B + \vec{a})$' and '$A - B$' is a translation-term.]
2. Not a theorem. ['$-B$' is nonsense.]
3. Not a theorem. ['$\vec{a} + B$' is nonsense.]

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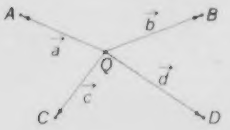
4. Theorem. ['$(a + c) - (b + d) = (a - b) + (c - d)$' is algebraically correct and each of '$(A + \vec{a}) - (B + \vec{b})$' and '$(A - B) + (\vec{a} - \vec{b})$' is a translation-term.]
5. Not a theorem. ['$a - (b + c) = c$' is not "algebraically" equivalent to '$b + c = a + c$'.]
6. Theorem. ['$(a + b) - (c + d) = a - (c - (b - d))$' is algebraically correct and each of '$(A + \vec{b}) - (C + \vec{d})$' and '$A - [C - (\vec{b} - \vec{d})]$' is a translation-term.]
7. Not a theorem. ['$\vec{a} + B$' is nonsense.]
8. Theorem. ['$(p + s) - [q - (q - r)] = (p - r) + s$' is algebraically correct and each of '$(P + \vec{p}) - [Q - (Q - R)]$' and '$(P - R) + \vec{p}$' is a translation-term.]
9. Theorem. ['$(b - c) - [(a - c) - (a - b)] = 0$' is algebraically correct and each of '$(B - \vec{a}) - [(A - \vec{a}) - (A - B)]$' and '$\vec{0}$' is a translation-term.]
10. Not a theorem. ['$b + [b - (a - (c - d))] = b$' is not algebraically correct.]

Answers for Part B

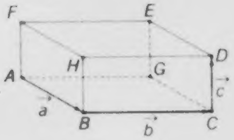
1. $B + (A - C)$; point
   [or: $A + (B - C)$; point]
2. $\vec{a}$; translation
3. C; point
4. $\vec{0}$; translation
5. $B + \vec{a}$; point
6. $B + \vec{c}$; point
7. $C - B$; translation
8. A; point
9. $\vec{0}$; translation
10. $\vec{0}$; translation

1. Given: $A = Q + \vec{a}, B = Q + \vec{b}, C = Q + \vec{c}, D = Q + \vec{d}$



(a) $B - A =$ ____________
(b) $D - B =$ ____________
(c) $C - D =$ ____________
(d) $A - C =$ ____________
(e) $(B - A) + (D - B) + (C - D) + (A - C) =$ ____________

2. Given: $B = A + \vec{a}, C = B + \vec{b}, D = C + \vec{c}, E = D - \vec{a},$
$F = E - \vec{b}, G = E - \vec{c}, H = B + \vec{c}$



(a) $C - A =$ ____________
(b) $D - A =$ ____________
(c) $G - C =$ ____________
(d) $G - A =$ ____________
(e) $G - B =$ ____________
(f) $E - B =$ ____________
(g) $F - D =$ ____________
(h) $C - E =$ ____________
(i) $(H - A) + (G - H) + (E - G) + (A - E) =$ ____________

3.08 A New Look at Postulates 1 and 2

Recall our Postulates 1 and 2:

1(a) $B - A \in \mathscr{T}$ (b) $A + \vec{a} \in \mathscr{E}$
2(a) $A + (B - A) = B$ (b) $\vec{a} = (A + \vec{a}) - A$

Now suppose that O is a point. Then [by 1(a)] for any point A,

$$A - O \in \mathscr{T}$$

So, we can define a mapping [which we shall call 'T_o'] of $\mathscr{E}$ into $\mathscr{T}$:

$$T_o(A) = A - O$$

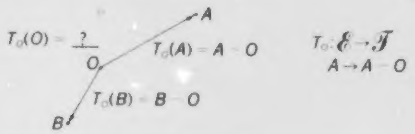

A
$T_O(O) = \underline{\ ?\ }$
O
$T_O(A) = A - O$
$T_O: \mathscr{E} \to \mathscr{T}$
$A \to A - O$
$T_O(B) = B - O$
B

Fig. 3–2

Also, [by 1(b)] for any translation $\vec{a}$,

$$O + \vec{a} \in \mathscr{E}$$

Answers for Part C

1. (a) $\vec{b} - \vec{a}$
(b) $\vec{d} - \vec{b}$
(c) $\vec{c} - \vec{d}$
(d) $\vec{a} - \vec{c}$
(e) $\vec{0}$ [or: A - A]

2. (a) $\vec{c} - \vec{a}$
(b) $\vec{a} + \vec{b} + \vec{c}$
(c) $-\vec{a}$
(d) $\vec{b}$
(e) $\vec{b} - \vec{a}$
(f) $\vec{b} + \vec{c} - \vec{a}$
(g) $-\vec{a} - \vec{b}$
(h) $\vec{a} - \vec{c}$
(i) $\vec{0}$ [or: A - A]

[It is recommended that you construct a stick model to aid the discussion of this exercise. Pencils and lumps of clay work nicely for this purpose. You can label points with flags taped to tooth picks.]

TC 138, 139 (1)

This section shows that, on the basis of Postulates 1 and 2, given an origin O in ℰ, there is determined a one-to-one correspondence between points and translations. This result is the basis for introducing coordinates in ℰ and, more directly, for the usual uses of vectors [*qua* directed segments] for solving problems in geometry.

This result also suggests a more special kind of geometry which, when vectors are mentioned, sometimes becomes confused with euclidean geometry. This geometry is euclidean geometry *plus* a chosen origin, O, and is properly called *centered euclidean geometry*. As the result of this section shows, there is no formal difference between vector algebra and *centered* euclidean geometry. Confusion of centered euclidean geometry with "homogenized" [ordinary] euclidean geometry leads to identifying points with vectors and to statements to the effect that euclidean geometry is just the study of vector algebra. Such statements are false. What is true is that euclidean geometry is just the study of how translations — which constitute a vector space — operate on the set ℰ of points, in accordance with Postulates 1 and 2. [One way to appreciate the difference between centered euclidean geometry and ordinary euclidean geometry is to note that in the former it is possible, as indicated on page 139, to introduce geometrically meaningful operations of addition and oppositing of points. Also, in such a geometry one is barred from considering motions under which the origin O is not fixed.]

The relation between 𝒯 and ℰ which are made explicit in this section can be used to good effect in gaining a more nearly complete understanding of notions concerning translations. Translations can be regarded from two points of view. In the first place, each translation is a mapping of ℰ onto itself. As in the case of any mapping, there are various graphical tricks which help us to concentrate our attention on what might be called the "structure" or the "nature" of a given translation. For example, we can picture a given translation by drawing a lot of arrows, all having the same sense and the same length, and interpret our drawing as a picture of 3-dimensional space in which, by the arrows, we indicate the images of selected points under the given translation. If we felt it necessary, we could construct a more "realistic" picture by hanging actual arrows [as used in archery] from the ceiling of a room. Other methods — the use of tracing sheets is an example — are illustrated in Chapter 1.

and we can define a mapping [which we shall call 'P_O'] of $\mathscr{T}$ into $\mathscr{E}$:

$$P_O(\vec{a}) = O + \vec{a}$$

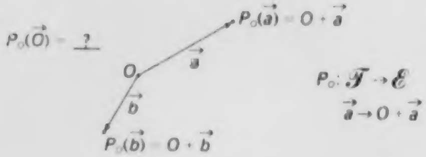

$P_O(\vec{O}) = \underline{?}$
$P_O(\vec{a}) = O + \vec{a}$
O
$\vec{a}$
$\vec{b}$
$P_O(\vec{b}) = O + \vec{b}$
$P_O: \mathscr{T} \to \mathscr{E}$
$\vec{a} \to O + \vec{a}$

Fig. 3–3

Now consider the composition of the mappings P_O and T_O. Complete the following:

$$\begin{aligned}[P_O \circ T_O](A) &= P_O(T_O(A))\\ &= P_O(\underline{\qquad\quad})\\ &\left.\begin{aligned}&= \underline{\qquad\qquad\qquad}\\ &= A\end{aligned}\right\}\ \text{[Postulate \underline{\quad}]}\end{aligned}$$

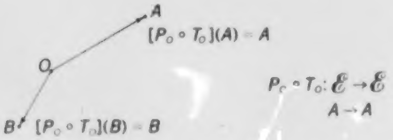


Fig. 3–4

We have seen that if we adopt a definition:

> **Definition 3–2**
> (a) $T_O(A) = A - O$ (b) $P_O(\vec{a}) = O + \vec{a}$

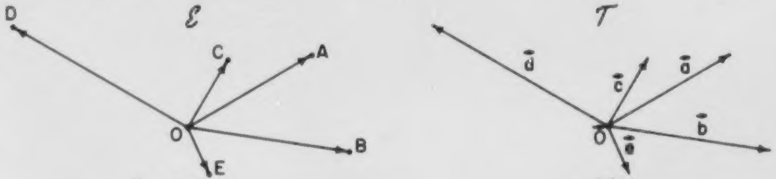
we have, as a consequence of Postulate 2:

> **Theorem 3–14**
> (a) $P_O \circ T_O = i_{\mathscr{E}}$ (b) $T_O \circ P_O = i_{\mathscr{T}}$

This theorem says, exactly, that the mappings P_O and T_O are inverses of one another. [Explain.] In other words, given any point O, the mappings P_O and T_O spell out a natural one-to-one correspondence between the points of $\mathscr{E}$ and the translations of $\mathscr{T}$.

The second way of regarding translations is characteristic of contemporary mathematics. Translations are objects of a certain kind and, as such, are related to one another in various ways. We recognize this when we speak of the set $\mathcal{T}$ whose members are just the translations of $\mathcal{E}$ and when, for example, we recognize that one translation may be the resultant obtained by composing other translations. Indeed, as is summarized in Postulate 4‴, the members of $\mathcal{T}$ are subject to certain operations, and these operations have certain properties. Later we shall find other operations on, and relations among, the members of $\mathcal{T}$. Our discovery of these operations and relations results from our study of the "structure" of individual translations. That is, it comes about from adopting the point of view referred to in the preceding paragraph. But, having made these discoveries, it now becomes profitable to concentrate our attention on them. In order to do so it is helpful to have some way of "visualizing" $\mathcal{T}$, itself.

The present section suggests how this can be done. If we choose a point of $\mathcal{E}$ then there is a one-to-one correspondence between the other points of $\mathcal{E}$ and the non-$\vec{0}$ translations. We may then represent each of these translations by an arrow from the chosen point to the point which is its image under the chosen translation. The result will be a picture like (a). It is a picture of $\mathcal{E}$, supplemented with arrows indicating certain translations. Re-labeling (a) gives us (b), which we can take as a "picture" of $\mathcal{T}$. In (b), the arrows are pictorial representations of certain non-$\vec{0}$ translations and the translation $\vec{0}$ is pictured by a dot. It may help in clarifying the distinction between (a) and (b) to point out that in (a) there is the possibility of indicating a given translation by drawing any one of many arrows. For example, the translation from A to C may be indicated by an arrow from A to C and, also, by an arrow from O to a properly chosen point. In fact each arrow in (a) is merely a representative chosen from a set of "equivalent" arrows, and it is this set which, primarily, we think of as "representing" a translation. In (b), on the other hand, only arrows initiating at the dot marked '$\vec{0}$' have any meaning, and each such arrow is to be interpreted to be, itself, the unique pictorial representation of some member of $\mathcal{T}$.



(a) (b)

The result of the present section, which justifies such pictorial representations as (b) of $\mathcal{T}$, can be interpreted intuitively in terms of (a) and (b). What it says in these terms is that, as suggested by the similarity between the two pictures, $\mathcal{T}$ can be "set down" on $\mathcal{E}$ in such a way that $\vec{0}$ corresponds with any point O we care to choose.

*

The blanks are filled in as follows: A - O; O + (A - O); 2(a)

The existence of such natural one-to-one correspondences will turn out to be very useful. Because of them, anything we learn about $\mathscr{T}$ furnishes knowledge of $\mathscr{E}$ also, and *vice versa*. In later chapters we shall be more concerned than we have been up to now with geometrical figures – triangles, etc. – that is, with subsets of $\mathscr{E}$. Because of the natural one-to-one correspondences we can study analogous things in $\mathscr{T}$ and then transfer what we learn to $\mathscr{E}$. The advantage of doing so is that translations are easier to deal with than are points. The reason for this is that $\mathscr{T}$ is a commutative group and, as we shall see, a very special kind of commutative group. So, we can apply all sorts of algebraic techniques to the study of $\mathscr{T}$.

The fact that we can move back and forth so easily between $\mathscr{T}$ and $\mathscr{E}$ suggests a way in which we *might* define oppositing and addition for points:

$$-A = P_O(-T_O(A)), \quad A + B = P_O(T_O(A) + T_O(B))$$

Notice, however, that these definitions refer to a point O which we must specify if we are to know what is meant by oppositing and addition of points. We have not really defined oppositing and addition of points. Instead, we have defined "oppositing with respect to O" and "addition with respect to O". To see the difference, note that if we choose a different point – say, O' – we get a different opposite for a given point A and a different sum for given points A and B.

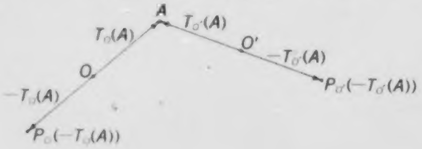


Fig. 3–5

Exercises

1. Draw a figure to show that if $O' \neq O$ then the sum of A and B with respect to O' is not the sum of A and B with respect to O.
2. **(a)** Which part of Theorem 3 – 14 says that T_O is one-to-one?
   (b) Which part says that P_O is one-to-one?

*3. Show that if we were to adopt as postulates:

1′ (a) $T_O(A) \in \mathscr{T}$ (b) $P_O(\vec{a}) \in \mathscr{E}$
2′ (a) $P_O \circ T_O = i_{\mathscr{E}}$ (b) $T_O \circ P_O = i_{\mathscr{T}}$

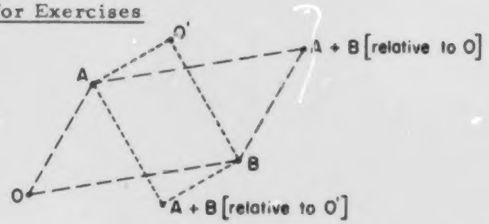
and as definitions:

(a) $B - A = T_A(B)$ (b) $A + \vec{a} = P_A(\vec{a})$

then Postulates 1 and 2 and Definition 3 – 2 would be theorems.

Answers for Exercises

1.



2. (a) Theorem 3-14(a) says that T_O is one-to-one [and that P_O is onto].

   (b) Theorem 3-14(b) says that P_O is one-to-one [and that T_O is onto].

   [In general, if $g \circ f$ is a one-to-one mapping of a set S onto a set T then no two members of S can have the same image under f — if they did, they would, by definition, have the same image under $g \circ f$ — and each member of T must be in the range of g — if it weren't it would, by definition, not be in the range of $g \circ f$. Since $i_{\mathscr{E}}$ is a one-to-one mapping of $\mathscr{E}$ onto itself it follows from Theorem 3-14(a) that T_O — which, by definition has domain $\mathscr{E}$ and range contained in $\mathscr{T}$ — is a one-to-one mapping of $\mathscr{E}$ into $\mathscr{T}$; and that P_O — which, by definition has domain $\mathscr{T}$ and range contained in $\mathscr{E}$ — maps $\mathscr{T}$ on all of $\mathscr{E}$.]

Due to the unfamiliar notation, Exercise 3 may be more appropriate as a class discussion exercise.

3. Postulate 1(a) follows from 1′(a) and part (a) of the definition; Postulate 1(b) follows from 1′(b) and part (b) of the definition. The proofs of Postulate 2(a) and 2(b) are as follows:

$$A + (B - A) = P_A(B - A) = P_A(T_A(B)) = [P_A \circ T_A](B) = i_{\mathscr{E}}(B) = B$$

$$(A + \vec{a}) - A = T_A(A + \vec{a}) = T_A(P_A(\vec{a})) = [T_A \circ P_A](\vec{a}) = i_{\mathscr{T}}(\vec{a}) = \vec{a}$$

3.09 Chapter Summary

Vocabulary Summary

translation of $\mathcal{E}$
postulate
binary operation
commutative group
constant mapping of $\mathcal{E}$
definition
singulary operation
commutative group operation

Postulates

1. (a) $B - A \in \mathcal{T}$ (b) $A + \vec{a} \in \mathcal{E}$

2. (a) $A + (B - A) = B$ (b) $\vec{a} = (A + \vec{a}) - A$

3. $(B - A) + (C - B) = C - A$

4_0. (a) $\vec{a} + \vec{b} \in \mathcal{T}$ (b) $\vec{0} \in \mathcal{T}$ (c) $-\vec{a} \in \mathcal{T}$
4_1. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
4_2. $\vec{a} + \vec{0} = \vec{a}$
4_3. $\vec{a} + -\vec{a} = \vec{0}$
4_4. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
$4'''$. $\mathcal{T}$ is a commutative group with respect to composition.

Definitions

3–1. (a) $A - \vec{a} = A + -\vec{a}$ (b) $\vec{a} - \vec{b} = \vec{a} + -\vec{b}$

3–2. (a) $T_O(A) = A - O$ (b) $P_O(\vec{a}) = O + \vec{a}$

Other Theorems

2–1. $A + \vec{a} = B \Longleftrightarrow \vec{a} = B - A$

2–2. $A + \vec{a} = A + \vec{b} \Longleftrightarrow \vec{a} = \vec{b}$

2–3. $A - C = B - C \Longleftrightarrow A = B$

2–4. $\vec{a} + \vec{b} = [(A + \vec{a}) + \vec{b}] - A$

2–5. (a) $\vec{a} + \vec{b} \in \mathcal{T}$ (b) $A + (\vec{a} + \vec{b}) = (A + \vec{a}) + \vec{b}$

3–1. (a) $A + \vec{0} = A$ (b) $A - A = \vec{0}$

3–2. (a) $A + \vec{a} = A \Longleftrightarrow \vec{a} = \vec{0}$ (b) $B = A \Longleftrightarrow B - A = \vec{0}$

3–3. $A + \vec{a} = B + \vec{a} \Longleftrightarrow A = B$

3–4. $C - A = C - B \Longleftrightarrow A = B$

3–5. (a) $-(B - A) = A - B$ (b) $-\vec{a} = A - (A + \vec{a})$

3–6. $(A + \vec{a}) + \vec{b} = A \Longleftrightarrow \vec{b} = -\vec{a}$

3–7. $(B - C) - (A - C) = B - A$

3–8. $(A - B) + \vec{c} = (A + \vec{c}) - B$

3–9. $A + (B - C) = B + (A - C)$

3–10. $(A - B) + \vec{c} = A - (B - \vec{c})$

3–11. $\vec{a} - (B - C) = C - (B - \vec{a})$

3–12. $A - (B - C) = C - (B - A)$

3–13. $(A - B) - (C - D) = (A - C) - (B - D)$

Corollary. $A - B = C - D \Longleftrightarrow A - C = B - D$

3–14. (a) $P_O \circ T_O = i_{\mathcal{E}}$ (b) $T_O \circ P_O = i_{\mathcal{T}}$

Rules of Inference

Any inference of either of the forms:

$$\frac{p \Longleftrightarrow q}{q \Longleftrightarrow p} \qquad \frac{p \Longleftrightarrow q \quad q \Longleftrightarrow r}{p \Longleftrightarrow r}$$

is valid.

Chapter Test

1. Simplify.
   - (a) $[B - (B - A)] + (B - C)$
   - (b) $[(P + \vec{p}) + (\vec{q} - \vec{r})] - [(R + \vec{r}) + (\vec{q} - \vec{r})]$
   - (c) $(C + \vec{d}) - [(B - \vec{b}) + (C - (B - \vec{b}))]$
   - (d) $(\vec{a} - \vec{b}) + (\vec{a} - \vec{c}) - (\vec{a} + \vec{b})$
   - (e) $[(A - C) - (C - A)] + (C - D)$
   - (f) $[(P - Q) - (Q - R)] + (Q - R)$

2. Which of the following are theorems and which are not? Justify each of your answers.
   - (a) $P + (Q - Q) = P$
   - (b) $(P + \vec{p}) - P = P$
   - (c) $(P + \vec{p}) - [Q + (P - Q)] = \vec{p}$
   - (d) $(A + B) - (B + C) = A - C$
   - (e) $[(A - B) + (\vec{a} - \vec{b})] + (\vec{b} - \vec{a}) = A - B$
   - (f) $\vec{b} + (A + [(\vec{a} - A) - \vec{b}]) = \vec{a}$

3. Given the set $\{1, -1\}$ and the operation multiplication $[\cdot]$ defined on this set in the usual way (i.e., $1 \cdot 1 = 1$, $1 \cdot -1 = -1$, $-1 \cdot 1 = -1$, $-1 \cdot -1 = 1$), demonstrate that this set is [or, is not] a commutative group under the given operation.

4. In each of the following, decide whether the given expression is one for a point, for a translation, or is meaningless.
   - (a) $(A + \vec{a}) + (B + \vec{b})$
   - (b) $[(A + \vec{a}) + (B - C)] - D$
   - (c) $(A - [B - (C - B)]) + (C - A)$
   - (d) $[(A + \vec{a}) + (A + \vec{a})] - B$
   - (e) $A - ([B - (C - D)] - E$
   - (f) $(A - B) - [(C - B) + (C - A)]$

5. Prove the following theorem:

$$(B - A) - [B - (A + \vec{a})] = \vec{a}$$

Key to Chapter Test

1. (a) $A + (B - C)$ [or: $B + (A - C)$]
   (b) $(P + \vec{p}) - (R + \vec{r})$ [or: $(P - R) + (\vec{p} - \vec{r})$]
   (c) $\vec{d}$
   (d) $\vec{a} - \vec{b} - \vec{b} - \vec{c}$ [cannot accept '$\vec{a} - \vec{b}2 - \vec{c}$' as yet]
   (e) $(A - C) + (A - D)$
   (f) $P - Q$

2. (a) Theorem. ['$p + (q - q) = p$' is true, and '$P + (Q - Q)$' and 'P' are point-terms. Another justification might be a proof of the theorem.]

   (b) Not a theorem. ['$(P + \vec{p}) - P$' is a translation-term and 'P' is a point-term. So, the given expression is not (even) an equation.]

   (c) Theorem. ['$(p + a) - [q + (p - q)] = a$' is true, and '$(P + \vec{p}) - [Q + (P - Q)]$' and '$\vec{p}$' are translation-terms. Another justification might be a proof of the theorem.]

   (d) Not a theorem. ['$A + B$' is nonsense. So is '$B + C$'.]

   (e) Theorem. ['$[(a - b) + (p - q)] + (q - p) = a - b$' is true, and '$[(A - B) + (\vec{a} - \vec{b})] + (\vec{b} - \vec{a})$' and '$A - B$' are translation-terms. Another justification might be a proof of the theorem.]

   (f) Not a theorem. ['$\vec{a} - A$' is nonsense.]

3. The set $\{1, -1\}$ is a commutative group under multiplication. 1 is the identity element, and each element of $\{1, -1\}$ is its own (multiplicative) inverse. Associativity and commutativity of multiplication are easily verified. [Another check is that closure of this set under multiplication together with the "facts" that $\{1, -1\} \subseteq$ Integers and multiplication is both associative and commutative over this "larger" set guarantees associativity and commutativity of the operation over the given subset.]

4. (a) meaningless
   (b) translation
   (c) translation
   (d) meaningless
   (e) point
   (f) translation

5. Here is a proof of the theorem:

$$(B - A) - [B - (A + \vec{a})] = (B - A) - [(B - A) - \vec{a}] =$$
$$[(B - A) - (B - A)] + \vec{a} = \vec{0} + \vec{a} = \vec{a}.$$

Chapter Four
Real Numbers

4.01 A Review

Up to now we have referred to the ordinary algebra of real numbers only for the purpose of giving examples [in Chapter 1] and analogies with our algebra of points and translations [in Chapters 2 and 3]. In Chapter 5 we shall begin making use of the real numbers in our algebra. This will then be an algebra of points, translations, and real numbers. Although you know a good deal about the algebra of real numbers, it will be worthwhile to review its foundations and put on record the postulates we need as a basis for this algebra. These postulates will deal with the operations of addition, oppositing, subtraction, multiplication, reciprocating, and division, and with the order relation greater than and its converse, less than.

Before stating our postulates – which we shall combine into a single Postulate 5 – it is necessary to say a few words about reciprocating. You are certainly familiar with the definition according to which the reciprocal of a nonzero real number is the quotient of 1 by that number. For example, the reciprocal of 2 is 1 ÷ 2. Given the operation of division this is a perfectly satisfactory definition of reciprocating. For our purposes, however, it is more advantageous to think of reciprocating as a "fundamental" operation and to define division in terms of it. [This is entirely analogous to our previous definition of subtraction in terms of oppositing.] Just as we need an operator ['—'] to use in referring to oppositing, we need, then, an operator to use in referring to reciprocating. For various reasons we choose to use '/'. For example, $/2 = 0.5$, $/-2.5 = -/2.5 = -0.4$, and, according to the definition we shall adopt for division, $2 \div 3 = 2 \cdot /3$. [Since, as is customary, we shall adopt the convention of omitting multiplication dots in most cases, fractions such as '2/3' will, by this convention, be abbreviations of expressions like '$2 \cdot /3$' and, so, will, as they should, be numerals for quotients like $2 \div 3$.]

Although reciprocating is analogous to oppositing there is one important difference of which we must take account. As you learned in the last chapter, addition of real numbers is a commutative group

For the most part we shall, in this course, take for granted students' knowledge of the algebra of real numbers. Since, however, the real numbers enter formally into our geometry in the next chapter it is merely honesty to include among our postulates one which asserts that the operations on real numbers [and the special real numbers 0 and 1] have the properties which we require. Such a postulate consists of the parts 5_0 - 5_7 on page 145 and 5_8 - 5_{12} [on order] which appear later in the chapter. [The last, 5_{12}(b), is stated in Part B on page 158.] More succinctly, the postulate in question is to the effect that the real number system is an ordered field. [In dealing, later, with arc-measure and the circular functions we shall need to postulate completeness — i.e., that each nonempty bounded set of real numbers has a least upper bound.]

The restrictions, '[$a \neq 0$]', etc., on 5_0(f), 5_3(b), and 5_7(b) have, of course, to do with "division by 0". They are discussed in section 4.02.

The derivations presented in the text and exercises are intended mainly to introduce or illustrate new rules of reasoning. As mentioned above, we are taking the student's knowledge of the real numbers for granted. This includes the student's knowledge of derivations for most real number theorems. The teacher should exercise caution in the treatment of this chapter lest valuable time be lost.

operation, with 0 as the corresponding identity element and oppositing as the corresponding inversing operation. On the other hand, multiplication of real numbers is only "almost" a commutative group operation. Although there is an identity element, 1, for multiplication, only nonzero real numbers have "multiplicative inverses". The number 0 has no inverse with respect to multiplication because the product of 0 by any real number is 0 and, so, is not 1. The multiplicative inverse of a *nonzero* real number is its reciprocal and, since 0 has no multiplicative inverse it is customary not to define a reciprocal for 0. This custom has some drawbacks, but we shall adopt it. One consequence is that our postulates concerning reciprocating will have restrictions added to them. How to deal with these restrictions will be taken up in the next section.

We shall now list the parts of Postulate 5 which do not concern the order relations.

5_0. (a) $a + b \in \mathscr{R}$ (b) $0 \in \mathscr{R}$ (d) $a \cdot b \in \mathscr{R}$ (e) $1 \in \mathscr{R}$
(c) $-a \in \mathscr{R}$ (f) $/a \in \mathscr{R}\ [a \neq 0]$
5_1. (a) $(a + b) + c = a + (b + c)$ (b) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
5_2. (a) $a + 0 = a$ (b) $a \cdot 1 = a$
5_3. (a) $a + -a = 0$ (b) $a \cdot /a = 1\ [a \neq 0]$
5_4. (a) $a + b = b + a$ (b) $a \cdot b = b \cdot a$
5_5. $0 \neq 1$
5_6. $(a + b) \cdot c = a \cdot c + b \cdot c$
5_7. (a) $a - b = a + -b$ (b) $a \div b = a \cdot /b\ [b \neq 0]$

[The *restriction* '$[a \neq 0]$' is to be read as 'for a different from 0'.] Of these, 5_0(a) – (c) and 5_1(a) – 5_4(a) say that $\mathscr{R}$ is a commutative group with respect to addition; 5_0(d) – (f) and 5_1(b) – 5_4(b) are analogous to the preceding, but with the significant difference that two of them are accompanied by restrictions. Since it follows from these postulates that products and reciprocals of nonzero numbers are not 0 it is not difficult to see that these postulates together with 5_5 imply that the nonzero real numbers form a group with respect to multiplication. [They say more than this because they also give us information about products in which 0 is a factor.] Postulate 5_6 links the operations of addition and multiplication. Postulate 5_7(a) is a definition; in view of the restriction, 5_7(b) does not quite define division as an operation on the real numbers.

Exercises

Part A

When dealing with algebraic terms which may involve division by 0 it is a good idea to indicate what restrictions are needed in order to avoid this impossible operation. For example, the term:

$$\frac{a^2 - b^2}{a - b}$$

In connection with our postulates for real numbers it is in order to make some remarks concerning nomenclature. First, for reasons which need not be gone into here, we find it convenient to consider that the right hand factor in an indicated product refers to the multiplier. It is because of this convention that we refer to the "right distributive law" as the distributive principle for multiplication over addition. Second, although we make no point of this distinction in the present text, we use 'principle' when referring to a true statement — which may or may not be a generalization. [For example, '2 + 2 = 4' is, under this usage, a principle.] Any principles are, of course, candidates for postulate-hood. There is no harm if, through preference or inadvertence, you read 'APA', say, as 'associative postulate for addition' rather than as 'associative principle for addition'. Third, definition principles are principles which might serve as definitions — 5_7(a) is an example of a definition principle which we have chosen to use as a definition; '$a + b = a - -b$' is also an example of a definition principle which in our postulational basis for real algebra happens not to be used as a definition. Finally, introduction principles are principles which might be used in lieu of definitions to introduce operators. For example, 5_3(a) introduces the oppositing operator '$-$' in a way that allows all required theorems about oppositing to be proved. [For consistency of nomenclature the PA0 and the PM1 should be called introduction principles.]

Of more moment than the preceding is some discussion of the conventions to which we adhere concerning the use of grouping symbols [for short, 'parentheses', whatever their shapes may be]. In the first place, parentheses are as much a part of the language we use when speaking about points, translations, and real numbers as are variables, operators, '0', '1', and '$\vec{0}$'. In a proper description of the grammar of the language parentheses occur in meaningful expressions according to strict rules — they are not, for example, merely inserted whenever the writer thinks their presence might be an aid to reading. Different languages will, of course, have different rules as regards to the use of parentheses. Our language has two very simple rules. Somewhat loosely stated they are these:

(1) There must be a pair of parentheses delimiting the total scope of any binary operator. [Examples: '$(a + 0)$', '$((A - B) + \vec{a})$', '$(\vec{a} + -\vec{b})$']

(2) There must be a pair of parentheses delimiting the scope of each suffixed singulary operator. [Examples: '$(-a)^2$', '$-(a)^2$']

Notice that no parentheses are associated with such prefixed singulary operators as, for example, oppositing operators. [The parentheses in both the examples for rule (2) "belong to" the exponent, a suffixed operator.] It can be proved that these rules are sufficient for the avoidance of all ambiguity. More easily, though, a little experience in using them will convince one of this and, at the same time, obviate any strangeness that such expressions as '$(a \cdot -a)$' may have.

Rules (1) and (2) are about as simple as punctuation rules can be and still prevent ambiguity. A competing system which may seem more familiar requires parentheses delimiting each of the two parts of the scope of a binary operator ['$(a) + (c)$', '$((A) - (B)) + (\vec{a})$'] and parentheses delimiting the scope of any singulary operator ['$(\vec{a}) + (-(\vec{b}))$', '$(-(a))^2$', '$(-(a)^2)$']. These rules are no more complicated than are (1) and (2), but, obviously, require the use of more parentheses.

TC 145 (2)

Rules (1) and (2) are part of the formal grammar of our language. A properly formed term will contain exactly those parentheses which are specified by these rules — no more and no fewer. Nevertheless, the simplicity in statement of rules (1) and (2) has been gained at the expense of requiring more parentheses than are actually needed and it is customary to regularize the use of ''slang'' by adopting additional rules which permit one to omit parentheses on some occasions where they are required by (1) and (2). In other words, one usually adopts rules for abbreviating terms. Such rules must be chosen with some care since, when one is faced with a term which has been abbreviated there must be no question as to how to replace the omitted parentheses.

The simplest such abbreviating rule concerns the omission of ''outermost parentheses'':

> (3) If a term is not part of another term and consists of an expression enclosed in parentheses then these parentheses may be omitted.

This rule concerns the removal of parentheses associated with a binary operator and could have been incorporated in rule (1). The next is similarly related to rule (2). Its purpose is to allow omission of parentheses from terms like '$((a + b))^2$', where the inner parentheses are required by (1) and the outer by (2), and like '$(a + (b)^2)$' in which the scope of the suffixed singulary operator is clear without the use of parentheses.

> (4) Parentheses required by (2) may be omitted if the expression they enclose either consists of an expression enclosed by parentheses or contains no operator.

For example '$-a^2$' must, if obtained by application of (4), be an abbreviation of '$-(a)^2$' since (4) does not permit the omission of parentheses from the only other possibility, '$(-a)^2$'.

In addition to these general rules (3) and (4) one usually adopts various rules which refer to parentheses associated with particular binary operators. One, which may be adopted for each binary operator permits the abbreviation of, say, '$(a + b) + c$' to '$a + b + c$'. The rule can also be extended to allow its application to strings of specified pairs binary operators, say, the pair ('+', '-') and the pair ('·', '÷'). So, '$((a - b) + c) - d$' may be abbreviated to '$a - b + c - d$'.

Still more special rules introduce conventions as to the relative ''strengths'' of various binary operators. One such rule is the one which allows, say, '$(a \cdot b) + (c \cdot d)$' to be abbreviated to '$ab + cd$'. Generally, as indicated, '·' is considered to be stronger than '+' [or '-'] and so does not to need parentheses to hold its operands together against the ''pull'' of '+'.

Finally, one can choose to omit one binary operator — usually '·' — and to represent the corresponding operation by mere juxtaposition. Strictly speaking, this rule can be adopted without any restrictions as long as no symbol is used both as binary operator and a prefixed singulary operator. Since, however, in our notation, the binary operator '-' and the singulary operator '$-$' might be confused, such a rule needs, practically, to be restricted.

The preceding discussion of grouping symbols has dealt with their use in constructing terms. It applies, however, equally well to the construction of compound sentences. Here, there is a rule like (1) with 'binary operator' replaced by 'binary sentence connective'. Due to the absence of analogues of suffixed singulary operators, there is no need for a rule analogous to (2). There is also a rule analogous to (3) and one may — but we shall not — make use of other abbreviating rules based on the notion of the strength of various connectives. In some systems, one binary connective — usually 'and' — is omitted in favor of juxtaposition. On the basis of the analogues of (1) and (2), sentences of the forms, say, '[(p and q) $\Longrightarrow$ r]' and '(p and [q $\Longrightarrow$ r])' are well-formed and may be abbreviated to similar sentences of the forms '(p and q) $\Longrightarrow$ r' and 'p and [q $\Longrightarrow$ r]', respectively. [On an informal basis, we tend to use brackets with '$\Longrightarrow$' and '$\Longleftrightarrow$' and parentheses with other binary connectives.] Also, a sentence of the form '[not p $\Longrightarrow$ q]' is a conditional sentence whose antecedent is a denial sentence, while one of the form 'not [p $\Longrightarrow$ q]' is the denial of some conditional sentence. The brackets may be omitted from sentences of the first type — when it is not part of a longer sentence — but not from one of the second type.

TC 145 (3)

TC 146 (1)

Answers for Part A

1. (a) $p + 13q$ (b) $10r^2 - 14r$ (c) $6a^2 + 13ab - 5b^2$

(d) $\frac{a}{a + b}$ $[a + b \neq 0]$ (e) $\frac{p - 3}{(p + 3)(p + 2)}$ $[-3 \neq p \neq -2]$

(f) 0 (g) -2 (h) 0 $[a \neq 0]$

[Note that oppositing and reciprocating of nonzero real numbers are permutable operations and that each is its own inverse.]

2. (a) 7/2 (b) 2 (c) $-1/12$ (d) 4

(e) 9, -3 (f) -9, 3 (g) [no solutions] (h) [no solutions]

[In each of (g) and (h), students may produce arguments which show that no numbers other than 3 [or, in the case of (h), perhaps, none other than 3 and -3] can be solutions. Such arguments contain steps which lead from one equation to a nonequivalent equation [multiplying on both sides by a factor which has 0 as one of its values].]

3. (a) $t = 2$ (b) $t = 6$

(c) $t = 12$ (d) $t = 4$

4. The error occurs at the word 'consequently'. Since $b = a - 1$, $(a - b) - 1 = 0$ and it does not follow from the previous equation that $a + b = 1$.

can, as you know, be simplified:

$$\frac{a^2 - b^2}{a - b} = \frac{(a + b)(a - b)}{a - b} = a + b \quad [a \neq b]$$

The bracketed restriction calls attention to the fact that the value of the given term, for given values of the variables 'a' and 'b', is the same as the corresponding value of '$a + b$' in case the given values of 'a' and 'b' are different. So, *subject to the restriction* one may replace the given term by the simpler one.

1. Simplify each of the following, noting any necessary restrictions.

(a) $7(p + q) - 6(p - q)$ **(b)** $(5r - 7)(r + 3) + (5r - 7)(r - 3)$

(c) $(3a - b)(2a + 5b)$ **(d)** $\frac{2a + b}{a + b} - 1$

(e) $\frac{6}{p + 3} + \frac{-5}{p + 2}$ **(f)** $a(b - 3) + a(3 - b)$

(g) $/-/2$ **(h)** $-/a - /-a$

2. Solve these equations.

(a) $5r - 2 + 7(3 - r) = 12$ **(b)** $2 - (1 - s) + 7s = 5 - 3(s - 6)$

(c) $t + 2(5t - 1) = -t - 3$ **(d)** $6 + (b - 3)^2 = (b + 1)^2 - 2$

(e) $(p - 9)(p + 3) = 0$ **(f)** $q^2 + 6q - 27 = 0$

(g) $a - \frac{3}{3 - a} = 2 + \frac{a}{a - 3}$ **(h)** $a - \frac{1}{a - 3} = \frac{3a^2 - a - 30}{a^2 - 9}$

3. For each of the following, write a sentence which has the same roots and which begins with '$t =$'. [In other words, solve for 't'.]

(a) $4t - (12 - 3t) = -5t + 12$ **(b)** $\frac{t}{2} - 2 = \frac{t}{6}$

(c) $3 - t/3 = -1$ **(d)** $t(t - 3) + t^2 = 2(t^2 - 6)$

4. Criticize the following argument.

2 is the only even number. For, suppose that m is even. It follows that $m = a + (b + 1)$ where $b = a - 1$. Since $a = b + 1$ it follows that $a^2 - 2a = (b + 1)^2 - 2(b + 1) = b^2 - 1$ and, so, that $a^2 - b^2 = 2a - 1$. From this it follows that $(a^2 - b^2) - (a + b) = (2a - 1) - (a + b) = (a - b) - 1$. On the other hand, $(a^2 - b^2) - (a + b) = (a + b)(a - b) - (a + b) = (a + b)((a - b) - 1)$. So, $(a + b)((a - b) - 1) = (a - b) - 1$ and, consequently, $a + b = 1$. Since $m = a + (b + 1) = (a + b) + 1$, $m = 2$. Hence, if m is an even number then $m = 2$.

Part B

Give column proofs of each of these real number-theorems.

1. $a + a = a \longrightarrow a = 0$
2. $0 \cdot a = 0$ [*Hint:* $0 \cdot a = (0 + 0) \cdot a = \ldots$.]
3. $a \cdot (b + c) = a \cdot b + a \cdot c$ [This is the left distributive principle for multiplication over addition.]

Answers for Part B

1.	(1)	$a + a = a$	[assumption]*
	(2)	$a + -a = a + -a$	[valid sentence]
	(3)	$(a + a) + -a = a + -a$	[(1), (2)]
	(4)	$(a + b) + c = a + (b + c)$	[APA]
	(5)	$(a + a) + -a = a + (a + -a)$	[(4)]
	(6)	$a + (a + -a) = a + -a$	[(5), (3)]
	(7)	$a + -a = 0$	[IPO]
	(8)	$a + 0 = 0$	[(7), (6)]
	(9)	$a + 0 = a$	[PA0]
	(10)	$a = 0$	[(9), (8)]
	(11)	$a + a = a \Longrightarrow a = 0$	[(10), *(1)]
2.	(1)	$0 + 0 = 0$	[PA0]
	(2)	$0 \cdot a = 0 \cdot a$	[valid]
	(3)	$(0 + 0) \cdot a = 0 \cdot a$	[(1), (2)]
	(4)	$(a + b) \cdot c = a \cdot c + b \cdot c$	[DPMA]
	(5)	$(0 + 0) \cdot a = 0 \cdot a + 0 \cdot a$	[(4)]
	(6)	$0 \cdot a + 0 \cdot a = 0 \cdot a$	[(5), (3)]
	(7)	$a + a = a \Longrightarrow a = 0$	[Exercise 1]
	(8)	$0 \cdot a + 0 \cdot a = 0 \cdot a \Longrightarrow 0 \cdot a = 0$	[(7)]
	(9)	$0 \cdot a = 0$	[(6), (8)]
3.	(1)	$a \cdot b = b \cdot a$	[CPM]
	(2)	$a \cdot (b + c) = (b + c) \cdot a$	[(1)]
	(3)	$(a + b) \cdot c = a \cdot c + b \cdot c$	[DPMA]
	(4)	$(b + c) \cdot a = b \cdot a + c \cdot a$	[(3)]
	(5)	$a \cdot (b + c) = b \cdot a + c \cdot a$	[(4), (2)]
	(6)	$a \cdot (b + c) = a \cdot b + c \cdot a$	[(1), (5)]
	(7)	$c \cdot a = a \cdot c$	[(1)]
	(8)	$a \cdot (b + c) = a \cdot b + a \cdot c$	[(7), (6)]

4.02 More about Reciprocating

As you can easily prove – by using the theorem of Exercise 2 of Part B and the postulate '$0 \neq 1$' – the product of 0 by a real number is never 1. In short, 0 has no multiplicative inverse. On the other hand, the introduction principle for reciprocating:

5_3(b) $\quad a \cdot /a = 1 \; [a \neq 0]$

tells us that the reciprocal of any *nonzero* real number is that number's multiplicative inverse. Since the only use for reciprocals is as multiplicative inverses, it is customary not to assign any meaning to '/0'; and – principally to avoid argument – we shall follow this custom. [Note, however, that while no one could "define into existence" a multiplicative inverse of 0, there is no reason aside from custom for not assigning a reciprocal to 0. The worst that would happen if, say, one defined /0 to be 0 would be that the reciprocal of 0 would not have many of the interesting properties which are shared by the reciprocals of other numbers.]

Since we shall need to use 5_3(b) in proving theorems, we need to decide how to take account of restrictions such as the one in this postulate. There are several possibilities. We shall adopt the simplest, which is to consider 5_3(b) as an abbreviation for the conditional sentence:

(1) $\quad a \neq 0 \Longrightarrow a \cdot /a = 1$

There are objections which can be raised to any method of dealing with restrictions, and the objection to this one is that from (1) we can derive the substitution-instance:

(2) $\quad 0 \neq 0 \Longrightarrow 0 \cdot /0 = 1$

Because we have assigned no meaning to '/0', this sentence is without meaning. We need not, however, take this objection too seriously. For, on the one hand, (2) is not a sentence which is likely to interest us and, on the other, we could easily insure that (2) would make sense – although remaining uninteresting – merely by choosing a meaning for '/0'. In any case, we cannot use (2) to prove '$0 \cdot /0 = 1$' since we certainly have no way of proving '$0 \neq 0$'. In short, whatever decision we might make about '/0', we should never be able to prove that 0 has a multiplicative inverse.

As (2) illustrates, '/0' is going to show up in our theorems. So, there is no point in keeping the restriction in 5_0(f), and *we shall drop it.*

Although any singulary multiplying operation — multiplying by 2, say, or multiplying by 0 — is a mapping of the set ℜ of real numbers into itself, the case in which the multiplier is 0 differs fundamentally from that in which the multiplier is some nonzero number. Multiplication by a nonzero real number is a one-to-one mapping of ℜ onto itself and, so, has an inverse. For example, the inverse of multiplying by 2 is multiplying by 0.5. On the other hand, since multiplying by 0 maps each real number on 0, this mapping has no inverse. Put in another way, for any nonzero real number a, there is a number y — the multiplicative inverse of a — such that $a \cdot y = 1$. But, there is no number y such that $0 \cdot y = 1$ — 0 has no multiplicative inverse. [Note that the word 'inverse' is used with two quite different, although closely related, meanings.] In contrast, each singulary adding operation — i.e., each translation of ℜ — has an inverse. Equivalently, each real number has an additive inverse.

If we read '−' as 'the opposite of' then 5_3(a) says that the opposite of any real number is an additive inverse of that number. Since from the existence of an additive inverse we can show its uniqueness [by using the APA and the PA0], we can think of 5_3(a) as a characterization of oppositing. Similarly, if we read '/' as 'the reciprocal of' then 5_3(b) says that the reciprocal of a nonzero real number is its multiplicative inverse; and 5_3(b) may be thought of as a partial characterization of reciprocation.

If we let matters rest at this point then '/0' is, at best, undefined and, at worst, nonsense. The disadvantages which result from insisting that '/0' be nonsense are considerable. To see what these are, note that prior to the introduction of '/' [or of '÷'] it is possible to give formal rules of grammar for constructing real number terms so that each term constructed in accordance with these rules will have a real number as its value, whatever values are chosen for the variables which occur in the term. [These rules correspond with 5_0(a) - (e) on page 145.] As a result, it is possible to give further rules of grammar for constructing real number sentences so that any sentence obtainable from such a sentence by substituting numerals for its variables is meaningful [although it may, of course, be false]. Similarly, in our algebra of points and translations, Postulates 1 and 4_0 may be made the basis for definitions of 'point-term' and 'translation-term'. By requiring that the sides of an equation be terms of the same kind, and that substitution-instances of sentences be obtained by substituting for variables terms which are of the same kind as the variable, we can make sure that everything we write is meaningful. However — returning to the algebra of real numbers — this simple state of affairs is destroyed once '/' is introduced if '/0' is regarded as meaningless. Then, '/a', say, is a pattern for both meaningful and meaningless terms. It is also a pattern for terms which, like '/(a + b)', have values for some values of 'a' and 'b' but not for others, and for terms which, like '/(a - a)', has a value for no value of 'a'. "Most" sentences in which '/' occurs will, like (1) on page 147, have both meaningful and meaningless instances. The problem of formulating rules of grammar for sorting out the meaningful instances is the problem of programming a computer for solving arbitrary algebraic equations and, so, is unsolvable.

The preceding considerations show that if '/' is introduced and '/0' is held to be nonsense then one must accept as meaningful sentences which have meaningless instances, and that there is no way of restricting the substitution rule so as to insure that meaningful sentences shall not have meaningless consequences. [Similar results would follow were '÷' taken as primitive and expressions containing '÷ 0' ruled meaningless.]

In view of this it is almost necessary to agree that '/0' does have some meaning or other and, for simplicity, that it denotes some real number. [Of course, whatever number we might choose to let it denote, this number could not be a multiplicative inverse of 0.] Acceding to this necessity amounts, formally, to dropping the restriction '[a ≠ 0]' from 5_0(f). [Of course, the restriction on 5_3(b) must remain, but we may now construe 5_3(b), as in the text, as an abbreviation for the conditional sentence (1). And we need have no qualms as to the meaningfulness of (1) or of any of its instances.] Whether or not we keep the restriction in 5_7(b) is less important. As a matter of elegance, it should be dropped; but it is probably better pedagogy to preserve it. One result of preserving it will be discussed shortly. Finally, it would be more forthright to add to 5_3(b) a definition of '/0'. This might be either '/0 = 0' or '/0 = 1'. But pedogogical considerations again suggest that this is better left undone.

The result of this tampering with the meaning of 'reciprocal' — or, more specifically, with the meaning of '/' — [but not with the meaning of 'multiplicative inverse'] is that the logical notions expressed by 'consequence' and 'imply' are allowed to keep their formal character and our rules of logic need not be cluttered up with restrictions whose enforcement poses generally unsolvable problems as to what expressions are meaningful. It is perhaps unfortunate that the advantages of this can be fully appreciated only after one has strived to proceed logically on the basis of the assumption that '/0' is meaningless.

There is one subject for which the present procedure might be considered pedagogically disadvantageous in comparison with the usual informal treatment of meaningless expressions. [As has been pointed out, this informal treatment cannot be satisfactorily formalized.] The subject in question is that of teaching the solution of fractional equations. Supposing such an equation to have been transformed into the form 'a/b = 0', one customarily says that its solutions are those zeros of the numerator which are not zeros of the denominator — the latter clause because a fraction is meaningless if the value of its denominator is 0. What this amounts to is saying that when one is asked to solve an equation of the form 'a/b = 0', what one is expected to do is to solve the system 'a = 0, b ≠ 0'. Under the present procedure, this would have to be said explicitly. By 5_7(c) [and the theorem '0 · /b = 0'] each zero of the numerator which is not a zero of the denominator is a root of the fractional equation. But, there is nothing in our postulates which will help one to decide whether a zero of the denominator is or is not a root of the equation. Lack of utility rather than resultant lack of meaning must be given as a reason for ignoring such roots should they exist. Properly developed, the former reason can be made at least as appealing as is the latter.

The preceding discussion of reciprocation and division is more lengthy — and raises more questions — than classroom treatment of the subject warrants. Your reading of it should give you a better understanding of the background for section 4.02 and of the reasons for the conventions which are adopted there. In teaching, however, you should stick rather closely to the text and, if necessary, discourage students from sterile discussions of the varying results of defining '/0' in this or that way. We have tried to avoid such questions by refraining

from adopting any such definition in the text and by suggesting that although, in consequence, '/0' has no [definite] meaning, things become simpler, and no harm results, from pretending that it does have some meaning unknown to us. And this, indeed, is the burden of the preceding discussion. The one part of this discussion which may be of explicit use in the classroom is the comparison in the second paragraph of 5_3(a) as a characterization of oppositing and 5_3(b) as an only partial characterization of reciprocation. The whole argument is, of course, predicated on a conceptual distinction between the notion of reciprocal and that of multiplicative inverse.

Turning now to the text, itself, it is perhaps worth noting that the reason for asserting that (2) on page 147 is an instance of (1) is Postulate 5_0(b). Since '/' is already present in (1), the restriction in 5_0(f) is of no help in ruling (2) out as an instance of (1). The moral is that if we use '/' in sentences, there is no way of avoiding the appearance of '/0' due to substitution and, perhaps, subsequent algebraic simplifications.

We shall, however, keep the restriction in $5_7(b)$ [the defining principle for division]. In view of our decision as to how to interpret restrictions, $5_7(b)$ is, then, an abbreviation for:

$$(3) \qquad b \neq 0 \Longrightarrow a \div b = a \cdot /b$$

Since multiplying by 0 is not a one-to-one mapping – and, so, has no inverse – there is no "dividing by 0" operation in the usual sense of 'dividing by'. Although from (3) we may infer:

$$0 \neq 0 \Longrightarrow a \div 0 = a \cdot /0$$

we cannot prove '$a \div 0 = a \cdot /0$'. So, the question as to whether '$\div\, 0$' has a meaning in some unusual sense is left open. This need not concern us since, whatever meaning we might choose to give it, we should never be able to prove '$(ab) \div b = a$' without the assumption '$b \neq 0$'.

As an example, let's see how we can now prove:

$$(*) \qquad b \neq 0 \Longrightarrow (ab) \div b = a$$

To shorten the proof we shall not state the postulates on which the conclusion depends but only the instances of them which are used in the proof.

(1) $b \neq 0$	[assumption]*
(2) $b \neq 0 \Longrightarrow (ab) \div b = (ab) \cdot /b$	[5_7(b)]
(3) $(ab) \div b = (ab) \cdot /b$	[(1), (2)]
(4) $(ab) \cdot /b = a(b \cdot /b)$	[5_1(b)]
(5) $b \neq 0 \Longrightarrow b \cdot /b = 1$	[5_3(b)]
(6) $b \cdot /b = 1$	[(1), (5)]
(7) $(ab) \cdot /b = a \cdot 1$	[(6), (4)]
(8) $a \cdot 1 = a$	[5_2(b)]
(9) $(ab) \cdot /b = a$	[(8), (7)]
(10) $(ab) \div b = a$	[(9), (3)]
(11) $b \neq 0 \Longrightarrow (ab) \div b = a$	[(10), *(1)]

We can safely abbreviate proofs like this one by introducing restrictions as illustrated in the following:

(1) $(ab) \div b = (ab) \cdot /b$	[$b \neq 0$]	[5_7(b)]
(2) $(ab) \cdot /b = a(b \cdot /b)$		[5_1(b)]
(3) $b \cdot /b = 1$	[$b \neq 0$]	[5_3(b)]
(4) $(ab) \cdot /b = a \cdot 1$	[$b \neq 0$]	[(3), (2)]
(5) $a \cdot 1 = a$		[5_2(b)]
(6) $(ab) \cdot /b = a$	[$b \neq 0$]	[(5), (4)]
(7) $(ab) \div b = a$	[$b \neq 0$]	[(6), (1)]

The second proof of (☆) which is given on page 148 may be interpreted in two ways. It is an abbreviation of the first proof in that it can be looked on as "shorthand" instructions for reproducing this proof. On the other hand, it can be thought of as an actual, unabbreviated, proof in which the use of stronger rules of inference allow fewer steps than needed in the first proof. In the first proof, '$b \neq 0$' is adopted as an assumption under which the remainder of the argument is carried out. What happens in the second proof can be illustrated by considering the inference of step (4) from steps (3) and (2). Recalling that (3), say, is an abbreviation for '$b \neq 0 \Longrightarrow b \cdot /b = 1$' the rule which justifies inferring (4) from (3) and (2) may be stated as follows:

If an inference of the form:

$$\frac{r \qquad s}{q}$$

is valid then so is any corresponding inference of the form:

$$\frac{r \qquad p \Longrightarrow s}{p \Longrightarrow q}$$

In short, argue as though the restriction in line (3) were not there, but then transfer this restriction to the conclusion in line (4). The rule stated above is easily justified:

$$\cfrac{\cfrac{r \qquad \cfrac{\overset{*}{p} \qquad p \Longrightarrow s}{s}\,(\text{M. P.})}{q} \longleftarrow \text{assumed valid}}{p \Longrightarrow q}\ {*}\,(\text{Ded. Rule})$$

Other such rules [for example, one with 'r' replaced by '$p \Longrightarrow r$'] are equally easily justified.

In the scheme given above it is assumed that the given inference is such that the deduction rule may be applied as indicated. This is not always the case. For example:

$$\frac{a \cdot /a = 1}{b \cdot /b = 1} \text{ is valid by (Subst), but: } \frac{a \neq 0 \Longrightarrow a \cdot /a = 1}{a \neq 0 \Longrightarrow b \cdot /b = 1} \text{ is not valid.}$$

This should cause students no difficulty for, in a column proof with restrictions in which (Subst) is used, the natural thing is to make the substitution in the restriction. One would write:

$$\begin{array}{lll} (n) & a \cdot /a = 1 & [a \neq 0] \\ (n+1) & b \cdot /b = 1 & [b \neq 0] \end{array};\ \text{not:}\ \begin{array}{lll} (n) & a \cdot /a = 1 & [a \neq 0] \\ (n+1) & b \cdot /b = 1 & [a \neq 0] \end{array}$$

Up to now, substitution-inferences are the only ones which lie outside the scope of the general rule we have been discussing. For these the rule holds only if the variable which is substituted for does not occur in the sentence which replaces 'p'. Other exceptions will occur when quantifiers are introduced in Chapter 6.

By our convention as to the meaning of restrictions, the restricted conclusion (7) is an abbreviation of (*). In a paragraph, this shorter proof would be:

For $b \neq 0$, $(ab) \div b = (ab) \cdot /b$. Now, since $(ab) \cdot /b = a(b \cdot /b)$ and, for $b \neq 0$, $b \cdot /b = 1$ it follows that, for $b \neq 0$, $(ab) \cdot /b = a \cdot 1$. Also, since $a \cdot 1 = a$ it follows that, for $b \neq 0$, $(ab) \cdot /b = a$. Consequently, for $b \neq 0$, $(ab) \div b = a$.

Notice that lines (2) – (6) of the preceding proof [and the second and third sentences of the paragraph] constitute a proof of the theorem:

$$(*)' \qquad b \neq 0 \Longrightarrow (ab) \cdot /b = a$$

Since the theorem '$(ab)c = (ac)b$' is an easy consequence of 5_1(b) and 5_4(b) it follows immediately that:

$$(**) \qquad b \neq 0 \Longrightarrow (a \cdot /b)b = a$$

is a theorem. From (**) and 5_7(b) it is easy to derive:

$$b \neq 0 \Longrightarrow (a \div b)b = a$$

[Since, for $b \neq 0$, $(a \cdot /b)b = a$ [by (**)], and since, for $b \neq 0$, $a \div b = a \cdot /b$, it follows that, for $b \neq 0$, $(a \div b)b = a$.]

We have already noted that the nonzero real numbers form a commutative group with respect to multiplication. [Establishing this fact requires the proof of two theorems which we shall discuss in the next section.] Since this is the case, any theorem about addition, 0, oppositing and subtraction which is a consequence of 5_1(a) – 5_4(a) and 5_7(a) can be transformed into a theorem about multiplication, 1, reciprocating, and division merely by changing '+'s to '·'s, '0's to '1's, '–'s to '/'s, '–'s to '÷'s, and adding "nonzero restrictions" for all variables. For example, since:

$$a + c = b + c \Longrightarrow a = b$$

is such a theorem, the sentence:

$$a \cdot c = b \cdot c \Longrightarrow a = b \quad [a \neq 0, b \neq 0, c \neq 0]$$

or, equivalently:

$$(a \neq 0 \text{ and } b \neq 0 \text{ and } c \neq 0) \Longrightarrow [a \cdot c = b \cdot c \Longrightarrow a = b]$$

is certainly a theorem. However, as you know from your previous study of algebra, we can do better. The sentence:

$$a \cdot c = b \cdot c \Longrightarrow a = b \quad [c \neq 0]$$

TC 148 (2)

There should be no need to explain this second interpretation of the proof to your students. They are likely to be happy enough to accept this way of presenting proofs as a welcome way of avoiding excess writing. All they need remember is that any sentence in a proof must be accompanied by all restrictions which affect any of the sentences from which it is inferred.

TC 149

In the paragraph-variant of the proof just discussed, the recurrent phrase 'for b ≠ 0' occurs more often than may be judged strictly necessary — or euphonious. The purpose here has been to ape the column proof; but, in practice, the phrase need be repeated only often enough to maintain awareness that it is still in force. For comparison, here is a paragraphing of the first of the two proofs of (☆):

> Suppose that $b \neq 0$. It follows that $(ab) \div b = (ab) \cdot /b$. Since $(ab) \cdot /b = a(b \cdot /b)$ and [since $b \neq 0$] $b \cdot /b = 1$ it follows that $(ab) \cdot /b = a \cdot 1$. Also, since $a \cdot 1 = a$ it follows that $(ab) \cdot /b = a$ and, so, that $(ab) \div b = a$. Hence, if $b \neq 0$ then $(ab) \div b = a$.

The cancellation principle for multiplication as it is stated first on page 149 illustrates the use of multiple restrictions. It would be consistent with our interpretation of single restrictions to think of this as an abbreviation for:

$$c \neq 0 \Longrightarrow [b \neq 0 \Longrightarrow [a \neq 0 \Longrightarrow [a \cdot c = b \cdot c \Longrightarrow a = c]]]$$

In view of the importation and exportation rules on page 101 this is equivalent to the interpretation in terms of 'and' which is given in the text.

is a theorem. Since your previous experience should be a reliable guide in judging whether a sentence is an "addition-subtraction theorem" and, also, in judging what restrictions are required for the corresponding "multiplication-division theorem", you will not often have occasion to prove theorems of these kinds.

Exercises

Part A

1. The theorem (☆) on page 148 is analogous to an "addition-subtraction" theorem:

$$(a + b) - b = a$$

Write a 7-line proof of this theorem like the proof of (☆) on page 148.

2. Compare the two proofs and note how, given *either*, you could easily obtain the other.

Part B

1. Show that the cancellation principle:

$$ac = bc \longrightarrow a = b \quad [c \neq 0]$$

is a consequence of the theorem (☆)′ on page 149. [*Hint:* As you know from your work in Chapter 2, '$a = b \longrightarrow ac = bc$' is a valid sentence. You can use an instance of this and two instances of (☆)′ to obtain a very short proof.]

2. The theorem proved in Exercise 1 is:

$$c \neq 0 \longrightarrow [ac = bc \longrightarrow a = b]$$

Two of the rules of logic you studied in Chapter 2 tell you that this sentence has the same meaning as another sentence which contains only one '$\longrightarrow$'. What rules? What other sentence?

Part C

Prove: $/a = 0 \longrightarrow 0 = 1 \quad [a \neq 0]$

[*Hint:* Use 5_3(b) and a theorem proved in an earlier set of exercises.]

4.03 Rules for 'not'

Up to now the word 'not' or abbreviations for it have not occurred very often in our theorems. Now that we have postulates '$0 \neq 1$', '$a \neq 0 \longrightarrow a \cdot /a = 1$' and '$b \neq 0 \longrightarrow a \div b = a \cdot /b$', in which '$\neq$' means what 'is *not* the same as' does, we need some new rules of logic

---

In Chapter 3 we have already expressed confidence in a student's ability to judge whether or not a sentence about real numbers is a theorem and, if so, to determine what postulates its proof is based on. If our confidence turns out to be misplaced, you may need to ask some true-false questions, and discuss how to go about attempting to prove some theorems. There is, however, no need to insist on mastery of these skills at this time. If practice is needed, opportunities for it will arise throughout the course.

Answers for Part A

1.	(1)	$(a + b) - b = (a + b) + -b$	[DPS]
	(2)	$(a + b) + -b = a + (b + -b)$	[APA]
	(3)	$b + -b = 0$	[IPO]
	(4)	$(a + b) + -b = a + 0$	[(3), (2)]
	(5)	$a + 0 = a$	[PA0]
	(6)	$(a + b) + -b = a$	[(5), (4)]
	(7)	$(a + b) - b = a$	[(6), (1)]

2. Given either proof changing from additive to multiplicative notation [or vice versa], and introducing [or deleting] "nonzero" restrictions, yields the other. [Also, one must revise the comments, but these are not — strictly — part of the proof.]

[These exercises show one advantage which this style of proof has over that exemplified by the first proof of (☆) given on page 148 To change that proof into a proof of the theorem of Exercise 1 is a more complicated task.]

Answers for Part B

1.	(1)	$ac = bc \Longrightarrow (ac)\cdot/c = (bc)\cdot/c$		[valid]
	(2)	$(ac)\cdot/c = a$	$[c \neq 0]$	[(☆)]
	(3)	$(bc)\cdot/c = b$	$[c \neq 0]$	[(☆)]
	(4)	$ac = bc \Longrightarrow a = b$	$[c \neq 0]$	[(2), (3), (1)]

[Two applications of the replacement rule for equations have been combined into a single step, thus saving a line in writing the proof.]

2. By importation and exportation [page 101] the given sentence has the same content as does:

$$(c \neq 0 \text{ and } ac = bc) \Longrightarrow a = b$$

Answers for Part C

(1)	$/a = 0$		[assumption]
(2)	$a \cdot 0 = 0$		[theorem]
(3)	$a \cdot /a = 0$		[(1), (2)]
(4)	$a \cdot /a = 1$	$[a \neq 0]$	[IPR]
(5)	$0 = 1$		[(3), (4)]
(6)	$/a = 0 \Longrightarrow 0 = 1$	$[a \neq 0]$	[(5), *(1)]

[A slight variant of this proof is given as a tree in Part E on page 156. The purpose of proving this theorem is to lay the ground work for a proof of '$/a \neq 0 \ [a \neq 0]$'. You can point this out while discussing the rule modus tollens given on page 152.]

to tell us how to deal with sentences in which 'not' [or an abbreviation for it] occurs.

One such rule which we need at this point can be illustrated by using the cancellation principle of Part B, above. For example, knowing that $3 \neq 0$, it follows from this theorem that

$$1768 \cdot 3 = 1766 \cdot 3 \Longrightarrow 1768 = 1766.$$

You know, of course, that 1768 is not 1766; hence—even if you had forgotten everything you know about multiplication—you can conclude at once that $1768 \cdot 3$ is not $1766 \cdot 3$.

What this boils down to is that because of what you mean by 'if . . . then ___' and 'not', you recognize that the inference:

$$\frac{1768 \cdot 3 = 1766 \cdot 3 \Longrightarrow 1768 = 1766 \qquad 1768 \neq 1766}{1768 \cdot 3 \neq 1766 \cdot 3}$$

is a valid one. The argument here is of the same kind as the one you would use to decide that because John does not live in Illinois, he doesn't live in Chicago:

$$\frac{\text{If John lives in Chicago then John lives in Illinois.} \qquad \text{John doesn't live in Illinois.}}{\text{John doesn't live in Chicago.}}$$

Before stating the rule which these two inferences illustrate, we need to come to terms with a matter of grammar. As you know, there are usually several ways to formulate a sentence in English which denies what is asserted by a given sentence. For example, to deny that John lives in Illinois we may say either:

John does not live in Illinois.

or:

It is not the case that John lives in Illinois.

For simplicity, we shall introduce another way which, while not good English, is perfectly satisfactory for the sentences of our language; to form the *denial* of a given sentence, we shall merely write 'not' in front of it. [Sometimes, in place of 'not' we shall use a wiggle, '~'.] Using this convention, the denial of the sentence 'John lives in Illinois,' is:

not [John lives in Illinois].

The denial of a given sentence is the sentence obtained from it by prefixing 'not' [or: '~']. Some authors use 'negation' instead of 'denial', and '−' and '¬' are not uncommon substitutes for '~'. One reason for preferring 'denial' to 'negation' is that derivatives of the word 'negative' already suffer from overwork.

A denial sentence is a sentence which is the denial of some [other] sentence.

Students are unlikely to have difficulty in accepting the rule modus tollens. The order in which the premisses are written in a modus tollens-inference is immaterial. Nevertheless, as in the case of other kinds of 2-premiss inferences, it is convenient to adopt one order and stay with it. The order chosen in the text corresponds to arguments of the form:

If p then q; but, not q. Therefore, not p.

and that of '1768 = 1766' is either:

$$\text{not } [1768 = 1766] \quad \text{or:} \quad \sim [1768 = 1766]$$

[We shall, however, continue to use '1768 ≠ 1766' as an abbreviation.]

We can now state the inference rule which has been illustrated above. We give it its Latin name.

> **Modus Tollens**
>
> Any inference of the form:
>
> $$\begin{array}{c} p \Longrightarrow q \quad \text{not } q \\ \hline \text{not } p \end{array}$$
>
> is valid.

For practice, tell which of the following can be completed to give examples of inferences of this new kind. Tell why the others cannot be made into such examples.

(a) $\begin{array}{c} A + \vec{a} = B \Longrightarrow \vec{a} = B - A \quad \vec{a} \neq B - A \\ \hline \end{array}$

(b) $\begin{array}{c} A = B \Longrightarrow C - A = C - B \\ \hline C - A \neq C - B \end{array}$

(c) $\begin{array}{c} B - A = \vec{0} \Longrightarrow A = B \\ \hline \end{array}$

(d) $\begin{array}{c} A = B \Longrightarrow B - A = \vec{0} \\ \hline A = B \end{array}$

(e) $\begin{array}{c} A \neq B \Longrightarrow A + \vec{a} \neq B + \vec{a} \\ \hline A = B \end{array}$

As you have probably decided, just two of (a) – (e) can be completed to give examples of modus tollens. If you completed (b) by introducing '$A \neq B$' as a second premiss, you are guilty of a fallacy called *denying the antecedent*. This is something like the fallacy of mistaking a conditional sentence for its converse because, from the *converse* of the premiss given in (b) together with '$A \neq B$' you may infer the giver conclusion. [By what rule?] You probably realized that (d) cannot be completed to give an example of modus tollens, but (e) may have fooled you. If, for example, you believe that:

$$\begin{array}{c} A \neq B \Longrightarrow A + \vec{a} \neq B + \vec{a} \quad A + \vec{a} = B + \vec{a} \\ \hline A = B \end{array}$$

is a *valid* inference, you are certainly right; if you think it is an example of modus tollens, you are wrong. If you are to use the sentence:

$$A \neq B \Longrightarrow A + \vec{a} \neq B + \vec{a}$$

Completions for illustrations on page 152:

(a) $A + \vec{a} \neq B$ [or: not $A + \vec{a} = B$, or: $\sim A + \vec{a} = B$]

(b) [Impossible to complete, since conclusion of inference must be the denial of the antecedent of the first premiss.]

(c) premiss: $A \neq B$; conclusion: $B - A \neq \vec{0}$

(d) [Impossible to complete.]

(e) [Impossible to complete.]

As explained in the text, (e) cannot be completed to a modus tollens-inference because its conclusion is not the denial of the antecedent of the first premiss. In fact, the conclusion of (e) is not the denial [or an abbreviation of the denial] of any sentence. That the completion for (e) which is given on page 152, is, by our rules, a valid inference is shown in Exercise 1 of Part A on page 153.

We don't recommend that you try to maintain this rigid interpretation of modus tollens very long. Students will quite naturally apply modus tollens and one of the double denial rules simultaneously, and after the exercises of Parts A - D, should be permitted to do so.

as the first premiss of a modus tollens inference then the second premiss must be the denial of the consequent '$A + \vec{a} \neq B + \vec{a}$'. But, the denial of this sentence is:

$$\text{not } A + \vec{a} \neq B + \vec{a}$$

or, removing the abbreviation '/':

$$\text{not not } A + \vec{a} = B + \vec{a}$$

Even though, intuitively, this sentence says just what '$A + \vec{a} = B + \vec{a}$' does, they are certainly different sentences. Evidently, in order to express what we mean by 'not', we need another rule:

> Rules for Double Denials
>
> Any inference of either of the forms:
>
> $$\frac{\text{not not } p}{p} \qquad \frac{p}{\text{not not } p}$$
>
> is valid.

Using this rule and modus tollens it is not difficult to show that the completion given above for (e) is a valid inference. [See Exercise 1 of Part A.]

Exercises

Part A

1. Complete the following derivation to show that the inference just mentioned is valid.

$$\cfrac{A \neq B \longrightarrow A + \vec{a} \neq B + \vec{a} \qquad \cfrac{A + \vec{a} = B + \vec{a}}{\qquad}}{\cfrac{\qquad}{A = B}} \text{(Modus Tollens)}$$

2. Write out a pattern for derivations like the one you gave in answer to Exercise 1 which will show that any inference of the form:

$$\frac{\text{not } q \longrightarrow \text{not } p \quad p}{q}$$

is valid.

3. Use modus tollens and the deduction rule to show that any inference of the form:

$$\frac{p \longrightarrow q}{\text{not } q \longrightarrow \text{not } p}$$

is valid.

Answers for Part A

1. not $A + \vec{a} \neq B + \vec{a}$ [which is an abbreviation of 'not not $A + \vec{a} = B + \vec{a}$']; not $A \neq B$

2.

$$\cfrac{\cfrac{\text{not } q \Longrightarrow \text{not } p \qquad \cfrac{p}{\text{not not } p}}{\text{not not } q}}{q}$$

3.

$$\cfrac{\cfrac{p \Longrightarrow q \qquad \overset{*}{\text{not}}\ q}{\text{not } p}}{\text{not } q \Longrightarrow \text{not } p}\,*$$

4. Exercise 3 should suggest another form for the rule you established in Exercise 2. State this other form.

Part B

1. Using the rule for double denials, the deduction rule, and a rule for biconditional sentences, show that any sentence of the form:

$$p \longleftrightarrow \text{not not } p$$

is a valid sentence.

2. (a) Must inferences of the form:

$$\frac{p \text{ and not not } q}{p \text{ and } q}$$

be valid?

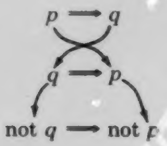
(b) How about those of the form:

$$\frac{p \text{ and } q}{p \text{ and not not } q}\ ?$$

3. In view of the result in Exercise 1 and the replacement rule for biconditional sentences, what can you say about a sentence and its "double denial"?

Part C

As you know, the sentence you obtain by interchanging the antecedent and consequent of a given conditional sentence is called the *converse* of the given sentence. If you make the further step of replacing the parts of this new sentence by their denials, you obtain another conditional sentence which is called the *contrapositive* of the given sentence:



$p \longrightarrow q$ [given sentence]

$q \longrightarrow p$ [converse of given sentence]

$\text{not } q \longrightarrow \text{not } p$ [contrapositive of given sentence]

For example, given the sentence:

(a) $$\vec{a} = B - A \longrightarrow A + \vec{a} = B$$

the converse of (a) is:

$$A + \vec{a} = B \longrightarrow \vec{a} = B - A$$

and the contrapositive of (a) is:

$$A + \vec{a} \neq B \longrightarrow \vec{a} \neq B - A$$

Answers for Part A [cont.]

4. Applying the deduction rule to an inference of the type described in Exercise 2 justifies the rule:

Any inference of the form:

$$\frac{\text{not q} \Longrightarrow \text{not p}}{\text{p} \Longrightarrow \text{q}}$$

is valid.

[Using the notion of the contrapositive of a conditional sentence (this is introduced in Part C); the rule of Exercise 3 says that any conditional sentence implies its contrapositive. The rule of Exercise 4 says that any conditional sentence is implied by its contrapositive. Using the two rules it is easy to show that any sentence of the form:

$$[\text{p} \Longrightarrow \text{q}] \Longleftrightarrow [\text{not q} \Longrightarrow \text{not p}]$$

is valid. For short, any conditional sentence is logically equivalent to its contrapositive. It follows by the replacement rule for biconditional sentences that either can be replaced by the other in any context.]

Note that 'contrapositive' is a grammatical term. For example, the contrapositive of a sentence of the form 'not q ⟹ not p' is the corresponding sentence of the form 'not not p ⟹ not not q'. Although the latter is logically equivalent to 'p ⟹ q' it is by no means the same sentence as this. To reiterate, 'p ⟹ q' is not the contrapositive of 'not q ⟹ not p'.]

Answers for Part B

1.

$$\cfrac{\cfrac{\cfrac{\overset{*}{\text{not not p}}}{\text{p}}}{\text{not not p} \Longrightarrow \text{p}}\,* \qquad \cfrac{\cfrac{\overset{*}{\text{p}}}{\text{not not p}}}{\text{p} \Longrightarrow \text{not not p}}\,*}{\text{p} \Longleftrightarrow \text{not not p}}$$

[Since all premisses of a derivation of this form are discharged, the conclusion of any such derivation is a valid sentence.]

2. (a) Yes. In detail, any inference of the form:

$$\frac{\text{q} \Longleftrightarrow \text{not not q} \qquad \text{p and not not q}}{\text{p and q}}$$

is valid by virtue of the replacement rule for biconditional sentences. Since, by Exercise 1, the first premiss of any such inference is a valid sentence, the conclusion is a consequence of the second premiss [taken by itself].

(b) Yes. [Similar argument.]

3. Any sentence "may" be replaced in any context by its double denial, and vice versa. [In this "technical sense", 'may' means that if one does as described, the given sentence implies the sentence which results.]

In this case both the given sentence (a) and its converse are theorems, but – as is usually the case with converses – they say quite different things. [Recall that (a) is essentially just another way of stating Postulate 2(a), while the converse of (a) is just another way of stating Postulate 2(b).] On the other hand, you know from Exercises 3 and 4 of Part A that any conditional sentence and its contrapositive are just different ways of making the same assertion. [Explain.]

1. Copy each of the given sentences; on the same line with the sentence, write its converse; beneath each of the two sentences write its contrapositive.

Sample. $\vec{a} = B - A \Longrightarrow A + \vec{a} = B$

Solution. $\vec{a} = B - A \Longrightarrow A + \vec{a} = B$ $\qquad A + \vec{a} = B \Longrightarrow \vec{a} = B - A$

$A + \vec{a} \neq B \Longrightarrow \vec{a} \neq B - A$ $\qquad \vec{a} \neq B - A \Longrightarrow A + \vec{a} \neq B$

(a) $A - C = B - C \Longrightarrow A = B$
(b) $a = b \Longrightarrow ac = bc$
(c) $A - B = C - D \Longrightarrow A - C = B - D$
(d) $B = A \Longrightarrow B - A = \vec{0}$
(e) $ab = 0 \Longrightarrow (a = 0$ or $b = 0)$

2. Any conditional sentence is the same as the converse of its converse.
   (a) What can you say about the converse of the contrapositive of a given sentence and the contrapositive of the converse of the given sentence?
   (b) What can you say about a given conditional sentence and the contrapositive of its contrapositive? [*Hint*: See Exercise 3 of Part A.]

*

Rules of Contraposition

Inferences of either of the forms:

$$\frac{p \Longrightarrow q}{\text{not } q \Longrightarrow \text{not } p} \qquad \frac{\text{not } q \Longrightarrow \text{not } p}{p \Longrightarrow q}$$

are valid.

Part D

1. Use modus tollens and the double denial rules to establish:

Symmetric Rules of Contraposition

Inferences of either of the forms:

$$\frac{p \Longrightarrow \text{not } q}{q \Longrightarrow \text{not } p} \qquad \frac{\text{not } p \Longrightarrow q}{\text{not } q \Longrightarrow p}$$

are valid.

Answers for Part C

[Sentence (a) which is used as an example here is the if-part of Theorem 2-1. Previous work with this theorem makes it easy to see that the converse of (a) and the contrapositive of (a) "say different things." If more examples are needed either part of any biconditional theorem proved up to now will do as well as (a). For example, the if-part of Theorem 2-2 is a valid sentence and its only if-part, together with Postulate 2(a), implies Postulate 2(b). So, the contrapositive of the if-part is valid, while the converse of the if-part is not. (This last because Postulate 2(a) does not by itself imply Postulate 2(b).)

The explanation asked for, just before the statement of Exercise 1, should refer to the results of Exercises 3 and 4 of Part A.]

1. (a) $A - C = B - C \Longrightarrow A = B$ $\qquad A = B \Longrightarrow A - C = B - C$
$A \neq B \Longrightarrow A - C \neq B - C$ $\qquad A - C \neq B - C \Longrightarrow A \neq B$

(b) $a = b \Longrightarrow ac = bc$ $\qquad ac = bc \Longrightarrow a = b$
$ac \neq bc \Longrightarrow a \neq b$ $\qquad a \neq b \Longrightarrow ac \neq bc$

(c) $A - B = C - D \Longrightarrow A - C = B - D$ $\qquad A - C = B - D \Longrightarrow A - B = C - D$
$A - C \neq B - D \Longrightarrow A - B \neq C - D$ $\qquad A - B \neq C - D \Longrightarrow A - C \neq B - D$

(d) $B = A \Longrightarrow B - A = \vec{0}$ $\qquad B - A = \vec{0} \Longrightarrow B = A$
$B - A \neq \vec{0} \Longrightarrow B \neq A$ $\qquad B \neq A \Longrightarrow B - A \neq \vec{0}$

(e) $ab = 0 \Longrightarrow (a = 0$ or $b = 0)$ $\qquad (a = 0$ or $b = 0) \Longrightarrow ab = 0$
not $(a = 0$ or $b = 0) \Longrightarrow ab \neq 0$ $\qquad ab \neq 0 \Longrightarrow$ not $(a = 0$ or $b = 0)$

[The given sentence of part (a) is a consequence of Postulate 1(a) and is implied by this postulate; in contrast, its converse is an equality principle and, so, is a valid sentence. The given sentence of part (b) is a valid sentence; its converse is a false sentence. The given sentence of part (c) is a theorem; its converse is a consequence of it and, also, implies it (see below for additional discussion of this). Both the given sentence of part (d) and its converse are theorems; neither is a valid sentence, and neither implies the other. A similar remark applies to part (e). The The sentence 'not (a = 0 or b = 0)' — "neither a nor b is 0" — may set students thinking about rules for the connective 'or'. These are discussed in section 4.05.)]

2. (a) As is illustrated by the answer for any part of Exercise 1, the converse of the contrapositive of a given conditional sentence is the same as the contrapositive of the converse of this sentence. [So, the converse of the contrapositive of a conditional sentence is equivalent to the contrapositive of the converse of this sentence.]

   (b) The contrapositive of the contrapositive of a given conditional sentence is equivalent [but different from] the given sentence. [By Exercise 3 of Part B (and the deduction rule) any sentence of the form:

$$[\text{not not } p \Longrightarrow \text{not not } q] \Longleftrightarrow [p \Longrightarrow q]$$

is a valid sentence. (For 'equivalent', see the discussion which follows.)]

The notion of logical equivalence has been introduced in the discussion of Exercise 4 of Part A. Briefly, sentences are logically equivalent if their biconditional is a valid sentence. More generally, sentences are equivalent [with respect to a deductive theory] if their biconditional is a theorem. Thus, for example, since the given sentence of Ex. 1(c) and its converse are both theorems, the sentences 'A - B = C - D' and 'A - C = B - D' are equivalent, but are not logically equivalent. Equivalence and logical equivalence are the basis for most applications of the replacement rule for biconditional sentences.

The converse of the given sentence of Ex. 1(c) can be inferred from this sentence by using the substitution rule. [This amounts to interchanging 'B' and 'C' by simultaneous substitution of 'C' for 'B' and of 'B' for 'C'. To carry this out by ordinary one-at-a-time substitutions, begin by substituting 'E', say, for 'B'. Then, substitute 'B' for 'C' and, finally, 'C' for 'E'.] Similarly, the given sentence can be inferred from its converse. Since each of the two sentences can be inferred from the other it follows that, intuitively, both sentences "say the same thing" or, in more technical terms, both sentences have the same content. Note, however, that since the derivation of one sentence from the other involves the use of the substitution rule we cannot go on to show, by applying the deduction rule, that the conditional sentence which has one of these as its antecedent and the other as its consequent is a valid sentence. In particular, although the two sentences have the same content they are not logically equivalent [nor are they even merely equivalent]. As a simpler example, each of the sentences 'a = 2' and 'b = 2' can be inferred from the other, but 'a = 2 ⟺ b = 2' is neither a valid sentence nor a theorem.

It is important to realize that having the same content is not a sufficient basis for applying the replacement rule for biconditional sentences.

*

The rules of contraposition stated between Part C and Part D summarize the discoveries of Part A. The discussion of Exercise 1 of Part C may have given students more feeling for the equivalence of a conditional sentence and its contrapositive.

*

Answers for Part D

1.

$$\dfrac{\dfrac{p \Longrightarrow \text{not } q \qquad \dfrac{\overset{*}{q}}{\text{not not } q}}{\text{not } p}}{q \Longrightarrow \text{not } p}\,*$$

$$\dfrac{\dfrac{\dfrac{\text{not } p \Longrightarrow q \qquad \overset{*}{\text{not } q}}{\text{not not } p}}{p}}{\text{not } q \Longrightarrow p}\,*$$

2. From Exercise 4 of Part C you know that any inference of the form:

$$\frac{\text{not } p \longrightarrow \text{not } q}{q \longrightarrow p}$$

is valid. Also, by conditionalizing, you know that any inference of the form:

$$\frac{\text{not } q}{\text{not } p \longrightarrow \text{not } q}$$

is valid. Hook these together, and supply an additional step, to justify:

Rule of Contradiction

Any inference of the form:

$$\frac{q \qquad \text{not } q}{p}$$

is valid.

Part E

1. In Part C on page 150 you were asked to prove:

$$(*) \qquad /a = 0 \longrightarrow 0 = 1 \quad [a \neq 0]$$

If put in tree-form, your proof might begin like this:

$$\cfrac{\cfrac{a \cdot 0 = 0 \qquad \cfrac{\overset{*}{/a = 0} \qquad \cfrac{a \neq 0 \qquad a \neq 0 \longrightarrow a \cdot /a = 1}{a \cdot /a = 1}}{a \cdot 0 = 1}}{0 = 1}}{/a = 0 \longrightarrow 0 = 1}\ *$$

To complete Part C you could use the deduction rule to discharge the assumption '$a \neq 0$' and then abbreviate the conclusion, '$a \neq 0 \longrightarrow [/a = 0 \longrightarrow 0 = 1]$', to obtain $(*)$. Instead of continuing in this way, introduce another postulate and apply modus tollens. Complete the argument to obtain a proof of:

$$/a \neq 0 \quad [a \neq 0]$$

2. The theorem proved in Exercise 1 is one of the two theorems which we needed to prove in order to show that the nonzero real numbers form a commutative group with respect to multiplication. State the other of these theorems.

2.

$$\cfrac{q \qquad \cfrac{\cfrac{\text{not } q}{\text{not } p \Longrightarrow \text{not } q}}{q \Longrightarrow p}}{p}$$

[One consequence of the rule of contradiction is that if a deductive theory is inconsistent then every sentence is a theorem. The rule is needed for another purpose in section 4.05.]

Answers for Part E

1. [Continuation of figure in text.]

$$\cfrac{\cfrac{/a = 0 \Longrightarrow 0 = 1 \qquad 0 \neq 1}{/a \neq 0}}{a \neq 0 \Longrightarrow /a \neq 0}\ *$$

[An asterisk should now appear over the '$a \neq 0$' in the text.] The conclusion depends only on theorems and, so is a theorem. By our conventions it may be abbreviated to '$/a \neq 0 \ [a \neq 0]$'.

2. $ab \neq 0 \ [a \neq 0,\ b \neq 0]$ [or: $b \neq 0 \Longrightarrow [a \neq 0 \Longrightarrow ab \neq 0]$, or: $(b \neq 0 \text{ and } a \neq 0) \Longrightarrow ab \neq 0$ (see the discussion in Part F)]

*

Sample Quiz

1. Simplify: $\frac{1}{p - 1} + \frac{1}{(p - 1)(p - 2)}$
2. Solve this equation: $1 + 4/t = 5t$
3. Which of the following is an example of modus tollens?

(a) $\dfrac{A - \vec{a} = B - \vec{a} \Longrightarrow A = B \qquad A \neq B}{A - \vec{a} \neq B - \vec{a}}$ (b) $\dfrac{\text{not } A \neq B}{\text{not } A = B}$

(c) $\dfrac{A - \vec{a} \neq B - \vec{a} \Longrightarrow A \neq B \qquad A = B}{A - \vec{a} = B - \vec{a}}$ (d) $\dfrac{\text{not not } A = B}{A = B}$

4. Which of the inferences given in 3 is an example of the rule for double denials?
5. Write (a) the converse of, and (b) the contrapositive of the sentence:

$$A - \vec{a} = B - \vec{a} \Longrightarrow A = B$$

Answers for Sample Quiz

1. $\frac{1}{p - 2}$ $[p \neq 1, 2]$ 2. $1, -4/5$
3. (a) [(c) is not; see the discussion on pages 152-153]
4. (d)
5. (a) $A = B \Longrightarrow A - \vec{a} = B - \vec{a}$ (b) $A \neq B \Longrightarrow A - \vec{a} \neq B - \vec{a}$

Part F

Since multiplication of real numbers is associative, multiplication of nonzero real numbers is certainly associative—that is, Postulate 5_1(b) certainly implies:

$$(ab)c = a(bc)\ [a \neq 0,\ b \neq 0,\ c \neq 0]$$

Similar remarks apply to 5_2(b) and 5_4(b); and 5_3(b) is already restricted. So, to show that the nonzero real numbers form a commutative group with respect to multiplication all that remains is to establish analogues to 5_0(d) – (f). We need to show that a product of nonzero real numbers is a nonzero real number, that 1 is a nonzero real number, and that the reciprocal of a nonzero real number is a nonzero real number. In each case the "real number" part is covered by 5_0(d), (e), or (f). So, what we need to establish is the "nonzero" part. For reciprocals, this has been done in Exercise i of Part E. That 1 is nonzero is a postulate. Consequently, all that remains is to prove:

(1) $$ab \neq 0\ [b \neq 0,\ a \neq 0]$$

which is short for:

(2) $$(b \neq 0 \text{ and } a \neq 0) \longrightarrow ab \neq 0$$

By the rules of exportation and importation [see page 101] sentence (2) is equivalent to:

$$b \neq 0 \longrightarrow [a \neq 0 \longrightarrow ab \neq 0]$$

which may be abbreviated to:

(3) $$a \neq 0 \longrightarrow ab \neq 0\ [b \neq 0]$$

The contrapositive of (3) follows easily from an instance of the cancellation principle of Part B on page 150 and a theorem about 0.

Use these hints to write a proof of (1).

4.04 Order

The postulates 5_0 through 5_7 constitute an adequate basis for developing the properties of the *operations* of addition, multiplication, etc. of real numbers. We still have to take account of the fact that there is a natural and useful way of ordering the real numbers according to which, for example, $2 < 5$ and $3 > -9$. [As you no doubt recall, '<' is read as 'is less than', and '>' is read as 'is greater than'.] We need to

Answers for Part F

(1) $ab = 0b \Longrightarrow a = 0$ $[b \neq 0]$ [theorem]

(2) $0b = 0$ [theorem]

(3) $ab = 0 \Longrightarrow a = 0$ $[b \neq 0]$ [(2), (1)]

(4) $a \neq 0 \Longrightarrow ab \neq 0$ $[b \neq 0]$ [(3)]

[The conclusion, (4), may be abbreviated to '$ab \neq 0\ [b \neq 0,\ a \neq 0]$'.]

It is not quite correct to speak of the restricted sentence (4) as the contrapositive of the restricted sentence (3). Explicitly, the contrapositive of:

(3) $$b \neq 0 \Longrightarrow [ab = 0 \Longrightarrow a = 0]$$

is certainly not the sentence:

(4) $$b \neq 0 \Longrightarrow [a \neq 0 \Longrightarrow ab \neq 0]$$

Actually, (4) is inferred from (3) by a rule like that discussed on TC 148(1). In this case the rule is:

If an inference of the form:

$$\frac{q}{r}$$

is valid then so is any corresponding inference of the form:

$$\frac{p \Longrightarrow q}{p \Longrightarrow r}$$

[The justification of this rule is, again, by using modus ponens and the deduction rule.] To justify inferring (4) from (3), replace 'q' by '$ab = 0 \Longrightarrow a = 0$', 'r' by '$a \neq 0 \Longrightarrow ab \neq 0$', and 'p' by '$b \neq 0$'.]

adopt some postulates which formulate the basic properties of these relations, and others which tell how they are related to addition and multiplication.

The basic properties of the greater than relation itself are that
- (i) of any two real numbers, one is greater than the other,
- (ii) no real number is greater than itself, and
- (iii) if a first number is greater than a second, while the second is greater than a third, then the first is greater than the third.

These are easy to formulate in our language:

5_8. $a > b$ or $b > a$ $[a \neq b]$

5_9. $a \not> a$ [Read as 'a is *not* greater than a'.]

5_{10}. $(a > b$ and $b > c) \Longrightarrow a > c$

The less than relation has exactly the same properties; but, instead of adopting three more postulates like 5_8 - 5_{10} it is more sensible to adopt a definition:

5_{11}. $a < b \Longleftrightarrow b > a$

Using the replacement rule for biconditional sentences, any theorem about less than can be derived from 5_{11} and a corresponding theorem about greater than.

To find postulates relating greater than to addition and multiplication, it is convenient to recall some of our work with linear functions in Chapter 1, in particular the diagrams like Fig. 1-2 on page 16 which we used to give graphical descriptions of such functions. Fig. 1-2 shows a certain translation of the number line—the translation which maps any number a on $a + 3$. We can also think of this figure as a picture of the singulary operation "adding 3". The figure makes it clear that, for any real numbers a and b such that $a > b$, $a + 3 > b + 3$. For short, we can say that the pictured translation of the number line—or the operation adding 3—*preserves* order. From your experience with translations it should be clear that any translation of the number line—and, so, any singulary adding operation—preserves order:

5_{12}. (a) $a > b \Longrightarrow a + c > b + c$

We need a part (b) for Postulate 5_{12} which tells how multiplication is related to order. Do you think that singulary multiplying operations—for example, multiplying by 2—preserve order?

Exercises

Part A

1. Make a sketch like Fig. 1-2 on page 16 to describe the linear function with slope 2 and intercept 0—that is, $\{(x, y): y = 2x\}$.
2. Explain how your answer for Exercise 1 indicates that multiplying by 2 does [or does not] preserve order.

Students are presumably acquainted with the greater-than and less than relations and the solution of simple inequations [see Part C on page 159]. So, the present section should turn out to be largely review.

The word 'or' in Postulate 5_8 — an abbreviation for:

$$a \neq b \Longrightarrow (a > b \text{ or } b > a)$$

— suggests the need for a discussion of the logic of 'or'. This, with some applications to inequality, is undertaken in section 4.05.

Postulates 5_8, 5_9, and 5_{10} may be paraphrased by saying that greater than is *connected*, *irreflexive*, and *transitive*, respectively. Any relation which has these properties — less than is another — is called *an irreflexive order relation* or *a simple order relation*. [A connected and transitive relation which, like greater than or equal to is reflexive rather than irreflexive is called *a reflexive order relation* or *a total order relation*. To round out this summary of terminology we note that set inclusion, '$\subseteq$', is a relation which, besides being transitive and reflexive is *antisymmetric*:

$$(S \subseteq T \text{ and } T \subseteq S) \Longrightarrow S = T$$

Such a relation — which need not be connected — is called *a partial order relation*.]

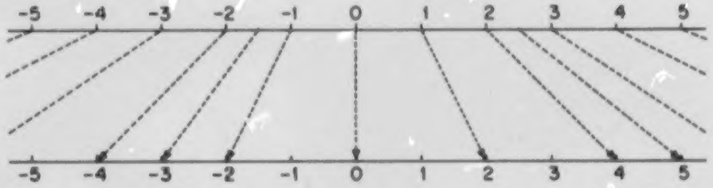
Postulates 5_{12}(a) and [see Part B on page 159] 5_{12}(b) are "monotony principles" which relate greater than with addition and multiplication.

Some fairly representative theorems are presented for proof in Part B. It is not of great importance that students work out these proofs for themselves, although that for Exercise 8 is intended as motivation for the dilemma rule of section 4.05. Liberal hints are supplied and it might be sufficient to sketch some of the proofs — with help from students — in class, paying extra attention to Exercise 8.

Of course, multiplying by 2 preserves order, but multiplying by 0 or by a negative number does not.

Answers for Part A

1.



2. Since none of the dashed arrows crosses another, this mapping preserves order.

3. Make a sketch like that of Exercise 1 to describe multiplying by 0. Does multiplying by 0 preserve order?
4. Make a third sketch to describe oppositing. [What has this to do with multiplying?]
5. Does oppositing preserve order? [If not, what would be a good word to use in place of 'preserves' to describe the effect of oppositing on order?]
6. (a) What kind of multiplying operations preserve order?
   (b) What kind of multiplying operations reverse order?

Part B

The exercises of Part A will have reminded you of our (almost) final postulate for real numbers:

5_{12}. (b) $a > b \Longrightarrow a \cdot c > b \cdot c \; [c > 0]$

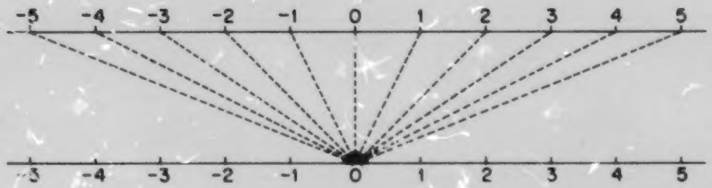
[Almost final postulate; much later in the course we shall need to introduce one more important postulate.] As a review of what you presumably know about inequations, prove the following theorems. You may – as in the remainder of this course – use as premisses any theorems which you are sure follow from Postulates $5_1 - 5_7$.

1. $a + c > b + c \Longrightarrow a > b$ [*Hint*: By 5_{12}(a), if $a + c > b + c$ then $(a + c) + -c > \ldots$.]
2. $a > b \Longrightarrow -b > -a$ [*Hint*: If you see why we suggested '$-c$' in the hint for Exercise 1, you should have no trouble here.]
3. $a < b \Longrightarrow -a > -b$ [*Hint*: Use Exercise 2 and 5_{11}.]
4. $-a > -b \Longrightarrow a < b$ [*Hint*: This is "almost an instance" of what theorem?]
5. $a > b \Longrightarrow bc > ac \; [c < 0]$ [*Hint*: Suppose that $c < 0$ and $a > b$. Since $c < 0$ it follows by Exercise 3 that $-c > -0 =$ ____. Since $-c > 0$ and $a > b \ldots$.]
6. $a > b \Longleftrightarrow a - b > 0$ [*Hint*: By 5_{12}(a) and Exercise 1, '$a > b \Longleftrightarrow a + c > b + c$' is a theorem.]
7. $(a > b \text{ and } c > d) \Longrightarrow a + c > b + d$
8. $a^2 > 0 \; [a \neq 0]$ [*Hint*: For $a \neq 0$ it follows by 5_8 that either $a > 0$ or $0 > a$. In the first case it follows by 5_{12}(b) that $\ldots$.]
9. $a^2 + b^2 > 2ab \; [a \neq b]$ [*Hint*: Use the theorem of Exercise 8. Note that $a \neq b$ if and only if $a - b \neq 0$.]

Part C

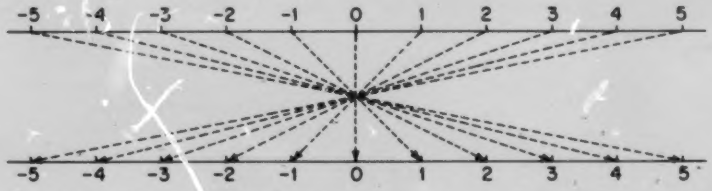
1. For each of the following, write a sentence which has the same roots and which begins with '$a >$' or '$a <$'. [In other words, solve for 'a'.]
   (a) $3a + 2 > 5a - 3$
   (b) $6a + 9 < 7a - 8$
   (c) $-(2a + 5) < 3a$
   (d) $-2a + 5 < 3a$
2. Solve the following.
   (a) $2a + 5 > a + 3$
   (b) $3(7 - b) > 7(3 + 2b)$
   (c) $(3c - 5)(2c + 7) > (2c - 5)(3c + 7)$
   (d) $(3d + 5)(2d - 7) > (2d + 5)(3d - 7)$

3.



Multiplying by zero does not preserve order. [Because of connectivity and irreflexivity, a mapping f which preserves order must be one-to-one. Briefly, suppose that $a \neq b$. Then $a > b$ or $b > a$ and, so, if f preserves order than $f(a) > f(b)$ or $f(b) > f(a)$. In either case, due to irreflexivity, $f(a) \neq f(b)$. Hence, if $a \neq b$ then $f(a) \neq f(b)$.]

4. [Oppositing is, of course, multiplying by -1; '$-a = a \cdot -1$' is a theorem.]



5. No. Oppositing reverses order.
6. (a) Multiplying by a positive number preserves order.
   (b) Multiplying by a negative number reverses order.

Answers for Part B

[Postulate 5_{12}(b) formulates the answer for Exercise 6(a) of Part A. The answer for Exercise 6(b) then becomes a theorem. See Exercise 5, below. To satisfy your curiosity, the remaining postulate, 5_{13}, is:

Each nonempty bounded subset of $\mathcal{R}$ has a least upper bound.

This postulate will not be needed during the first half of this course.]

1. By 5_{12}(a), if $a + c > b + c$ then $(a + c) + -c > (b + c) + -c$. But, $(a + c) + -c = a$ and $(b + c) + -c = b$. Hence, if $a + c > b + c$ then $a > b$.
2. By 5_{12}(a), if $a > b$ then $a + -(a + b) > b + -(a + b)$. But, $a + -(a + b) = -b$ and $b + -(a + b) = -a$. Hence, if $a > b$ then $-b > -a$.
3. By Exercise 2, if $b > a$ then $-a > -b$. But, by 5_{11}, $a < b$ if and only if $b > a$. Hence, if $a < b$ then $-a > -b$. ['Hence' by an application of the replacement rule for biconditional sentences.]
4. By Exercise 2, if $-a > -b$ then $--b > --a$. So, since $--b = b$ and $--a = a$ it follows that if $-a > -b$ then $b > a$. Hence, by 5_{11}, if $-a > -b$ then $a < b$.

5. Suppose that $c < 0$ and $a > b$. Since $c < 0$ it follows by Exercise 3 that $-c > -0 = 0$. Hence, by 5_{12}(b), since $a > b$, $a \cdot -c > b \cdot -c$. But, $a \cdot -c = -(a \cdot c)$ and $b \cdot -c = -(b \cdot c)$. So, by Exercise 4, $ac < bc$ and, by 5_{11}, $bc > ac$. Hence, for $c < 0$, if $a > b$ then $bc > ac$.

6. By 5_{12}(a) and Exercise 1, $a > b$ if and only if $a + -b > b + -b$. But, $a + -b = a - b$ and $b + -b = 0$. Hence, $a > b$ if and only if $a - b > 0$.

7. Suppose that $a > b$ and $c > d$. Since $a > b$, $a + c > b + c$. Since $c > d$, $b + c > b + d$. So, [since $a + c > b + c$ and $b + c > b + d$] it follows by 5_{10} that $a + c > b + d$. Hence, if $a > b$ and $c > d$ then $a + c > b + d$

8. For $a \neq 0$ it follows by 5_{8} that $a > 0$ or $0 > a$. If $a > 0$ then by 5_{12}(b), $a \cdot a > 0 \cdot a = 0$. If $0 > a$ then [by 5_{11}] $a < 0$ and, by Exercise 5, $a \cdot a > 0 \cdot a = 0$. Since, for $a \neq 0$, $a > 0$ or $0 > a$ and since, in either case, $a \cdot a > 0$ it follows that, for $a \neq 0$, $a \cdot a > 0$. Since, by definition, $a \cdot a = a^2$, $a^2 > 0$ $[a \neq 0]$.

9. For $a \neq b$, $a - b \neq 0$. So, by Exercise 8, for $a \neq b$, $(a - b)^2 > 0$. But, $(a - b)^2 = a^2 + b^2 - 2ab$ and, by 5_{12}(a), if $a^2 + b^2 - 2ab > 0$ then $a^2 + b^2 > 2ab$. [Note that $(a^2 + b^2 - 2ab) + 2ab = a^2 + b^2$.] Hence, for $a \neq b$, $a^2 + b^2 > 2ab$.

[You may wish to remind students (without proof) of some other theorems concerning greater than. One which follows at once by modus tollens from the instance '$(a > b \text{ and } b > a) \Longrightarrow a > a$' of 5_{11}, and 5_{12}, is 'not $(a > b$ and $b > a)$'. This asserts that greater than is asymmetric. Two important theorems whose proofs are tedious (and require the rules given in the next section) are:

$$ab > 0 \iff ((a > 0 \text{ and } b > 0) \text{ or } (a < 0 \text{ and } b < 0))$$

$$ab < 0 \iff ((a > 0 \text{ and } b < 0) \text{ or } (a < 0 \text{ and } b > 0))$$

Students should be reminded of these 'factoring transformation principles for inequations'. Two others which are worth noting are:

$$/a > 0 \iff a > 0 \quad \text{and:} \quad /a > 1 \iff 0 < a < 1$$

Students should be led to guess at the latter in the guise '$/a > 1 \iff (a > 0 \text{ and } a < 1)$' and be reminded of the convention according to which '$a < b$ and $b < c$' is abbreviated to '$a < b < c$'.]

Answers for Part C

1. (a) $a < 5/2$ (b) $a > 17$ (c) $a > -1$ (d) $a > 1$

2. ['Solve', here, means to give the solution-set.]

(a) $\{x\colon x > -2\}$ (b) $\{x\colon x < 0\}$ (c) $\{x\colon x > 0\}$

(d) $\{x\colon x < 0\}$

[What letter students may use in place of 'x' in these answers is immaterial. Strictly, it should be an index, rather than a variable; but there is no need to go into this now.]

4.05 Rules for 'or'

In order to solve Exercise 8 of Part B, above, you had to use a new kind of argument whose validity depends on the meaning of the word 'or' in Postulate [illegible]. A sentence which is of the form:

$$p \text{ or } q$$

is called an *alternation sentence*. Here is a typical argument in which an alternation sentence is used as a premiss:

> [This afternoon,] I go to the circus or I go to the drugstore. If I go to the circus then I must carry some money [to pay for admission]. If I go to the drugstore then I must carry some money [to buy a magazine]. Therefore [in either case], I must carry some money.

Such an argument is called a *simple dilemma* and is of the form:

$$\frac{p \text{ or } q \quad p \Longrightarrow r \quad q \Longrightarrow r}{r}$$

[What sentences in the preceding argument should be put in place of '*p*', '*q*', and '*r*'?] Here is one way of using a dilemma to prove the theorem:

$$a \neq 0 \Longrightarrow a^2 > 0$$

of Exercise 8. We begin by assuming that $a \neq 0$. It follows by Postulate 5_8 that

(1) $$a > 0 \text{ or } 0 > a.$$

This sentence will do as the first premiss of a dilemma. By Postulate 5_{12}(b),

$$a > 0 \Longrightarrow [a > 0 \Longrightarrow a \cdot a > 0 \cdot a]$$

and it follows from this that

(2) $$a > 0 \Longrightarrow a \cdot a > 0 \cdot a.$$

In a similar way, using the theorem of Exercise 5 of Part B and Postulate 5_{11}, it follows that

(3) $$0 > a \Longrightarrow a \cdot a > 0 \cdot a.$$

The basic rules for alternation sentences which are given on page 161 are nearly as transparent as those for conjunction sentences on page 100. These rules together with those for denial sentences justify various well-known equivalences involving 'not', 'and' and 'or' which are established in the exercises of Part A on page 162. They also furnish a basis for establishing results about the greater than *or* equal to relation, some of which are investigated in Parts B and C.

The "values" of 'p', 'q', and 'r' for the argument on page 160 are

p: I go to the circus. q: I go to the drugstore.

r: I must carry some money.

In distinguishing the two meanings of 'or' it is customary to speak of the exclusive meaning *vs* the inclusive meaning. To some students, however, 'inclusive' seems to suggest more than is intended — possibly that the alternatives *must* "overlap". For this reason we prefer the more exact term 'non-exclusive'. The example has been chosen to show that the alternatives *may* overlap.

One occasional source of confusion is the mistaken belief that which of the two meanings 'or' has in a given sentence depends on whether the alternatives stated in the sentence do or do not exclude one another. This is putting the cart before the horse. The meaning of 'or' must be decided on independently of the sentences which it connects.

It is, of course, possible to say in our language what could be said more simply if we used 'or' in the exclusive sense. The obvious way is by writing:

$$(p \text{ or } q) \text{ and not } (p \text{ and } q)$$

A much simpler [but less obvious] means to the same end is to write:

$$p \not\Leftrightarrow q$$

Now, (1), (2), and (3) are the premisses of a dilemma whose conclusion is:

$$(4) \qquad a \cdot a > 0 \cdot a$$

[What are 'p', 'q', and 'r' for this dilemma?] Discharging the assumption '$a \neq 0$', we obtain:

$$a \neq 0 \Longrightarrow a \cdot a > 0 \cdot a$$

as a consequence of 5_8, 5_{12}(b), the theorem of Exercise 5 and 5_{11}. The desired theorem now follows from this, the definition '$a^2 = a \cdot a$', and the theorem '$0 \cdot a = 0$'.

The rule of the simple dilemma gives us a way for using alternation sentences in proofs [an elimination rule for 'or']. Our experience with '$\Longrightarrow$' and 'and' suggests that we need, also, an introduction rule. Like many words, the word 'or' may be used in more than one way. Sometimes when we use 'or' we mean "one or the other, but not both", and sometimes we mean "one or the other, possibly both". The first is called the *exclusive* meaning of 'or'. The second is called the *non-exclusive* meaning of 'or' and is the one which is *always* used in mathematics. With this second meaning of 'or', any sentence—for example:

I am going to the movies.

—implies any alternation sentence of which it is a part—for example:

I am going to the movies or I am going to eat popcorn.

This suggests the appropriate introduction rule, and we can now state all our basic rules for alternation sentences together.

Rules for Alternation Sentences

Inferences of any of the forms:

$$\text{(a)}\ \frac{p \text{ or } q \quad p \Longrightarrow r \quad q \Longrightarrow r}{r}$$

$$\text{(b)}\ \frac{p}{p \text{ or } q} \qquad \text{(c)}\ \frac{q}{p \text{ or } q}$$

are valid.

As an example of how these rules may be used we shall show that any inference of the form:

$$\frac{\text{not } (p \text{ or } q)}{\text{not } p}$$

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is valid. Here's how:

$$\dfrac{\dfrac{\dfrac{\overset{*}{p}}{p \text{ or } q}}{p \Longrightarrow (p \text{ or } q)}^{\,*} \quad \text{not } (p \text{ or } q)}{\text{not } p}\ (\text{MT})$$

Exercises

Part A

1. Using the second introduction rule for 'or' you can show, just as above, that any inference of the form:

$$\dfrac{\text{not } (p \text{ or } q)}{\text{not } q}$$

is valid. Do so.

2. Combine the tree-diagram in the text with yours from Exercise 1 to show that any inference of the form:

$$\dfrac{\text{not } (p \text{ or } q)}{\text{not } p \text{ and not } q}$$

is valid.

3. Use the result of Exercise 2 and the result of Part B on page 154 to show that any inference of the form:

$$\dfrac{\text{not (not } p \text{ or not } q)}{p \text{ and } q}$$

is valid.

4. In Exercise 2 you used the introduction rules for 'or', the deduction rule, modus tollens, and the introduction rule for 'and' to justify one important kind of inference. In much the same way you can show that any inference of the form:

$$\dfrac{\text{not } p \text{ or not } q}{\text{not } (p \text{ and } q)}$$

is valid. Use, this time, the elimination rules for 'and', the deduction rule, a rule of contraposition, and the elimination rule for 'or'

5. The rule of Exercise 3 [and the deduction rule] shows that any sentence of the form:

$$\text{not (not } p \text{ or not } q) \longrightarrow (p \text{ and } q)$$

Answers for Part A

1, 2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{*}{p}}{p \text{ or } q}}{p \Longrightarrow (p \text{ or } q)}^{\,*} \quad \text{not } (p \text{ or } q)}{\text{not } p} \qquad \dfrac{\dfrac{\dfrac{\overset{*}{q}}{p \text{ or } q}}{q \Longrightarrow (p \text{ or } q)}^{\,*} \quad \text{not } (p \text{ or } q)}{\text{not } q}}{\text{not } p \text{ and not } q}$$

3. [Begin by replacing 'p' and 'q' throughout the result of Exercise 2 by 'not p' and 'not q'. [This is allowable, since the result in question concerns any sentences p and q and, for any sentences p and q, not p and not q are also sentences.)]

$$\dfrac{q \Longleftrightarrow \overset{*}{\text{not}} \text{ not } q \quad \dfrac{p \Longleftrightarrow \overset{*}{\text{not}} \text{ not } p \quad \dfrac{\text{not (not } p \text{ or not } q)}{\text{not not } p \text{ and not not } q}}{p \text{ and not not } q}}{p \text{ and } q}$$

replacement rule
for biconditional
sentences

[The '*'s indicate that the premisses in question are valid sentences.]

4.

$$\dfrac{\text{not } p \text{ or not } q \quad \dfrac{\dfrac{\dfrac{\overset{*}{p \text{ and } q}}{p}}{(p \text{ and } q) \Longrightarrow p}^{\,*}}{\text{not } p \Longrightarrow \text{not } (p \text{ and } q)} \quad \dfrac{\dfrac{\dfrac{\overset{*}{p \text{ and } q}}{q}}{(p \text{ and } q) \Longrightarrow q}^{\,*}}{\text{not } q \Longrightarrow \text{not } (p \text{ and } q)}}{\text{not } (p \text{ and } q)}$$

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5. By Exercise 4 [and the deduction rule] any sentence of the form:

$$(\text{not } p \text{ or not } q) \Longrightarrow \text{not } (p \text{ and } q)$$

is valid. Combining this result with that stated in this exercise we see that any sentence of the form:

$$(\star) \qquad \text{not } (p \text{ and } q) \Longleftrightarrow (\text{not } p \text{ or not } q)$$

is valid.

is valid. So, by a symmetric rule of contraposition, any sentence of the form:

$$\text{not } (p \text{ and } q) \longrightarrow (\text{not } p \text{ or not } q)$$

is valid. Obtain a better result by combining this with the result of Exercise 4.

6. Show that sentences of the following forms are valid:

(a) $\text{not } (p \text{ or } q) \longleftrightarrow (\text{not } p \text{ and not } q)$

(b) $(p \text{ and } q) \longleftrightarrow \text{not } (\text{not } p \text{ or not } q)$

(c) $(p \text{ or } q) \longleftrightarrow \text{not } (\text{not } p \text{ and not } q)$

Part B

Recall Postulates 5_8 – 5_{10}:

5_8. $a \neq b \longrightarrow (a > b \text{ or } b > a)$

5_9. $a \not> a$

5_{10}. $(a > b \text{ and } b > c) \longrightarrow a > c$

1. Use 5_{10} and 5_9 to prove: not $(a > b$ and $b > a)$
2. Prove: $a \not> b$ or $b \not> a$
3. The instance of 5_{10} which you used in Exercise 1 can be transformed by exportation [page 101] into:

$$a > b \longrightarrow [b > a \longrightarrow a > a]$$

Using this and 5_9, prove: $a > b \longrightarrow b \not> a$.

4. Show that Postulate 5_9 implies: $a = b \longrightarrow b \not> a$.
5. Use the definition:

$$a \geqslant b \longleftrightarrow (a > b \text{ or } a = b)$$

to prove:

$$a \geqslant b \longrightarrow b \not> a$$

Part C

Rules for Denying an Alternative

Inferences of either of the forms:

$$\frac{p \text{ or } q \quad \text{not } p}{q} \qquad \frac{p \text{ or } q \quad \text{not } q}{p}$$

are valid.

6. For (a), replace 'p' and 'q' throughout (☆) by 'not p' and 'not q' and proceed as in Exercise 3. The result is the converse of (a), but it is known that any biconditional sentence is equivalent to its converse.

For (b) and (c), note that by the replacement rule for biconditional sentences any sentences of the form '$p \Leftrightarrow q$' implies the corresponding sentence of the form '$\sim p \Leftrightarrow \sim q$':

$$\frac{p \Leftrightarrow q \qquad \sim p \Leftrightarrow \sim p}{\sim p \Leftrightarrow \sim q}$$

So, (☆) implies:

$$\text{not not } (p \text{ and } q) \Leftrightarrow \text{not } (\text{not } p \text{ or not } q)$$

and (☆) implies:

$$\text{not not } (p \text{ or } q) \Leftrightarrow \text{not } (\text{not } p \text{ and not } q)$$

From these, (b) and (c) are obtained as in Exercise 3.

Answers for Part B

1. By 5_{10}, if $a > b$ and $b > a$ then $a > a$. Since, by 5_9, $a \not> a$ it follows that not both $a > b$ and $b > a$.
2. By Exercise 1, not both $a > b$ and $b > a$. So, by Exercise 5 of Part A, either $a \not> b$ or $b \not> a$.

3.	(1) $a > b \Rightarrow [b > a \Rightarrow a > a]$	$[5_{10}]$
	(2) $a > b$	[assumption]*
	(3) $b > a \Rightarrow a > a$	[(2), (1)]
	(4) $a \not> a$	$[5_9]$
	(5) $b \not> a$	[(3), (4)]
	(6) $a > b \Rightarrow b \not> a$	[(5), *(2)]
4.	(1) $a = b$	[assumption]*
	(2) $a \not> a$	$[5_9]$
	(3) $b \not> a$	[(1), (2)]
	(4) $a = b \Rightarrow b \not> a$	[(3), *(1)]

5. By Exercises 3 and 4, if either $a > b$ or $a = b$ then $b \not> a$. So, by definition, if $a \geq b$ then $b \not> a$.

[The first sentence conceals the use of the dilemma and the deduction rule:

$$\cfrac{\overset{*}{p \text{ or } q} \qquad p \Rightarrow r \qquad q \Rightarrow r}{\cfrac{r}{(p \text{ or } q) \Rightarrow r}\,*}$$

The kind of 2-premiss inference justified by this scheme is often used — as is illustrated in the preceding answer.]

1. Show that any inference of the first of these two kinds is valid. [*Hint*: Use a dilemma. By the rule of contradiction (Part D on page 156), not p implies $p \longrightarrow q$. Also, $q \longrightarrow q$ is a valid sentence.]
2. Show that Postulate 5_e implies:

$$(a \ngtr b \text{ and } a \neq b) \longrightarrow b > a$$

3. Prove:

$$\text{not } (a > b \text{ or } a = b) \longrightarrow b > a$$

4. Prove:

$$a \geq b \longleftrightarrow b \ngtr a$$

[*Hint*: Recall Exercise 5 of Part B.]

5. Prove:

$$(a \geq b \text{ and } b \geq a) \longrightarrow a = b$$

Part D

Law of Noncontradiction

Any sentence of the form:

$$\text{not } (p \text{ and not } p)$$

is valid.

This law is sometimes taken as one of the basic rules of logic. We can derive it by using the rule of contradiction and a symmetric rule of contraposition. Here is the first step for doing so:

$$\cfrac{\cfrac{\cfrac{\cfrac{p \overset{*}{\text{ and}} \text{ not } p}{p} \quad \cfrac{p \overset{*}{\text{ and}} \text{ not } p}{\text{not } p}}{\text{not } q}}{(p \text{ and not } p) \longrightarrow \text{not } q}\,{}^{*}}{q \longrightarrow \text{not } (p \text{ and not } p)}$$

So far we have shown that any sentence of the form:

$$q \longrightarrow \text{not } (p \text{ and not } p)$$

is valid. To finish the job all we need do is choose for q any sentence which is itself a valid sentence. For example:

$$\frac{p \longrightarrow p \quad [p \longrightarrow p] \longrightarrow \text{not } (p \text{ and not } p)}{\text{not } (p \text{ and not } p)}$$

[Since both premisses of this modus ponens-inference are valid sentences, so is the conclusion.]

Answers for Part C

1.

$$\cfrac{p \text{ or } q \qquad \cfrac{\cfrac{\overset{*}{p} \quad \text{not } p}{q}}{p \Longrightarrow q}\,{}^{*} \qquad q \overset{\dagger}{\Longrightarrow} q}{q}$$

[Conversely, the rule for denying an alternative (and the introduction rule for 'or') implies the rule of contradiction:

$$\cfrac{\cfrac{p}{p \text{ or } q} \qquad \text{not } p}{q}$$

This fact, together with the intuitive appeal of the rule for denying an alternative, may help in gaining acceptance of the — to some — less appealing rule of contradiction. The rule of contradiction can, of course, be justified very easily by using conditionalizing and a symmetric rule of contraposition:

$$\cfrac{\text{not } p \qquad \cfrac{\cfrac{p}{\text{not } q \Longrightarrow p}}{\text{not } p \Longrightarrow q}}{q}$$

Incidentally, in case conditionalizing — without discharging a premiss — still disturbs students, it may help to point out that it is easy enough to introduce a premiss which can be discharged:

$$\cfrac{\cfrac{\cfrac{\overset{*}{p} \quad q}{p \text{ and } q}}{q}}{p \Longrightarrow q}\,{}^{*}$$

So, if one accepts the deduction rule and the rules for 'and' then he must, willy-nilly, accept conditionalizing.]

2. Suppose that $a \ngtr b$ and $a \neq b$. Since $a \neq b$ it follows by 5_e that either $a > b$ or $b > a$. So, since $a \ngtr b$ it follows that $b > a$. Hence, if $a \ngtr b$ and $a \neq b$ then $b > a$.

3. Suppose that it is not the case that either $a > b$ or $a = b$. By (a) of Exercise 6, Part A, it follows that both $a \ngtr b$ and $a \neq b$. So, by Exercise 2, $b > a$. Hence if neither $a > b$ nor $a = b$ then $b > a$.

4. By definition and the result of Exercise 3 it follows that if $a \ngeq b$ then $b > a$. Hence, if $b \ngtr a$ then $a \geq b$. The converse of this was proved in Exercise 5 of Part B. Hence, $a \geq b$ if and only if $b \ngtr a$.

Use the law of noncontradiction and your discoveries in Part A to establish the

Law of the Excluded Middle

Any sentence of the form:

not *p* or *p*

is valid.

*

Since, for any real number a, it is not the case that both $a < 0$ and $a > 0$, and since $a > 0$ if and only if $-a < 0$, it is not the case that both $a < 0$ and $-a < 0$. Hence, either $a \not< 0$ or $-a \not< 0$. In words, for any real number, either it or its opposite is nonnegative.

What has just been shown is that, for any a,

there exists a number x such that $x \not< 0$ and ($x = a$ or $x = -a$).

There could be two such numbers only if $a \not< 0$, $-a \not< 0$, and $a \neq -a$. To show that this is impossible, suppose that $a \not< 0$ and $-a \not< 0$. Since $-a \not< 0$, $0 \not< a$. So, it follows that $a \not< 0$ and $0 \not< a$. From this it follows that $a \geqslant 0$ and $0 \geqslant a$ and, so, that $a = 0$. Since if $a = 0$ then $a = -a$ it follows that if $a \not< 0$ and $-a \not< 0$ then $a = -a$. Hence, there cannot be two nonnegative numbers each of which is either a or $-a$.

For any real number a, the unique nonnegative real number which is either a or $-a$ is the absolute value of a—for short, $|a|$. *Using* '$\geqslant$' in place of '$\not<$' we have:

$$(*) \qquad |a| \geqslant 0 \text{ and } (|a| = a \text{ or } |a| = -a)$$

and (because of uniqueness):

$$(**) \qquad (b \geqslant 0 \text{ and } (b = a \text{ or } b = -a)) \longrightarrow |a| = b.$$

Note that if $a \geqslant 0$ then ($a \geqslant 0$ and ($a = a$ or $a = -a$)). So, by (**),

$$a \geqslant 0 \longrightarrow |a| = a.$$

On the other hand, if $a \leqslant 0$ then $-a \geqslant 0$ and, by an argument like that above,

$$a \leqslant 0 \longrightarrow |a| = -a.$$

TC 164 (2)

5. Suppose that $a \geq b$ and $b \geq a$. It follows by Exercise 4 [or by Exercise 5 of Part B] that $b \not> a$ and $a \not> b$. In other words [(a) of Exercise 6, Part A], it is not the case that either $b > a$ or $a > b$. So, by 5_B, it is not the case that $a \neq b$ — that is, $a = b$. Hence, if $a \geq b$ and $b \geq a$ then $a = b$.

   [As remarked in the commentary for section 4.04, this theorem can be paraphrased by saying that the relation greater than or equal to is antisymmetric.]

TC 165 (1)

Answer for Part D

By (c) of Exercise 6, Part A,

$$(\text{not } p \text{ or } p) \Longleftrightarrow \text{not } (\text{not not } p \text{ or not } p).$$

Since, by the law of noncontradiction, not (not not p or not p) it follows that not p or p.

Absolute valuing enters briefly in the final exercise of Part A on page 179. Consequently, it seems appropriate to remind students now of this operation. You may prefer to do just this, ignoring the discussion which precedes Part E.

The purpose of the discussion is to indicate that one might go about introducing absolute valuing postulationally by adopting the introduction principle (☆) as a postulate. If one does this then (☆☆) is an easily proved theorem and, on the basis of (☆) and (☆☆) it is rather easy to develop the theory of absolute valuing. For example, as shown in the text the alternative definition:

$$[a \geq 0 \Longrightarrow |a| = a] \text{ and } [a \leq 0 \Longrightarrow |a| = -a]$$

follows easily by using (☆☆). The theorems:

$$|a| = |-a| \quad \text{and:} \quad |ab| = |a| \cdot |b|$$

can be proved as follows:

> By (☆), $|-a| \geq 0$ and ($|-a| = -a$ or $|-a| = --a$). So, since $--a = a$, it follows by (☆☆) [with '$|-a|$' for 'b'] that $|a| = |-a|$.
>
> Since, by (☆), $|a| \geq 0$ and $|b| \geq 0$ it follows that $|a| \cdot |b| \geq 0$. Since, also, $|a| = a$ or $|a| = -a$) and ($|b| = b$ or $|b| = -b$) it follows that $|a| \cdot |b|$ is $a \cdot b$, $a \cdot -b$, $-a \cdot b$, or $-a \cdot -b$ and so, in any case, is $a \cdot b$ or $-(a \cdot b)$. Hence, by (☆☆) [with '$|a| \cdot |b|$' for 'b'], $|ab| = |a| \cdot |b|$.

It is also easy to prove:

$$-b \leq a \leq b \Longrightarrow |a| \leq b \quad \text{and:} \quad |a| \leq b \Longrightarrow -b \leq a \leq b$$

For the first, we note that, by (☆), $|a| = a$ or $|a| = -a$. Assuming that $-b \leq a$ and $a \leq b$ it follows, in the first case, that, since $a \leq b$, $|a| \leq b$ and, in the second, since $-b \leq a$, that $-b \leq -|a|$. So, in either case, $|a| \leq b$. Hence, if $-b \leq a$ and $a \leq b$ then $|a| \leq b$. For the second theorem we begin by noting that, since $|a| \geq 0$, $-|a| \leq |a|$. So, since $a = |a|$ or $a = -|a|$, $-|a| \leq a \leq |a|$. Suppose now that $|a| \leq b$. Since $a \leq |a|$ it follows that $a \leq b$; and since $-|a| \leq a$ it follows [since $-b \leq -|a|$] that $-b \leq a$. Hence, if $|a| \leq b$ then $-b \leq a \leq b$.

Part E

1. Evaluate each of the following.
   (a) $|2 - 3|$ **(b)** $|2| - |3|$ **(c)** $|-4|$ **(d)** $|-3 \cdot 2|$ **(e)** $|-3| \cdot |2|$
2. Absolute valuing is a mapping of $\mathscr{R}$ into $\mathscr{R}$. Draw a picture like Fig. 1-2 on page 16 which describes this mapping.
3. Prove:
   (a) $|a| = |-a|$ [*Hint*: Use an instance of (**).]
   (b) $|ab| = |a| \cdot |b|$

☆ 4.06 Fields

You learned in Section 3.05 that a set on which there is an associative and commutative binary operation and [with respect to this and an identity element] a singulary inversing operation is called *a commutative group*. For example, the parts $4_0 - 4_4$ of Postulate 4 tell us that $\mathscr{T}$ together with the operations of composition and inversing of translations, and the identity mapping $\overrightarrow{0}$, is a commutative group. Similarly, the parts 5_0(a) - (c) and 5_1(a) - 5_4(a) of Postulate 5 tell us that $\mathscr{R}$ is a commutative group with respect to addition [and oppositing and the identity element 0].

You have also seen that the set $\mathscr{R}_0$ which consists of the *nonzero* members of $\mathscr{R}$ is a commutative group with respect to multiplication [and reciprocating and the identity element 1]. In order to show this it was necessary to show that

(0) $\mathscr{R}_0$ is closed with respect to multiplication and reciprocating; $1 \in \mathscr{R}_0$;
(1) $(ab)c = a(bc)$, for a, b, and c in $\mathscr{R}_0$;
(2) $a \cdot 1 = a$, for $a \in \mathscr{R}_0$;
(3) $a \cdot /a = 1$, for $a \in \mathscr{R}_0$; and
(4) $ab = ba$, for a and b in $\mathscr{R}_0$.

The first part of (0) was proved in Section 4.03, the last part follows from 5_0(e) and 5_3, and each of (1) - (4) follows at once from the corresponding one of 5_1(b) - 5_4(b).

What we have seen so far is that $\mathscr{R}$ is a commutative group with respect to addition and that the members of $\mathscr{R}$ different from the identity element for addition form a commutative group with respect to multiplication. Besides this we know—among other things—that multiplication is a binary operation on $\mathscr{R}$ [not just on $\mathscr{R}_0$] and that multiplication is distributive with respect to addition.

Saying just this much about the real numbers tells us nearly all that Postulate 5 does. [We can ignore the definitions 5_7(a) and 5_7(b).] That $0 \neq 1$ follows from our statement that 1 is the identity element in the "multiplicative group" of *nonzero* elements of $\mathscr{R}$. All there is left to say is that, for any real number a,

$$(*) \qquad a \cdot 0 = 0 \text{ and } 0 \cdot a = 0.$$

Finally, one can prove:

$$|a + b| \leq |a| + |b|$$

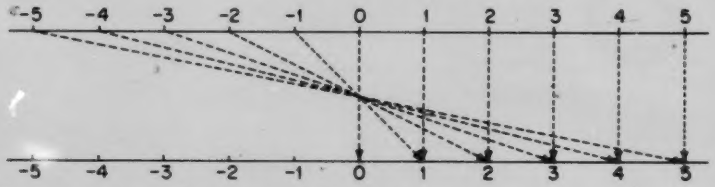
as follows:

> Since, as previously shown, $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$ it follows that $-(|a| + |b|) \leq a + b \leq |a| + |b|$. Hence, by the preceding theorem, $|a + b| \leq |a| + |b|$.

TC 166

Answers for Part E

1. (a) 1 (b) -1 (c) 4 (d) 6 (e) 6
2.



3. [Proofs have been given on TC 165.]

For, if we are told this then we can, for example, retrieve all of the commutative principle 5_4(b) from its special case, (4), above. [If neither a nor b is 0 then $ab = ba$ by (4); if either a or b is 0 then $ab = 0 = ba$ by (*).] The same applies to 5_1(b) and 5_2(b).

Now, it is useful to note that the second part of (*) follows from the group property of $\mathscr{R}$ with respect to addition and the distributive principle 5_6. To see this, recall that, in Exercise 1 of Part B, page 146, you have proved that the sentence:

$$a + a = a \Longrightarrow a = 0$$

is a theorem merely because $\mathscr{R}$ is a group with respect to addition. Using this theorem, we see that, because

$$0 \cdot a = (0 + 0) \cdot a = 0 \cdot a + 0 \cdot a$$

[by 5_2(a) and 5_6], it follows that $0 \cdot a = 0$. Moreover, using the result of Exercise 3, Part B, page 146, it turns out to follow that $a \cdot 0 = 0$.

The upshot of all this is that 5_0 through 5_6 can be restated as follows:

(F) There are two binary operations, addition and multiplication, on $\mathscr{R}$. $\mathscr{R}$ is a commutative group with respect to addition [and an appropriate identity element and inversing operation]. The members of $\mathscr{R}$ different from the identity element for addition form a commutative group with respect to multiplication [and an appropriate identity element and inversing operation]. Finally, multiplication is both distributive and left-distributive with respect to addition.

The value of stating $5_0 - 5_6$ in this form is that it relates the algebra of real numbers very directly to the important notion of a group.

Just as there are many groups, so there are many mathematical systems which, like the real number system, satisfy (F). Any such system is called a *field*. One way to find examples of fields is to look for subsets of $\mathscr{R}$ which contain 0 and 1 and are closed with respect to addition, oppositing, multiplication, and reciprocating. [Explain why any such subset of $\mathscr{R}$ is a field with respect to those operations.] For example, your answers for Exercises 9 and 10 of Part C on page 130 show that the system of rational real numbers is a field. In the exercises which follow you will discover another *subfield* of the real number field.

A field for which there is an ordering relation, like greater than, with the properties formulated in $5_8 - 5_{12}$ is called an *ordered field*. Evidently, any subfield of the field of real numbers is an ordered field.

Numbers play an essential role in many modern treatments of geometry. This is the case, for example, in treatments based on coordinates, and in treatments like that in High School Mathematics, Course 2, in which measures of segments or distances between points are of basic importance. Numbers are also introduced as a matter of course in the usual treatments of proportionality and similarity [but the theory of proportion developed by Eudoxus and expounded by Euclid shows that this use of numbers is avoidable]. It is customary in such treatments to take the theory of real numbers for granted as a subject which, while essential to the course, is, in some sense, outside it.

Since the real numbers play as essential a role in our organization of geometry as do points and translations, it is appropriate to include among our postulates one which summarizes the properties of these numbers. For this purpose, a postulate consisting of the parts $5_0 - 5_{12}$ will serve. Since, however, there is standard terminology for radically abbreviating such a postulate it seems worthwhile to introduce it. As it is not very important to do so this section bears a '☆' to indicate that it is not essential to the course.

A field is sometimes defined as a commutative group with respect to addition whose nonzero elements constitute a commutative group with respect to multiplication, and which is such that multiplication distributes over addition. To be correct, this must be interpreted somewhat liberally. [See (F) on page 167.] To begin with, multiplication must, like addition, be defined for all elements of the set and not, as one might guess, only for its nonzero elements. Also, multiplication must have the property expressed by (☆) on page 166. This will be the case if 'distributes' in the first sentence, above, is assumed to mean distributes "from both sides". The results of weaker assumptions are pointed out in "Another Remark Concerning the Definition of a Field", by H. E. Vaughan, Mathematics Magazine, v. 39(3), 1966, pp. 161, 162.

If S is a subset of $\mathscr{R}$ which contains 0 and 1 and is closed with respect to addition, oppositing, multiplication, and reciprocating then it is a commutative group with respect to addition and, as on page 166 with 'S' in place of '$\mathscr{R}_0$', its nonzero members form a commutative group with respect to addition. Also, distributivity — both ways — obtains in S because it obtains in $\mathscr{R}$. So, S is a field. [Furthermore, S is ordered by greater than because $\mathscr{R}$ is. So, S is an ordered field.]

In the exercises you will find an example of a field which is not an ordered field.

Using this terminology, the postulates we have adopted in this chapter can be collected into:

Postulate 5′ $\mathcal{R}$ is an ordered field.

[The "′" is to "leave room for" the one additional postulate which, as mentioned in Part B on page 159, we shall need later in this course.]

Exercises

Part A

In the following exercises you will investigate a field which is quite different from the field of real numbers. For one thing, this field has only two members; for another, no ordering relation for this field has the properties expressed by $5_8 - 5_{11}$.

Consider the set consisting of just the real numbers 0 and 1. Since we shall wish to mention it frequently in the exercises, let's call it 'S' : $S = \{0, 1\}$.

1. Is S closed with respect to addition? [Give a reason for your answer.]
2. Is S closed with respect to multiplication?
3. We are going to define a new binary operation on S. Since it will be a little like addition of real numbers we shall call it 'S-addition' and we shall use '$+_s$' as an operator for writing "S-sums". Here is the definition:

$$0 +_s 0 = 0, 0 +_s 1 = 1, 1 +_s 0 = 1, 1 +_s 1 = 0$$

Notice that, for any a and b in S, $a +_s b$ is just $a + b$ *unless* a and b are both 1. In any case, $a +_s b \in S$.
   - **(a)** Is there an identity element for S-addition?
   - **(b)** Does each member of S have an inverse with respect to S-addition? [Can you define an "S-oppositing" operation?]
   - **(c)** Is S-addition commutative?
   - **(d)** Is S-addition associative? [*Hint*: Notice that, for any b and c in S, $(0 +_s b) +_s c = b +_s c$ and $0 +_s (b +_s c) = b +_s c$. What happens if you substitute '0' for one of the other variables in "the associative principle for S-addition"? What other case of associativity do you need to check?]
4. **(a)** Is S a commutative group with respect to S-addition?
   - **(b)** Is the set of nonzero members of S a commutative group with respect to [ordinary] multiplication?
   - **(c)** Is S a field with respect to S-addition and multiplication?

Answers for Part A

1. No.; $1 + 1 = 2 \notin S$, since $2 \neq 0$ and $2 \neq 1$.
2. Yes.
3. (a) Yes.; 0. (b) Yes.; $-0 = 0$, $-1 = 1$.
   - (c) Yes.; as may be seen by checking instances.
   - (d) Yes.; the instances in which one addend is 0 are trivial, the instance in which each addend is 1 is easily seen to be true.
4. (a) Yes., by Exercise 3.
   - (b) Yes.; $\{1\}$ contains 1 and is closed with respect to multiplication and reciprocating.
   - (c) Yes.; distributivity of multiplication over S-addition is easily checked and multiplication is commutative in S.

TC 169 (1)

5. [Answer for 'Why?': By the postulate '$a >_s b \Longrightarrow a + c >_s b + c$'.]
   - (a) It has been shown that if $1 >_s 0$ then $1 +_s 1 >_s 0 +_s 1$ and, so, $0 >_s 1$. Similarly, if $0 >_s 1$ then $0 +_s 1 >_s 1 +_s 1$ and, so, $1 >_s 0$. Hence, in any case, both $1 >_s 0$ and $0 >_s 1$.
   - (b) Since S-greater than is assumed to be transitive $[5_{10}]$ it follows from part (a) that $1 >_s 1$ [and that $0 >_s 0$]. This contradicts the assumption that S-greater than is irreflexive $[5_9]$.

5. For S to be an *ordered* field it must be possible to define an S-greater than relation, '$>_s$', which would have properties like those expressed in Postulates $5_8 - 5_{12}$. In particular, since $0 \neq 1$ it will have to be the case either that $0 >_s 1$ or $1 >_s 0$. If it is the case that $1 >_s 0$ then it will also have to be the case that $1 +_s 1 >_s 0 +_s 1$. [Why?]
   - **(a)** Complete the argument to show that in either case [$1 >_s 0$ or $0 >_s 1$] it will have to be the case that *both* $1 >_s 0$ and $0 >_s 1$.
   - **(b)** Use the result of part (a) to show that there is no S-greater than relation which satisfies postulates like $5_8 - 5_{12}$.

* 6. Consider the sentence:

$$(**) \qquad a + a = 0 \longrightarrow a = 0$$

- **(a)** Show that (**) cannot be derived from $5_0 - 5_6$.
- **(b)** Show that (**) and 5_5 imply '$1 + 1 \neq 0$'.
- **(c)** Show that (**) is a consequence of $5_0 - 5_6$ and '$1 + 1 \neq 0$'.
- **(d)** Show that (**) is a theorem for any ordered field.

Part B

Since 0 and 1 are rational numbers and since the set of all rational numbers is closed with respect to addition, oppositing, multiplication, and reciprocating, the set of rational numbers is a field with respect to these operations.

1. Explain [but do not try to *prove*] why any subfield of the real number system must contain all the rational numbers.
2. Consider the set S of all real numbers "of the form" $a + b\sqrt{2}$, where a and b are rational numbers. $0 \in S$ because $0 = 0 + 0\sqrt{2}$ and 0 is a rational number. S is closed with respect to addition because

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}$$

and the set of rational numbers is closed with respect to addition.
   - **(a)** Show that $1 \in S$.
   - **(b)** Show that S is closed with respect to oppositing.
   - **(c)** Show that S is closed with respect to multiplication.
   - **(d)** Show that the reciprocal of any nonzero member of S belongs to S. [*Hint*: To show that a real number b is the reciprocal of a nonzero real number a it is sufficient to show that $a \cdot b = 1$. (For $a \neq 0$, $a \cdot /a = 1$. So, if $ab = 1$ then $a \cdot b = a \cdot /a$ and, by a cancellation principle, for $a \neq 0$, $b = /a$.)]
3. Is the set S of Exercise 2 an ordered field?

☆6. (a) Since $1 +_s 1 = 0$ and $1 \neq 0$, '$a +_s a = 0 \Longrightarrow a = 0$' is false. So, (☆☆) cannot be derived from any sentences which are true for all fields.

(b) (☆☆) implies its instance '$1 + 1 = 0 \Longrightarrow 1 = 0$'. So, by modus tollens, '$1 + 1 \neq 0$'.

(c) By 5_2(b), 4_4(b), and 5_6, $a + a = a(1 + 1)$. Hence, if $a + a = 0$ then $a(1 + 1) = 0$. But, by a theorem, if $a(1 + 1) = 0$ then $a = 0$ or $1 + 1 = 0$. So, assuming that $1 + 1 \neq 0$ it follows that $a = 0$. Hence, if $a + a = 0$ then $a = 0$.

(d) By part (c) it is sufficient to show that $1 + 1 \neq 0$ in an ordered field. For this, because of irreflexivity, it is sufficient to show that $1 + 1 > 0$. If we can show that $1 > 0$ it will follow that $1 + 1 > 1 + 0 = 1$ and, by transitivity, that $1 + 1 > 0$. Since $1 \neq 0$, either $1 > 0$ or $0 > 1$. So, it is sufficient to show that $0 \not> 1$. Suppose that $0 > 1$. Since $1 < 0$ and $0 > 1$ it follows [by a known theorem] that $1 \cdot 1 > 0 \cdot 1$ — that is, that $1 > 0$. Hence, if $0 > 1$ then both $0 > 1$ and $1 > 0$. Since this is not the case, $0 \not> 1$.

[Other finite fields can be defined by choosing a prime number p and taking $S = \{0, 1, \ldots, p - 1\}$. For $a, b \in S$, define $a +_s b$ and $a \times_s b$ to be the remainders on dividing $a + b$ and ab by p. For an example, and further discussion of fields, see High School Mathematics, Course 3, pages 292 and 293. Also, see the commentary for page 277 of this reference for a proof that the field discussed in Part B, below, can be ordered in either of two ways!]

Answers for Part B

1. A subset of $\mathcal{R}$ which contains 1 and is closed with respect to addition and oppositing must contain all integers. If it is also closed under multiplication and reciprocating it contains all quotients of integers by nonzero integers — that is, all rational numbers.
2. (a) $1 = 1 + 0\sqrt{2}$, and 1 and 0 are both rational.
   - (b) $-(a + b\sqrt{2}) = -a + -b\sqrt{2}$, and the set of rational numbers is closed with respect to oppositing.
   - (c) $(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}$, and $ac + 2bd$ and $ad + bc$ are rational if a, b, c, and d are.
   - (d) $(a + b\sqrt{2})(a - b\sqrt{2}) = a^2 - 2b^2$. So, unless $a^2 - 2b^2 = 0$, $(a + b\sqrt{2})(c + d\sqrt{2}) = 1$, where $c = a/(a^2 - 2b^2)$ and $d = -b/(a^2 - 2b^2)$, and c and d are rational if a and b are. But, since $\sqrt{2}$ is irrational, $a^2 - 2b^2 \neq 0$ for rational a and b unless $b = 0$ and $a = 0$.
3. Yes.; any subfield of the real field inherits the latter's order.

4.07 Chapter Summary

Vocabulary Summary

- reciprocal
- antecedent of a conditional
- converse of a conditional
- alternation sentence
- commutative group
- order relation
- denial sentence
- consequent of a conditional
- contrapositive of a conditional
- order-preserving mapping
- field
- ordered field

Additional Postulates

5_0. **(a)** $a + b \in \mathscr{R}$ **(b)** $-a \in \mathscr{R}$ **(c)** $0 \in \mathscr{R}$ **(d)** $a \cdot b \in \mathscr{R}$ **(e)** $1 \in \mathscr{R}$
(f) $/a \in \mathscr{R}$

5_1. **(a)** $(a + b) + c = a + (b + c)$ **(b)** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

5_2. **(a)** $a + 0 = a$ **(b)** $a \cdot 1 = a$

5_3. **(a)** $a + -a = 0$ **(b)** $a \cdot /a = 1\ [a \neq 0]$

5_4. **(a)** $a + b = b + a$ **(b)** $a \cdot b = b \cdot a$

5_5. $0 \neq 1$

5_6. $(a + b) \cdot c = a \cdot c + b \cdot c$

5_7. **(a)** $a - b = a + -b$ **(b)** $a \div b = a \cdot /b\ [b \neq 0]$

5_8. $a > b$ or $b > a\ [a \neq b]$

5_9. $a \not> a$

5_{10}. $(a > b \text{ and } b > c) \longrightarrow a > c$

5_{11}. $a < b \longleftrightarrow b > a$

5_{12}. **(a)** $a > b \longrightarrow a + c > b + c$
(b) $a > b \longrightarrow a \cdot c > b \cdot c\ [c > 0]$

$5'$. $\mathscr{R}$ is an ordered field

Additional Basic Rules of Logic

Dealing with denial sentences

Modus Tollens [See page 152.]

Any inference of the form:

$$\frac{p \longrightarrow q \quad \text{not } q}{\text{not } p}$$

is valid.

Double Denial Rules [See page 153.]
Inferences of either of the forms:

$$\frac{p}{\text{not not } p} \qquad \frac{\text{not not } p}{p}$$

are valid.

Dealing with alternation sentences

Elimination Rule [See page 161.]
Any inference of the form:

$$\frac{p \text{ or } q \quad p \longrightarrow r \quad q \longrightarrow r}{r}$$

is valid.

Introduction Rules [See page 161.]
Inferences of either of the forms:

$$\frac{p}{p \text{ or } q} \qquad \frac{q}{p \text{ or } q}$$

are valid.

Other Rules of Logic

Some valid sentences

Sentences of any of the forms:

$p \longleftrightarrow \text{not not } p$ [See page 154.]

$\text{not } (p \text{ and } q) \longleftrightarrow (\text{not } p \text{ or not } q)$ [DeMorgan's Laws; see page 163.]

$\text{not } (p \text{ or } q) \longleftrightarrow (\text{not } p \text{ and not } q)$

$\text{not } (p \text{ and not } p)$ [Law of Noncontradiction; see page 164.]

$\text{not } p \text{ or } p$ [Law of Excluded Middle; see page 165.]

are valid.

Rules of Contraposition [See page 155.]

Inferences of either of the forms:

$$\frac{p \longrightarrow q}{\text{not } q \longrightarrow \text{not } p} \qquad \frac{\text{not } q \longrightarrow \text{not } p}{p \longrightarrow q}$$

are valid.

Symmetric Rules of Contraposition [See page 155.]
Inferences of either of the forms:

$$\frac{p \longrightarrow \text{not } q}{q \longrightarrow \text{not } p} \qquad \frac{\text{not } p \longrightarrow q}{\text{not } q \longrightarrow p}$$

are valid.

Rule of Contradiction [See page 156.]
Any inference of the form:

$$\frac{q \quad \text{not } q}{p}$$

is valid.

Rules for Denying an Alternative [See page 163.]
Inferences of either of the forms:

$$\frac{p \text{ or } q \quad \text{not } p}{q} \qquad \frac{p \text{ or } q \quad \text{not } q}{p}$$

are valid.

Chapter Test

1. Give reciprocals [in simplest terms] for each of the following:
(a) $-3/2$ **(b)** 0.5 **(c)** $\sqrt{25}$ **(d)** $|-4|$

2. Solve the following.
(a) $\frac{p^2 - 9}{p + 3} = 7$ **(b)** $q(2q - 9) = 8(2q - 9)$
(c) $8r + 9 = 5(3 - 2r)$ **(d)** $|6s + 5| = 11$
(e) $-3(b - 5) > 2(2b - 3)$ **(f)** $|d + 1| \le 1$

3. Here are several inference schemes:

$$\text{(I) } \frac{p \quad p \longrightarrow q}{q} \qquad \text{(II) } \frac{p \longrightarrow q \quad \text{not } q}{\text{not } p} \qquad \text{(III) } \frac{\text{not not } p}{p}$$

$$\text{(IV) } \frac{p \text{ or } q}{q} \qquad \text{(V) } \frac{p \text{ and } q}{q} \qquad \text{(VI) } \frac{\text{not } p \longrightarrow \text{not } q}{\text{not not } q \longrightarrow \text{not not } p}$$

Match these schemes with the following.
(a) modus tollens **(b)** a rule for conjunctions
(c) modus ponens **(d)** a rule for double denial
(e) invalid inference **(f)** a rule for contraposition

4. Simplify.
(a) $1 - \frac{2}{p + 2}$ **(b)** $\frac{3}{p - 3} + \frac{2}{2 - p}$
(c) $\frac{10p^2 - 3p - 1}{2p - 1} - 5p$ **(d)** $\frac{5(p^2 - 1)}{p - 1} - (5p + 1)$

Answers for Chapter Test

1. (a) $-2/3$ (b) 2 (c) $1/5$ (d) $1/4$

2. (a) $\{10\}$ (b) $\{8, 9/2\}$
(c) $\{1/3\}$ (d) $\{1, -8/3\}$
(e) $\{x: x < 3\}$ (f) $\{x: -2 \le x \le 0\}$

3. (a) II (b) V
(c) I (d) III
(e) IV (f) VI

4. (a) $\frac{p}{p + 2}$ $[p \ne -2]$ (b) $\frac{p}{(p - 3)(p - 2)}$ $[p \ne 3, p \ne 2]$
(c) 1 $[p \ne 1/2]$ (d) 4 $[p \ne 1]$

Background Topic

[In these exercises on solving equations we shall use 'x' and 'y' as variables.]

Part A

Solve these systems of linear equations:

Sample 1.

$$\begin{aligned} 2x - 3y &= 5 \\ x + 4y &= 6 \end{aligned}$$

Reminder: A *solution* of this system is a pair (a,b) of real numbers such that each equation is satisfied when 'x' has the value a and 'y' has the value b. To *solve* the system is to find all of its solutions. There are several ways to solve a system like this one.

Solution. If $2x - 3y = 5$
and $x + 4y = 6$
then $8x - 12y = 20$ Also, $2x - 3y = 5$
and $3x + 12y = 18.$ $2x + 8y = 12$
If this is the case, then
$11x = 38$ $11y = 7$
and, so, $x = 38/11.$ $y = 7/11.$

Hence, if the system has any solution then it has the unique solution $(38/11, 7/11)$. Substituting in the given equations shows that this is a solution.

Answer. $(38/11, 7/11)$

Sample 2.

$$\begin{aligned} 3x - 4y &= 6 \\ 6x - 8y &= 12 \end{aligned}$$

Solution. If we proceed as in the solution for Sample 1, our first step is to obtain equations equivalent to the given ones:

$$\begin{aligned} 6x - 8y &= 12 \\ 6x - 8y &= 12 \end{aligned}$$

Evidently the two given equations are equivalent to one another; any solution of one is a solution of the other, and *vice versa*.

Answer. $\{(x,y)\colon 3x - 4y = 6\}$

Sample 3.

$$\begin{aligned} 3x - 4y &= 6 \\ 6x - 8y &= 13 \end{aligned}$$

Solution. An equivalent pair of equations is:

$$\begin{aligned} 6x - 8y &= 12 \\ 6x - 8y &= 13 \end{aligned}$$

The Background Topic is to remind students of how to solve simultaneous linear equations in two variables, and to introduce, very briefly, the second order determinant function. Solutions of such systems of equations are needed in Chapter 6, and determinants will be of considerable use in later chapters. Our use here of 'x' and 'y' — rather than earlier letters — as variables is to facilitate relating to students' past experience. [More will be said, in the text, about determinants before students need to use them.]

Obviously, these have no common solution.

Answer. No solution.

[Equations like those in Sample 1 are said to be *independent*; in Sample 2, *dependent*; in Sample 3, *inconsistent*.]

1. $5x + 6y = 10$; $3x - 2y = 6$

2. $6x - 9y = 15$; $4x - 6y = 10$

3. $x + y = 3$; $x - y = 4$

4. $6x - 9y = 15$; $6x - 9y = 10$

5. $\frac{1}{2}x + \frac{3}{4}y = 1$; $\frac{3}{4}x + \frac{2}{3}y = 2$

6. $6x - 10y = 5$; $9x - 15y = 4$

Part B

It is sometimes important to be able to tell at a glance whether or not a system:

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

is independent. In order to see how to do this, let's begin to solve this system for '*x*' and '*y*' as we did the system in Sample 1 of Part A. [You fill in the three blanks.]

$$\begin{array}{l|r} a_1b_2x + b_1b_2y = c_1b_2 & = a_2c_1 \\ a_2b_1x + b_2b_1y = c_2b_1 & = a_1c_2 \\ \hline (a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1 & = a_1c_2 - a_2c_1 \end{array}$$

1. If $a_1b_2 - a_2b_1 \neq 0$ then the system of equations has at most one solution.

(a) What is this possible solution?

(b) Check to see whether it actually is a solution.

2. If $a_1b_2 - a_2b_1 = 0$ then the system has no solution unless $c_1b_2 - c_2b_1 = 0$ and $a_1c_2 - a_2c_1 = 0$. Explain.

* **3.** Suppose that $a_1b_2 = a_2b_1$, $c_1b_2 = c_2b_1$, and $a_1c_2 = a_2c_1$.

(a) Show that, for any x and y,

(1) $a_1(a_2x + b_2y + c_2) = a_2(a_1x + b_1y + c_1)$

and **(2)** $b_1(a_2x + b_2y + c_2) = b_2(a_1x + b_1y + c_1)$

(b) Show that if either $a_1 \neq 0$ or $b_1 \neq 0$ then any solution of the first of the given equations is also a solution of the second.

(c) Show that if either $a_2 \neq 0$ or $b_2 \neq 0$ then any solution of the second of the given equations is a solution of the first.

4. Which of the following systems are systems of independent equations?

(a) $3x - 4y = 3$; $4x - 5y = 2$

(b) $152x + 37y = 63$; $19x + 5y = 14$

(c) $6x + y = x + 5y$; $5x - y = y - 5x$

(d) $20x + 10y = 7$; $12x + 6y = 5$

Answers for Part A

1.	(2, 0)	2.	$\{(x, y)\colon\ 2x - 3y = 5\}$
3.	(7/2, −1/2)	4.	No solution.
5.	(40/11, −12/11)	6.	No solution.

Answers for Part B

1. (a) For $a_1b_2 - a_2b_1 \neq 0$, the only possible solution is

$$\left(\frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}\right)$$

(b) That this is, indeed, a solution is shown as follows for the first equation:

$$\begin{aligned} & a_1 \cdot \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} + b_1 \cdot \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \\ &= \frac{a_1c_1b_2 - a_1c_2b_1 + b_1a_1c_2 - b_1a_2c_1}{a_1b_2 - a_2b_1} \\ &= \frac{a_1c_1b_2 - b_1a_2c_1}{a_1b_2 - a_2b_1} = \frac{(a_1b_2 - a_2b_1)\cdot c_1}{a_1b_2 - a_2b_1} = c_1 \end{aligned}$$

The check for the second equation is equally simple.

2. If $a_1b_2 - a_2b_1 = 0$ then (x, y) is a solution of the given equations only if

$$\begin{aligned} 0 \cdot x &= c_1b_2 - c_2b_1 \\ 0 \cdot y &= a_1c_2 - a_2c_1 \end{aligned}$$

These equations have a solution only if $c_1b_2 - c_2b_1 = 0$ and $a_1c_2 - a_2c_1 = 0$.

3. (a) [Multiply out and cancel.]

(b) If $a_1 \neq 0$ and $a_1x + b_1y + c_1 = 0$ then, by (1), $a_2x + b_2y + c_2 = 0$. If $b_1 \neq 0$ and $a_1x + b_1y + c_1 = 0$ then, by (2), $a_2x + b_2y + c_2 = 0$.

(c) [Similar to (b).]

4. (a), (b) and (c) are systems of independent equations.

*

There is an easy way to remember the results you found in Part B. The principle results are that the system:

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

has a unique solution if and only if $a_1b_2 - a_2b_1 \neq 0$; and, in this case, the solution is given by the formulas:

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1},\ y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

Note the similarity of the expressions in the numerators and denominators of these two fractions. Expressions like these occur frequently in mathematics and it is customary to define:

$$(*) \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

The *operation* indicated by the vertical bars, whose value for the pairs (a_1, b_1), (a_2, b_2) is the number $a_1b_2 - a_2b_1$ is called the *determinant operation*. Equation $(*)$ is read as 'the determinant of (a_1, b_1), (a_2, b_2) is $a_1b_2 - a_2b_1$'.

The formulas for the solution of the given pair of equations can be written:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},\ y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

[assuming that the value of the determinant indicated by the denominators is not 0].

*

5. Make use of the determinant operation to solve the following systems of equations.

(a) $3x + 2y = 6$

$5x - 6y = 12$

(b) $\frac{1}{3}x + \frac{4}{3}y = 2$

$\frac{3}{4}x + \frac{3}{2}y = 3$

Answers for Part B [cont.]

5. (a) $x = \dfrac{\begin{vmatrix} 6 & 2 \\ 12 & -6 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 5 & -6 \end{vmatrix}} = \dfrac{-36 - 24}{-18 - 10} = \dfrac{60}{28} = \dfrac{15}{7}$;

$y = \dfrac{\begin{vmatrix} 3 & 6 \\ 5 & 12 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 5 & -6 \end{vmatrix}} = \dfrac{36 - 30}{-18 - 10} = \dfrac{6}{-28} = -\dfrac{3}{14}$

Solution: (15/7, -3/14)

(b) $x = \dfrac{\begin{vmatrix} 2 & \frac{4}{3} \\ 3 & \frac{3}{2} \end{vmatrix}}{\begin{vmatrix} \frac{1}{3} & \frac{4}{3} \\ \frac{3}{4} & \frac{3}{2} \end{vmatrix}} = \dfrac{3 - 4}{\frac{1}{2} - 1} = \dfrac{-1}{-\frac{1}{2}} = 2$; $y = \dfrac{\begin{vmatrix} \frac{1}{3} & 2 \\ \frac{3}{4} & 3 \end{vmatrix}}{\frac{1}{2} - 1} = \dfrac{1 - \frac{3}{2}}{\frac{1}{2} - 1} = \dfrac{-\frac{1}{2}}{-\frac{1}{2}} = 1$

Solution: (2, 1)

TC 176 (1)

In this chapter we complete our postulates in so far as affine geometry is concerned, except that we do not as yet specify the dimension of our space. Definitions of 'line', 'triangle', 'plane', etc. will be introduced, beginning in Chapter 7, and the course will then become more easily recognizable as one dealing with geometry. [A hint as to how 'line', 'half-line', 'segment', and 'ray' might be defined may be garnered from Exercise 2 on page 181.]

As far as our formalism is concerned, the purpose of this chapter is to show that the set $\mathcal{T}$ of translations can be given the structure of a vector space. [The word 'vector' is introduced on page 191.] The requisite additional postulates, 4_0(d) and 4_5 - 4_8, are adopted in section 5.02 after having been made intuitively reasonable in the concept-development section 5.01. Section 5.03 is devoted to proving the basic '$\vec{0}$-product theorem' [see Exercise 3 on page 232], and section 5.04 to the notion of a vector space and of its subspaces [see the exercises following the section]. Section 5.05 introduces — on a purely intuitive level — another vector space — the space of "measure vectors" and shows how it can be used in solving problems concerning directed trips, velocities, and forces similar to those discussed on pages 1 - 2 of the introduction. This section on "applications" has no connection with later parts of the course. [This is not to say that this section should be skipped in order to "get on with the course". This work has proved to be quite useful in reinforcing in the students' minds the relationships between the formal structure embodied in the postulate system under construction and the intuitions from which this structure is being fashioned.] Section 5.06 is an optional section which deals more formally with the notion of measure vectors.

Chapter Five

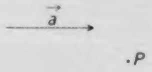
Extending Our List of Postulates

5.01 Multiplying Translations by Numbers

Earlier we said that we wish to study geometry by making use of the properties of translations. One of our big jobs will involve describing lines, planes, and other common geometric objects in terms of translations. As in the past, we shall use our intuitions about what these geometric objects are in order to see just what we need to add to our formal system to describe these objects.

Exploration Exercises

Consider the translation $\vec{a}$ and point P shown below.



Draw a picture something like this on your paper.

1. Locate the points $P + \vec{a}$ and $P + -\vec{a}$ on your picture.
2. Since addition of translations is a binary operation on $\mathscr{T}$, we know that

$$\vec{a} + \vec{a},\ (\vec{a} + \vec{a}) + \vec{a},\ [(\vec{a} + \vec{a}) + \vec{a}] + \vec{a},$$
$$-\vec{a} + -\vec{a},\ (-\vec{a} + -\vec{a}) + -\vec{a},\ [(-\vec{a} + -\vec{a}) + -\vec{a}] + -\vec{a}$$

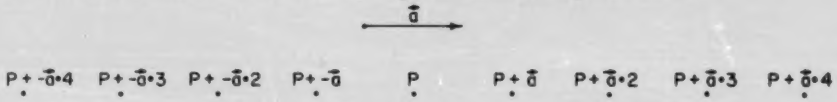
are translations. Locate on your paper the images of P under these translations.

3. **(a)** How many lines are there through P and $P + \vec{a}$? Do you think that $P + -\vec{a}$ is on any such line? Why?
   (b) Consider the translations described in Exercise 2. Do you think that the images of P under these translations are on a line through P and $P + \vec{a}$?

As remarked above, section 5.01 is, like sections 1.05 and 1.06, a concept-development section. [For a discussion of the role of such sections and the importance of student's distinguishing between them and "course-development" sections, see TC 26, 27.] Specifically, the operation of "multiplying a translation by a real number" is motivated by intuitive considerations which deal with the magnitude and sense of a translation. Neither of these latter notions has any place as yet in our formal theory. Since intuitive notions of sense help to motivate the introduction of the new operation it should come as no surprise when, in Chapter 7, the sense and direction of a translation are defined formally in terms of this operation. [For a quick preview, the direction of a translation $\vec{d}$ is the set $\ulcorner\vec{d}\,\urcorner$ defined in Part A on page 192.] You and your students should be quite clear that such a procedure does not involve any circularity — at least, none of a "vicious" nature. The motivation for adopting any formal definition of 'sense' must arise out of intuitive notions concerning the sense of a translation. That these intuitive notions should suggest defining something other than 'sense' in terms of which 'sense' can, ultimately, be defined is merely an example of the serendipitous way in which concepts develop.

Answers for Exploration Exercises

1, 2. Your students should have diagrams something like this:



3. (a) Exactly one.; Yes.; intuitions about how translations act on points [for example, a translation and its opposite "move" points along a line].

   (b) Yes, all of them.

(c) Do you think that you can use successive applications of $\vec{a}$ or $-\vec{a}$ to get from P to any point on a line through P and $P + \vec{a}$? If you think so, then tell how many times to apply $\vec{a}$ [or, $-\vec{a}$] to get from P to $P + [(\vec{a} + \vec{a}) + \vec{a}]$. To the point $P + (-\vec{a} + -\vec{a}) + -\vec{a}$. To the point midway between $P + (\vec{a} + \vec{a} + \vec{a} + \vec{a} + \vec{a})$ and $P + (\vec{a} + \vec{a} + \vec{a} + \vec{a} + \vec{a} + \vec{a} + \vec{a})$. To the point midway between $P + (\vec{a} + \vec{a})$ and $P + (\vec{a} + \vec{a} + \vec{a})$. To P.

*

Certain things should be intuitively clear at this point. Among them are:

(i) There is a line through P and $P + \vec{a}$.
(ii) The point midway between P and $P + \vec{a}$ ought to be on any line through P and $P + \vec{a}$.
(iii) The translation from P to the point midway between P and $P + \vec{a}$ moves points half as far as $\vec{a}$ does.

Of course, the only way that we can make use of $\vec{a}$ to get from P to the point midway between P and $P + \vec{a}$ is to somehow make use of "half" of $\vec{a}$. This suggests that we introduce:

$$\vec{a} \cdot \tfrac{1}{2}$$

as an abbreviation for:

> the translation that moves points in the sense of $\vec{a}$ and half as far as $\vec{a}$ does

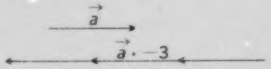
Similarly, we introduce:

$$\vec{a} \cdot -2$$

as an abbreviation for:

> the translation that moves points in the sense opposite to that of $\vec{a}$ and twice as far as $\vec{a}$ does

4. **(a)** Give similar abbreviations for the translations described in Exercise 2, above.
   (b) Can each point on the line through P and $P + \vec{a}$ be reached by successive applications of $\vec{a} \cdot \frac{1}{2}$ to P? Describe at least two points that cannot be so reached.
5. Draw arrows to describe each of the following:
   Sample. $\vec{a} \cdot -3$



Answers for Exploration Exercises [cont.]

(c) No. [three times; three times; six times; can't be done; zero times] [Note the relevance to part (c) of Theorem 2-5(b), page 141.]

*

Given any commutative group, it is both possible and convenient to define a multiplication of elements of the given group by integers. Such a definition can be formulated on the basis of mathematical induction, much as one defines powers with integral exponents. In the case of T, the definition would be:

$$\vec{a} \cdot 0 = \vec{0} \text{ and } \vec{a} \cdot (k + 1) = \vec{a} \cdot k + \vec{a} \text{ and } \vec{a} \cdot (k - 1) = \vec{a} \cdot k - \vec{a}$$

[where 'k' is a variable whose domain is the set I of integers]. On the basis of this definition one can prove various "laws of multiplication" corresponding precisely to the usual "laws of exponents". For example, the analogue of:

	$a^{j+k} = a^j a^k$	is:	$\vec{a} \cdot (j + k) = \vec{a} \cdot j + \vec{a} \cdot k$,
that of:	$(ab)^k = a^k b^k$	is:	$(\vec{a} + \vec{b}) \cdot k = \vec{a} \cdot k + \vec{b} \cdot k$, and
that of:	$(a^j)^k = a^{jk}$	is:	$(\vec{a} \cdot j) \cdot k = \vec{a} \cdot (jk)$

[cf. 4_6 - 4_8 of section 5.02]. ['$\vec{a} \cdot 0 = \vec{0}$' is the analogue of the exponent law '$a^0 = 1$'.]

Although multiplication by integers can be introduced into any commutative group, a multiplication by [non-integral] rational numbers which obeys similar laws is possible only for rather special groups. [When possible, the definition is analogous to that of powers with rational exponents.] In agreeing that there is a translation which can "reasonably" be referred to as $\vec{a} \cdot \frac{1}{2}$, we are accepting the existence of a translation $\vec{b}$ such that $\vec{b} + \vec{b} = \vec{a}$. That there is such a translation [for any translation $\vec{a}$] is intuitively obvious — just as "obvious" as it is that each segment has a midpoint. Nevertheless, the nonexistence of such a translation as $\vec{b}$, above, is consistent with the postulates we have adopted up to this point.

The commutative groups for which there is a "nice" multiplication by arbitrary real numbers are very special indeed and are called vector spaces. Our intuitions concerning the multiplication of translations by real numbers amount to something more than that it is possible to use real numbers as coordinates for the points on any geometric line — in itself a rather profound intuition.

Answers for Exploration Exercises [cont.]

4. (a) $\vec{a} \cdot 2$, $\vec{a} \cdot 3$, $\vec{a} \cdot 4$, $-\vec{a} \cdot 2$, $-\vec{a} \cdot 3$, $-\vec{a} \cdot 4$ [Note that by the conventions as to grouping symbols explained on TC 145(1-3), page 145, '$-\vec{a} \cdot 2$' refers to the product of the opposite of $\vec{a}$ by 2 — for short 'the opposite of $\vec{a}$ [hesitate] multiplied by 2'. On the other hand, to refer to the opposite of the product of $\vec{a}$ by 2, one should use '$-(\vec{a} \cdot 2)$' — to be read as 'the opposite of [hestiate] $\vec{a}$ multiplied by 2'.]
   (b) No. [For example, the point Q "one-third of the way from P to $P + \vec{a}$" cannot be reached in this way. But, Q - P is a translation which maps P on Q and it is natural to refer to it by '$\vec{a} \cdot \frac{1}{3}$'.]

(a) $\vec{a} \cdot \frac{3}{2}$ (b) $\vec{a} \cdot -2$

(c) $\vec{a} \cdot \frac{3}{4}$ (d) $\vec{a} \cdot -\frac{3}{4}$

6. Complete the following.

(a) $-\vec{a} = \vec{a} \cdot$ ____ (b) $-\vec{a} \cdot 3 = \vec{a} \cdot$ ____

(c) $\vec{a} \cdot 0 =$ ____ (d) $-\vec{a} \cdot -3 = \vec{a} \cdot$ ____

*

Although we have been dealing intuitively with a kind of multiplication – multiplication of translations by real numbers – we have no postulates as yet which refer to this kind of multiplication. So, even though you may have many insights into how this multiplication "works", in our formal development of the algebra of points and translations expressions like '$\vec{a} \cdot 2$' are, up to now, not part of our formal algebra.

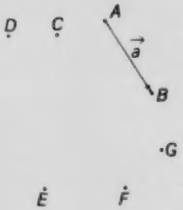
In order to get this sort of multiplication into our algebra, we shall have to add some postulates. We shall need, for example, a postulate that tells us that for each real number x, $\vec{a} \cdot x$ is a translation. Moreover, we shall need some postulates that tell us how this multiplication works.

The following exercises should help you to gain some further insights into how multiplication of translations by real numbers works. After we choose statements of some of the simpler properties as postulates, you will be able to prove many of the principles which you discover in these exercises.

Exercises

Part A

On your paper, draw a picture like the one at the right.



1. Draw arrows to help you locate these points:
   - (a) $C + \vec{a}2$
   - (b) $D + \vec{a}3$
   - (c) $E + -\vec{a} \cdot 2$
   - (d) $F + -\vec{a}$
   - (e) $G + \vec{a} \cdot 2 + -\vec{a} \cdot 2$

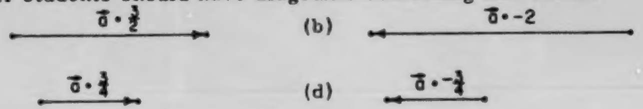
2. (a) Compare the length of $\overleftrightarrow{AB}$ with the lengths of the arrows you drew in Exercise 1.

   (b) Use your drawings from Exercise 1 to compare the sense of $\overleftrightarrow{AB}$ with the sense of each of the rays determined by a given point and its image under the given translation.

5. Your students should have diagrams something like these:

(a) (b)

(c) (d)

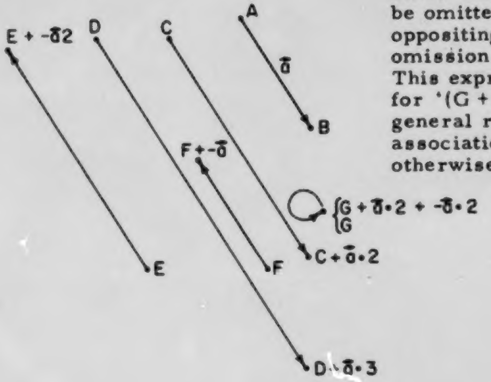

$\vec{a} \cdot \frac{3}{2}$
$\vec{a} \cdot -2$
$\vec{a} \cdot \frac{3}{4}$
$\vec{a} \cdot -\frac{3}{4}$

6. (a) -1 (b) -3

(c) $\vec{0}$ (d) 3

Answers for Part A

1.



[As illustrated in (a) and (b) we shall frequently omit multiplication dots. Such a dot may not, however, be omitted when it precedes an oppositing sign. Note, also, the omission of parentheses in (e). This expression is an abbreviation for '$(G + \vec{a} \cdot 2) + -\vec{a} \cdot 2$'. As a general rule, in repeated sums association is to the left unless otherwise indicated.]

2. (a) $\overleftrightarrow{AB}$ is one half the length of $\overleftrightarrow{C\ (C + \vec{a} \cdot 2)}$, one third the length of $\overleftrightarrow{D\ (D + \vec{a} \cdot 3)}$, one half the length of $\overleftrightarrow{E\ (E + -\vec{a} \cdot 2)}$, and the same length as $\overleftrightarrow{F\ (F + -\vec{a})}$. The segment $\overleftrightarrow{AB}$ is an "$\vec{a}$-length" longer that the (singleton) segment with end points G and $G + \vec{a} \cdot 2 + -\vec{a} \cdot 2$.

   (b) $\overleftrightarrow{AB}$ has the same sense as the rays from C through $C + \vec{a} \cdot 2$ and from D through $D + \vec{a} \cdot 3$. It is oppositely sensed from the rays from E through $E + -\vec{a} \cdot 2$ and from F through $F + -\vec{a}$.

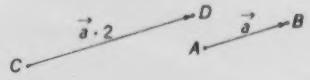
3. Complete each of the following.
   (a) $\vec{a} \cdot 5$ moves each point of $\mathscr{E}$ ____ times as far as does $\vec{a}$, and in the _______ [same/opposite] sense as does $\vec{a}$.
   (b) $\vec{a} \cdot -3$ moves each point of $\mathscr{E}$ ____ times as far as does $\vec{a}$, and in the _________ sense as does a.
   (c) $\vec{a} \cdot \sqrt{7}$ moves each point of $\mathscr{E}$ ____ times as far as does $\vec{a}$, and in the _______ sense as does $\vec{a}$.
4. Measure the length of $\overrightarrow{AB}$ in Exercise 1. Without measuring, determine the lengths of the segments

$$\overrightarrow{C\,(C + \vec{a} \cdot 2)},\ \overline{D\,(D + \vec{a} \cdot 3)},\ \text{and}\ \overline{E\,(E + \vec{a} \cdot -3)}$$

5. Suppose that $\vec{a}$ is any proper translation that is, that $\vec{a} \neq \vec{0}$, that a is any nonzero real number, and that $A + \vec{a} = A'$. Complete the following.
   (a) Given that a is a positive real number, then $\vec{a} \cdot a$ is the translation which maps A on the point X such that the measure of $\overrightarrow{AX}$ is _______ times that of _______ and the sense of $\overrightarrow{AX}$ is [the same as/opposite to] ___________ the sense of $\overrightarrow{AA'}$.
   (b) Given that a is a negative real number, then $\vec{a} \cdot a$ is the translation which maps A on the point X such that the measure of $\overrightarrow{AX}$ is _______ times that of _______ and the sense of $\overrightarrow{AX}$ is [the same as/opposite to] ___________ the sense of $\overrightarrow{AA'}$.
   (c) For any nonzero real number a, $\vec{a} \cdot a$ is the translation which moves each point of $\mathscr{E}$ _______ times as far as $\vec{a}$ does, in the [same, opposite] _________ sense as $\vec{a}$ if $a > 0$ and in the [same, opposite] _________ sense if $a < 0$.

*

Consider the segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ pictured below.



Since $D - C = \vec{a} \cdot 2$ and $B - A = \vec{a}$, it should be intuitively clear that

(1) the segment $\overrightarrow{CD}$ is twice as long as the segment $\overrightarrow{AB}$ *and*
(2) the ray from C through D has the same sense as the ray from A through B.

We can express *both* of the notions in the following manner:

(*) The ratio of the segment *from C to D* to the segment *from A to B* is 2.

We emphasize the (*) says the same thing as (1) and (2) together.

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3. (a) five; same
   (b) three; opposite
   (c) $\sqrt{7}$; same
4. The length of the segment $\overrightarrow{AB}$ is about 1 inch. So, the given segments are, respectively, 2, 3, and 3 inches long.
5. (a) a; $\overrightarrow{AA'}$; the same as
   (b) $|a|$; $\overrightarrow{AA'}$; opposite to
   (c) $|a|$; same; opposite

*

Exercise 5(c) is analogous to the usual definition of multiplication of "vectors" by "scalars". When we have completed Postulate 4 we shall be able to define 'sense' and 'magnitude' for translations in such a way that 5(c) will be a theorem.

TC 180 (1)

The notion of the ratio of sensed segments which is introduced in the discussion preceding Part B is treated formally in Chapter 7. This notion is basic for the discussion of points dividing segments in a given ratio and, so, for some familiar — and many less familiar — geometric theorems. [Among the familiar ones are those concerning the intersection of the diagonals of a parallelogram, and the intersection of the medians of a triangle.] The exercises of Part B [and of Part D on page 182] are intended to help prepare students for this later development.

In the present intuitive development students may, as is more or less suggested, assign ratios to parallel [or collinear] sensed segments by finding the quotient of their length measures and then taking the ratio to be the corresponding positive or negative real number according as the senses of the segments are the same or opposite. Alternatively, as suggested in Part D, the sensed segments ["directed trips"] may be assigned real numbers as measures, in which case the ratio of two sensed segments is precisely the quotient of these measures. [Note that, in any case, only sensed segments whose senses are the same or opposite have ratios to one another.]

Although ratios of objects are frequently described, as above, in terms of quotients of measures of the objects, it is important to recognize that "ratioing" is, actually, a prerequisite for measuring. For example, a child may make such judgments as that he is in the middle of his play-pen long before he has any notion of measuring distances. The ratio of two objects is, indeed, the quotient of their measures; but this is because the measure of an object is the ratio of that object to an arbitrarily chosen unit object, μ. The fundamental law of "ratioing" is, in fact:

(☆) $$\alpha:\beta = (\alpha:\mu) \div (\beta:\mu)$$

It is not too much to say that what makes measurement worthwhile is the fact that ratios can be defined [before one knows how to measure] in such a way that (☆) is true.

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Similarly, it is intuitively clear that

(3) the segment $\overrightarrow{DC}$ is twice as long as the segment $\overrightarrow{AB}$ *and*

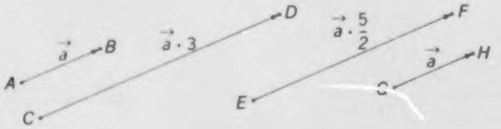
(4) the ray from D through C has the sense opposite to that of the ray from A through B.

We can express the notions (3) and (4) together in this way:

(**) The ratio of the segment *from D to C* to the segment *from A to B* is −2.

Part B

1. Given the segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ from the above discussion, complete the following sentences.
   - **(a)** The ratio of the segment from B to A to the segment from C to D is ____.
   - **(b)** The ratio of the segment from B to A to the segment from D to C is ____.
   - **(c)** The ratio of the segment from D to C to the segment from ______ is 2.
   - **(d)** $D - C = (\_\_\_\_) \cdot 2$ and $A - B = (D - C)$ ____.
2. Here is a picture of several segments.



Let's agree to write:

$(A$ to $B)$:$(C$ to $D)$ is 1/3

as an abbreviation for:

> The ratio of the segment from A to B to the segment from C to D is 1/3.

Complete the following.

(a) (A to B):(E to F) is ____

(b) (A to B):(G to H) is ____

(c) (E to F):(G to H) is ____

(d) (E to F):(A to B) is ____

(e) (E to F):(C to D) is ____

(f) (C to D): ______ is 6/5.

3. To say that (A to B):(C to D) is 1/3 is to say that the segment $\overrightarrow{AB}$ is ____ as long as the segment $\overrightarrow{CD}$ and that the sense of the ray from A through B is __________ the sense of the ray from C through D.
4. **(a)** In Exercise 3, replace the '1/3' by '−1/3' and repeat the problem.
   (b) In Exercise 3, replace the '1/3' by '−5/4' and repeat the problem.

In view of the preceding it is a worthwhile classroom activity to ask students to find ratios of sensed segments without measuring the segments. For example, draw two arrows — similarly sensed or oppositely sensed, as you wish — one of which is, say twice as long as the other. Then, ask students to find the ratio of one to the other without measuring either of them. Their procedure will probably amount to finding the measure of the longer with respect to the shorter as unit. [This is fine.] If students seem to need a hint, dangle an "unmarked yardstick" or a pointer in front of them. They can use this to "transfer" one arrow onto the other. Bring out the fact that the ratio of the longer arrow to the shorter is 2 [or −2] and that of the shorter to the longer is 1/2 [or −1/2]. Try to build up to an example in which the ratios are, say, 5/2 and 2/5 [or their opposites]. Students should arrive at the understanding that ratios can be found by direct comparison, and that finding ratios by dividing measures is only a handy short cut. [The preceding sentence applies equally well to the "incommensurable case" — only the method of making the comparison is more complex, and so is the notion of irrational measures. But, this is not the time to face students with this problem.]

Answers for Part B

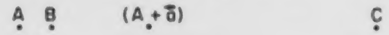
1. (a) −1/2 (b) 1/2 (c) B to A (d) B - A; $\cdot -\frac{1}{2}$
2. (a) 2/5 (b) 1
   (c) 5/2 (d) 5/2
   (e) 5/6 (f) E to F
3. one-third; the same as
4. (a) one-third; opposite to
   (b) five-fourths; opposite to

Answers for Part C

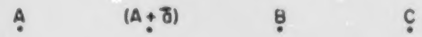
Note: The set notation used beginning in Exercise 2 frequently causes some problems. The problem usually centers around the use of 'X'. Students are unable to see how a particular point gets into a set. This problem can usually be solved by making an analogy to sets of real numbers such as $\{x: \exists_{y \geq 1}\ x = 5 + {}^{-}2y\}$ and asking questions like

(a) How do we know that $^{+}1$ is in the set?
(b) How does a number get into this set?

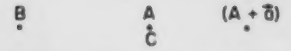
1. (a)



(b)



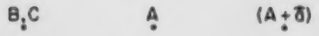
(c)



B is to the "left" of A on the line A,$(A + \vec{o})$.

[C = A, for $C = B + \vec{a} \cdot b = (A + \vec{a} \cdot a) + \vec{a} \cdot (-a) = A + \vec{a} \cdot (a + -a) = A + \vec{0} = A$. The students will be able to give arguments like this when additional postulates are introduced in section 5.02.]

(d)



[C = B, for $C = B + \vec{a}b$ and $b = 0$.]

Part C

1. For each part of this exercise, picture a point A and a *proper* translation $\overrightarrow{a}$ [that is, $\overrightarrow{a} \neq \overrightarrow{0}$]. Then, locate in your picture the points B and C such that $B = A + \overrightarrow{a}a$ and $C = B + \overrightarrow{a}b$.

   (a) $a = \frac{1}{4}, b = 2\frac{3}{4}$ **(b)** $a = 2, b = 1$
   (c) $a < 0, b = -a$ **(d)** $a < 0, b = 0$
   (e) $a < 0, b < 0$ **(f)** $a > 0, b < 0, |a| > |b|$

2. Picture a point A and a proper translation $\overrightarrow{a}$. Then, draw a picture of the given set and, if possible, describe the set by name.

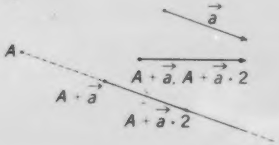
   Sample. $\{X: \exists_{y \geq 1}\, X = A = \overrightarrow{a}y\}$

   Solution. A point C belongs to the set in question if and only if there is a real number $y \geq 1$ such that $C = A + \overrightarrow{a}y$. For examples, since $1 \geq 1$, $2 \geq 1$, $3/2 \geq 1$, and $5 \geq 1$, the points

   $$A + \overrightarrow{a} \cdot 1,\ A + \overrightarrow{a} \cdot 2,\ A + \overrightarrow{a} \cdot \frac{3}{2}, \text{ and } A + \overrightarrow{a} \cdot 5$$

   belong to the set.

   Answer.



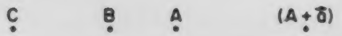
$\overrightarrow{a}$
A
$A + \overrightarrow{a}.\ A + \overrightarrow{a} \cdot 2$
$A + \overrightarrow{a}$
$A + \overrightarrow{a} \cdot 2$

   [In part (f), I is the set of all integers.]

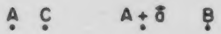
   (a) $\{X: \exists_y\, X = A + \overrightarrow{a}y\}$ **(b)** $\{X: \exists_{y>0}\, X = A + \overrightarrow{a}y\}$
   (c) $\{Z: \exists_{x<0}\, Z = A - \overrightarrow{a}x\}$ **(d)** $\{Z: \exists_x(-2 \leq x \leq 5 \text{ and } Z = A + \overrightarrow{a}x)\}$
   (e) $\{X: \exists_{x \leq 0}\, X = A + \overrightarrow{a}x\}$ **(f)** $\{X: \exists_{n \in I}\, X = A + \overrightarrow{a}n\}$

3. Describe the sets given in Exercise 2 in case $\overrightarrow{a}$ is the identity mapping.
4. Suppose that $\overrightarrow{a}$ and $\overrightarrow{b}$ are the proper translations in different directions [so that the lines through A and $A + \overrightarrow{a}$ and through A and $A + \overrightarrow{b}$ are not parallel]. Picture each of the following sets of points.
   (a) $\{X: \exists_x\, X = A + \overrightarrow{a}x\}$
   (b) $\{X: \exists_x\, X = A + \overrightarrow{b}x\}$
   (c) $\{X: \exists_{n \in I} \exists_x\, X = (A + \overrightarrow{a}n) + \overrightarrow{b}x\}$
   (d) $\{Z: \exists_y\, Z = A + (\overrightarrow{a}y + \overrightarrow{b}y)\}$
   (e) $\{X: \exists_x \exists_y\, X = A + (\overrightarrow{a}x + \overrightarrow{b}y)\}$
   (f) $\{X: \exists_x \exists_{y \geq 0}\, X = A + (\overrightarrow{a}x + \overrightarrow{b}y)\}$
5. Describe the sets given in Exercise 4 in case $\overrightarrow{a}$ and $\overrightarrow{b}$ are translations in the same direction.

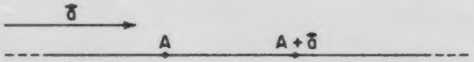
(e)



(f)



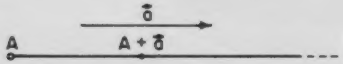
2. (a)



$\vec{a}$
A
$A + \vec{a}$

; $\overleftrightarrow{A(A + \vec{a})}$ a line

(b)

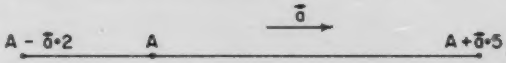


$\vec{a}$
A
$A + \vec{a}$

; $\overrightarrow{A(A + \vec{a})}$, a half-line

(c) [Answers are the same as for (b).]

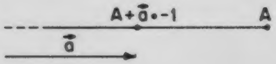
(d)



$\vec{a}$
$A - \vec{a} \cdot 2$
A
$A + \vec{a} \cdot 5$

; $\overline{(A - \vec{a}2)(A + \vec{a}5)}$, a segment

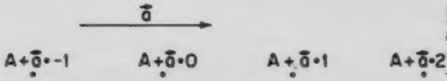
(e)



$A + \vec{a} \cdot -1$
A
$\vec{a}$

; $\overrightarrow{A(A - \vec{a})}$, a ray

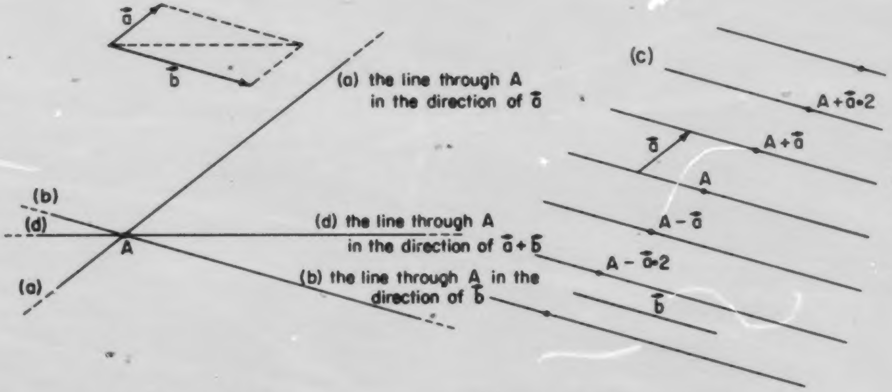
(f)



$\vec{a}$
$A + \vec{a} \cdot -1$
$A + \vec{a} \cdot 0$
$A + \vec{a} \cdot 1$
$A + \vec{a} \cdot 2$

[This is a set of points equally spaced along $\overleftrightarrow{A(A + \vec{a})}$.]

3. Each is $\{A\}$ — the set whose only member is A.
4.



$\vec{a}$
$\vec{b}$
(a) the line through A in the direction of $\vec{a}$
(b)
(d)
(a)
A
(d) the line through A in the direction of $\vec{a} + \vec{b}$
(b) the line through A in the direction of $\vec{b}$
(c)
$A + \vec{a} \cdot 2$
$A + \vec{a}$
$\vec{a}$
A
$A - \vec{a}$
$A - \vec{a} \cdot 2$
$\vec{b}$

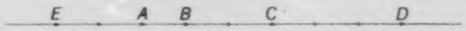
(e) This set is the plane containing the three noncollinear points A, $A + \vec{a}$, and $A + \vec{b}$.

(f) This set is the closed half-plane with edge $\overleftrightarrow{A(A + \vec{a})}$ and containing $A + \vec{b}$.

5. Each is the line $\overleftrightarrow{A(A + \vec{a})}$.

Part D

Suppose that A, B, C, and D are points on a line as illustrated in the figure below. Suppose, also, that $B = A + \vec{a}$, $C = B + \vec{a} \cdot 2$, $D = C + \vec{a} \cdot 3$, and $E = D + \vec{a} \cdot -8$.

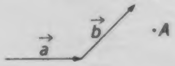


In your introduction to addition of real numbers you may have considered measures of trips along a road. Think of the line shown above as a road and consider trips along it. Suppose that you use positive numbers to measure trips in the same sense as the sense of $\vec{a}$ [and negative numbers for trips in the opposite sense], and that the distance between A and B is 1. Answer each of the following questions.

1. What are the measures of the following trips?
   - **(a)** from B to C **(b)** from C to D
   - **(c)** from A to C **(d)** from A to E
2. Using *only* your answers for Exercise 1, compute the measures of the following trips.
   - **(a)** from A to D **(b)** from C to A
   - **(c)** from B to A **(d)** from C to E
   - **(e)** from B to E **(f)** from D to B
3. Give each of the following ratios.
   - **(a)** (A to B):(B to C) **(b)** (A to B):(D to B)
   - **(c)** (A to C):(C to B) **(d)** (A to D):(D to B)
   - **(e)** (C to D):(D to E) **(f)** (A to B):(C to E)
4. Complete:
   - **(a)** $C = A + \vec{a} \cdot$ ____ **(b)** $D = A + \vec{a} \cdot$ ____
   - **(c)** $E = A + \vec{a} \cdot$ ____ **(d)** $E = C + \vec{a} \cdot$ ____

Part E

1. Here is a picture of a point A and translations $\vec{a}$ and $\vec{b}$.



   (a) Draw a single picture which illustrates the following:

   $A + \vec{a} = B$ $B + \vec{b} = C$
   $A + (\vec{a} \cdot 2 + \vec{b} \cdot 2) = E$ $A + \vec{b} = F$
   $G + (\vec{b} \cdot 2 + \vec{a} \cdot -2) = H$ $B + (\vec{a} + \vec{b}) = J$
   $A + (\vec{a} + \vec{b} \cdot 2) = D$ $C + (\vec{a} + \vec{b} \cdot -1) = G$

   (b) Use your picture from part (a) to help you to complete each of the following:

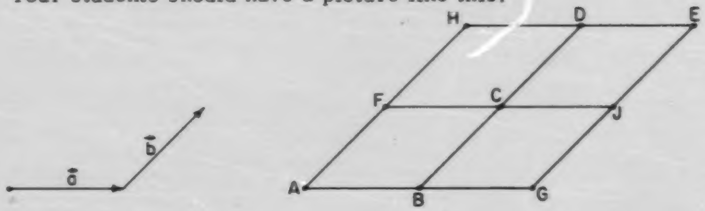
   $C + (\vec{a} + \vec{b}) =$ ____ $H + ($____ $+ \vec{b} \cdot -1) = J$
   $D + ($____ $+ \vec{a}) = G$ ____ $+ (\vec{a} + \vec{b} \cdot -1) = G$
   ____ $+ (\vec{b} \cdot -2 + \vec{a} \cdot 2) = G$ $A + (\vec{a} + \vec{b}) \cdot 2 =$ ____
   ____ $+ (\vec{a} \cdot 2 + \vec{b}) = J$ ____ $+ (\vec{a} \cdot 2 + \vec{b}) = E$

Answers for Part D

1. (a) 2 (b) 3
   (c) 3 (d) −2
2. (a) 6 (b) −3
   (c) −1 (d) −5
   (e) −3 (f) −5
3. (a) 1/2 (b) −1/5
   (c) −3/2 (d) −6/5
   (e) −3/8 (f) −1/5
4. (a) 3 (b) 6
   (c) −2 (d) −5

Answers for Part E

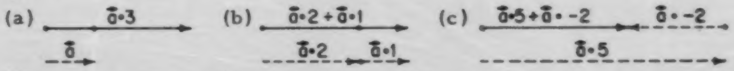
1. (a) Your students should have a picture like this:


$\vec{a}$
$\vec{b}$
H
D
E
F
C
J
A
B
G

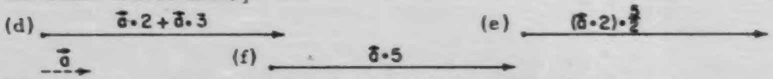
   (b) E ; $\vec{a}2$
   $\vec{b} \cdot -2$; C
   H ; E
   A ; F

TC 183 (1)

2. (a) 15 (b) 15 (c) 15
   (d) 6 (e) 6 (f) $3\sqrt{2}$
   (g) 1 (h) 0 (i) 3
3. (a) (b) (c)


(a)
$\vec{a} \cdot 3$
$\vec{a}$
(b)
$\vec{a} \cdot 2 + \vec{a} \cdot 1$
$\vec{a} \cdot 2$
$\vec{a} \cdot 1$
(c)
$\vec{a} \cdot 5 + \vec{a} \cdot -2$
$\vec{a} \cdot -2$
$\vec{a} \cdot 5$

   [The arrows drawn for (a), (b), and (c) should be the same length and have the same sense. In other words, they should describe the same translation.]


(d)
$\vec{a} \cdot 2 + \vec{a} \cdot 3$
$\vec{a}$
(e)
$(\vec{a} \cdot 2) \cdot \frac{5}{2}$
(f)
$\vec{a} \cdot 5$

   [The arrows drawn for (d), (e), and (f) describe the same translation.]

2. Suppose that $\vec{b}$ is the translation which maps P onto Q, and that the length of $\overrightarrow{PQ}$ is 3. Determine the distance between P and its image under each of the following translations.
(a) $\vec{b} \cdot 5$ **(b)** $\vec{b} \cdot -5$ **(c)** $\vec{b} \cdot -3 + \vec{b} \cdot -2$
(d) $\vec{b} \cdot 8 + \vec{b} \cdot -10$ **(e)** $\vec{b} \cdot (8 + -10)$ **(f)** $\vec{b} \cdot -\sqrt{2}$
(g) $\vec{b} \cdot -\frac{1}{3}$ **(h)** $\vec{b} \cdot 0$ **(i)** $\vec{b} \cdot 1$

3. Suppose that $\vec{a}$ is a proper translation. Draw arrows to describe the following translations.
(a) $\vec{a} \cdot 3$ **(b)** $\vec{a} \cdot 2 + \vec{a} \cdot 1$ **(c)** $\vec{a} \cdot 5 + \vec{a} \cdot -2$
[Compare the results in (a), (b), and (c).]
(d) $\vec{a} \cdot 2 + \vec{a} \cdot 3$ **(e)** $(\vec{a} \cdot 2) \cdot \frac{5}{2}$ **(f)** $\vec{a} \cdot 5$
[Compare the results in (d), (e), and (f).]

4. Suppose that $\vec{a}$ and $\vec{b}$ are proper translations in different directions. Draw arrows to describe the following translations.
(a) $\vec{a} \cdot \frac{3}{2} + \vec{b} \cdot \frac{3}{2}$ **(b)** $(\vec{a} \cdot 3 + \vec{b} \cdot 3) \cdot \frac{1}{2}$ **(c)** $(\vec{a} + \vec{b}) \cdot \frac{3}{2}$
[Compare the results in (a), (b), and (c).]
(d) $\vec{a} \cdot -2 + \vec{b} \cdot -2$ **(e)** $-\vec{a} \cdot 2 + -\vec{b} \cdot 2$ **(f)** $(\vec{a} + \vec{b}) \cdot -2$
[Compare the results in (d), (e), and (f).]

Part F

In each of the following, fill the blanks to make what seems intuitively to be a true sentence.

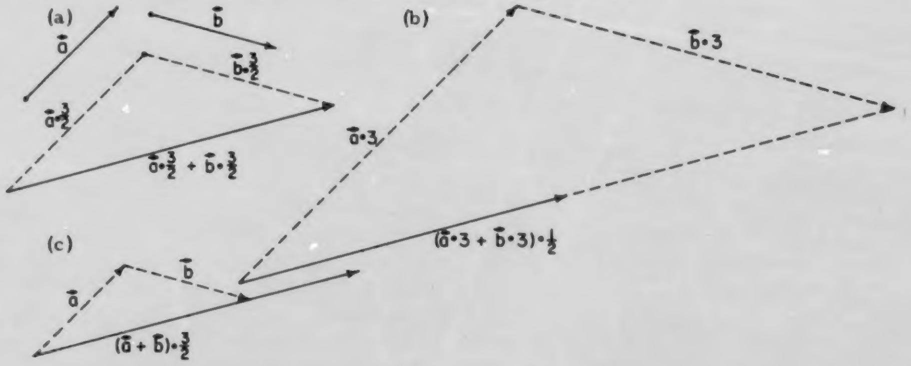
1. $\vec{a} \cdot a \in$ ____ [for any real number a]
2. $\vec{a} \cdot 1 =$ ____
3. $\vec{a} \cdot (a + b) = \vec{a} \cdot$ ____ $+ \vec{a} \cdot$ ____
4. $(\vec{a} + \vec{b}) \cdot b = \vec{a} \cdot$ ____ $+ \vec{b} \cdot$ ____
5. $(\vec{a} \cdot a) \cdot b = \vec{a} \cdot ($____$)$
6. $\vec{a} \cdot 0 =$ ____ and $\vec{0} \cdot a =$ ____
7. $-(\vec{a} \cdot a) = -\vec{a} \cdot$ ____ and $-(\vec{a} \cdot a) = \vec{a} \cdot ($____$)$

5.02 Admitting the Real Numbers as Operators

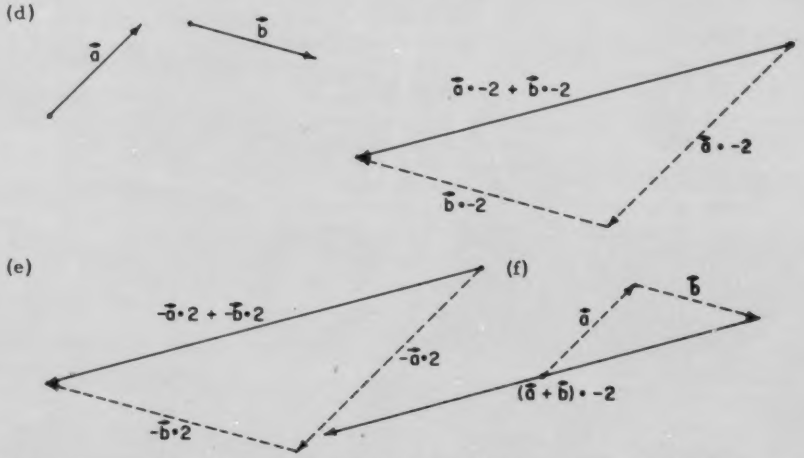
Up to now we have dealt with two kinds of things in our algebra – points and translations. We are now bringing in a third kind of thing – real numbers – and a procedure for "combining" a translation and a real number to produce a translation.

In Chapter 4 we have already adopted postulates which tell us all we need to know about the algebra of the real numbers alone. [These are the Postulates $5_0 - 5_{12}$ which we collected together in Postulate 5′.] In addition to these we need postulates describing the multiplication of translations by real numbers. One possibility would be to use Exercise 5(c) of Part A on page 179. The difficulty in doing this is that we would then need to adopt some postulates concerning distance and

4. Your students should have pictures something like this:


(a)
$\vec{a}$
$\vec{b}$
$\vec{b} \cdot \frac{3}{2}$
$\vec{a} \cdot \frac{3}{2}$
$\vec{a} \cdot \frac{3}{2} + \vec{b} \cdot \frac{3}{2}$
(b)
$\vec{b} \cdot 3$
$\vec{a} \cdot 3$
$(\vec{a} \cdot 3 + \vec{b} \cdot 3) \cdot \frac{1}{2}$
(c)
$\vec{a}$
$\vec{b}$
$(\vec{a} + \vec{b}) \cdot \frac{3}{2}$

[The arrows drawn for parts (a), (b), and (c) describe the same translation.]


(d)
$\vec{a}$
$\vec{b}$
$\vec{a} \cdot -2 + \vec{b} \cdot -2$
$\vec{a} \cdot -2$
$\vec{b} \cdot -2$
(e)
$-\vec{a} \cdot 2 + -\vec{b} \cdot 2$
$-\vec{a} \cdot 2$
$-\vec{b} \cdot 2$
(f)
$\vec{a}$
$\vec{b}$
$(\vec{a} + \vec{b}) \cdot -2$

[The arrows drawn for parts (d), (e), and (f) describe the same translation.]

Answers for Part F

1. T
2. $\vec{a}$
3. a; b
4. b; b
5. ab [or: a · b]
6. $\vec{0}$; $\vec{0}$
7. a; −a

sense. [Explain.] These notions are somewhat complicated to get at directly. It turns out that if we postulate some of the properties of multiplication which you noted down in Part F above, then we shall, later, be able to define distance and sense in such a way that Exercise 5(c) on page 179 becomes a theorem. This is what we shall do. Since these new postulates will deal with translations but not [explicitly] with points, we shall include them in Postulate 4.

To begin with, we need a closure postulate similar to 4_0(a)–(c):

> Postulate 4_0 (d) $\vec{a} \cdot b \in \mathcal{T}$
> The product of a translation by a real number is a translation.

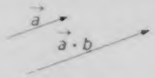


Fig. 5–1

Recall that we have already listed Postulates 4_1, 4_2, 4_3, and 4_4. We now continue this list as we introduce the following postulates into our algebra.

> Postulate 4_5 $\vec{a} \cdot 1 = \vec{a}$
> The product of a translation by 1 is that translation itself.
>
> Postulate 4_6 $\vec{a} \cdot (b + c) = \vec{a} \cdot b + \vec{a} \cdot c$
>
> Postulate 4_7 $(\vec{a} + \vec{b}) \cdot c = \vec{a} \cdot c + \vec{b} \cdot c$
>
> Postulate 4_8 $(\vec{a} \cdot b) \cdot c = \vec{a} \cdot (bc)$

Draw diagrams to illustrate Postulates 4_6, 4_7, and 4_8.

Postulates 4_0(d) and 4_5 – 4_8 are usually summarized in the following manner:

> The group $\mathcal{T}$ of translations *admits the real numbers as operators.*

Thus far, then, in our postulates about translations alone, we know that the set of translations is a commutative group and that this commutative group admits the real numbers as operators. We shall have more to say about this a little later.

Look at the exercises of Part F on page 183. One might ask, quite naturally, whether or not we need postulates that tell us that

$$\vec{a} \cdot 0 = \vec{0} = \vec{0} \cdot a$$

and that

$$\vec{a} \cdot -a = -(\vec{a} \cdot a) = -\vec{a} \cdot a.$$

Section 5.01 was a concept-development section. In the present section 5.02 we enlarge our formalism by introducing into it terms of the form '$\vec{a} \cdot b$' and postulating some of the properties of multiplication of translations by real numbers which have been discovered in the preceding section. From now on, all we "know" about this multiplication is what is contained — explicitly or implicitly — in our postulates. In particular, we can no longer justify our statements — as we did in section 5.01 — by arguments about sense and magnitude of translations. [Of course, our intuitive notions of sense and magnitude may still suggest statements which we may be able to show to be theorems — but only by deriving them from the postulates.]

As it turns out, these sentences can be derived from our extended list of postulates. We shall list these results in our next two theorems and leave the proofs as exercises. [We shall often omit multiplication dots just as we do in the case of multiplication of real numbers.]

Theorem 5-1 (a) $\vec{a}0 = \vec{0}$ (b) $\vec{0}a = \vec{0}$

Theorem 5-2
(a) $\vec{a} \cdot -a = -(\vec{a}a)$ (b) $-\vec{a} \cdot a = -(\vec{a}a)$

Since $\mathcal{T}$ is a commutative group with respect to addition, you can use analogues of the real number theorems:

$$a + a = a \longrightarrow a = 0 \text{ and: } a + b = 0 \longrightarrow b = -a$$

to prove that a given translation is $\vec{0}$ or that it is the opposite of some given translation.

Exercises

Part A

Prove each of the following:

1. (a) $\vec{a}0 = \vec{0}$ (b) $\vec{0}a = \vec{0}$
2. (a) $\vec{a} \cdot -a = -(\vec{a}a)$ (b) $-\vec{a} \cdot a = -(\vec{a}a)$
3. $-\vec{a} = \vec{a} \cdot -1$
4.

Theorem 5-3
(a) $\vec{a}(a - b) = \vec{a}a - \vec{a}b$ (b) $(\vec{a} - \vec{b})a = \vec{a}a - \vec{b}a$

5. (a) $(A + \vec{a}a) + \vec{a}b = A + \vec{a}(a + b)$
   (b) $(A + \vec{a}a) + \vec{b}a = A + (\vec{a} + \vec{b})a$

Part B

1. Simplify each of the following.
   (a) $\vec{a}2 + \vec{b}7 + \vec{a} \cdot -3 + \vec{b}4$ (b) $(\vec{a} + \vec{b})\frac{1}{2} + (\vec{a} - \vec{b}) \cdot -\frac{1}{2}$
   (c) $\vec{c}5 + \vec{c}3 - \vec{c} \cdot -2 + \vec{c} \cdot -10$ (d) $(\vec{a} - \vec{b})7 - (\vec{a} - \vec{b}) \cdot -7$
2. Solve for '$\vec{a}$':
   (a) $\vec{a} + \vec{b} \cdot -2 = \vec{0}$
   (b) $(\vec{a} - \vec{b})2 - \vec{c} \cdot -3 = \vec{d} - (\vec{a} + \vec{b}) \cdot -3$
   (c) $\vec{a}0 + \vec{b} - \vec{a} = \vec{0}$
   (d) $(\vec{a} + \vec{b})2 + \vec{c}3 = \vec{a}2 + (\vec{b}\frac{1}{3} - \vec{c})6$

Answers for Part A

1. (a) By Postulate 4_6, $\vec{a}0 + \vec{a}0 = \vec{a}(0 + 0) = \vec{a}0$. Now, since $\mathcal{T}$ is a commutative group, we know that if $\vec{a}0 + \vec{a}0 = \vec{a}0$ then $\vec{a}0 = \vec{0}$. Since $\vec{a}0 + \vec{a}0 = \vec{a}0$, it follows that $\vec{a}0 = \vec{0}$.

   Here is a tree-diagram of the proof of this theorem.

$$\dfrac{\dfrac{\dfrac{a + 0 = a}{0 + 0 = 0} \qquad \dfrac{\vec{a}a + \vec{a}b = \vec{a}(a + b)}{\vec{a}0 + \vec{a}0 = \vec{a}(0 + 0)}}{\vec{a}0 + \vec{a}0 = \vec{a}0} \qquad \dfrac{\vec{a} + \vec{a} = \vec{a} \Longrightarrow \vec{a} = \vec{0}}{\vec{a}0 + \vec{a}0 = \vec{a}0 \Longrightarrow \vec{a}0 = \vec{0}}}{\vec{a}0 = \vec{0}}$$

   (b) By Postulate 4_7, $\vec{0}a + \vec{0}a = (\vec{0} + \vec{0})a = \vec{0}a$. We know that if $\vec{0}a + \vec{0}a = \vec{0}a$ then $\vec{0}a = \vec{0}$. So [by modus ponens], $\vec{0}a = \vec{0}$.

2. (a) First note that $\vec{a}a + \vec{a} \cdot -a = \vec{a}(a + -a) = \vec{a}0 = \vec{0}$. Since $\mathcal{T}$ is a commutative group, we know that if $\vec{a} + \vec{b} = \vec{0}$ then $\vec{b} = -\vec{a}$. In particular, if $\vec{a}a + \vec{a} \cdot -a = \vec{0}$ then $\vec{a} \cdot -a = -(\vec{a}a)$. So, since $\vec{a}a + \vec{a} \cdot -a = \vec{0}$, it follows that $\vec{a} \cdot -a = -(\vec{a}a)$.

   (b) Since $\mathcal{T}$ is a commutative group, we know that if $\vec{a}a + -\vec{a}a = \vec{0}$ then $-\vec{a}a = -(\vec{a}a)$. Now, $\vec{a}a + -\vec{a}a = (\vec{a} + -\vec{a})a = \vec{0}a = \vec{0}$. Thus, $-(\vec{a}a) = -\vec{a}a$. So, $-\vec{a}a = -(\vec{a}a)$.

3. By Exercise 2(a), $\vec{a} \cdot -1 = -(\vec{a}1)$ and, by 4_5, $\vec{a}1 = \vec{a}$. So, $\vec{a} \cdot -1 = -\vec{a}$.

4. (a) By definition $[5_7(a)]$, $\vec{a}(a - b) = \vec{a}(a + -b)$. By 4_6, $\vec{a}(a + -b) = \vec{a}a + \vec{a} \cdot -b$. By Exercise 2(a), $\vec{a} \cdot -b = -(\vec{a}b)$. So, $\vec{a}(a - b) = \vec{a}a + -(\vec{a}b) = \vec{a}a - \vec{a}b$.

   (b) [Proof is similar, but uses Definition 3-1(b) [page 141], 4_7, and Exercise 2(b).]

5. (a) $(A + \vec{a}a) + \vec{a}b = A + (\vec{a}a + \vec{a}b) = A + \vec{a}(a + b)$, by Theorem 2-5(b) and 4_6.

Answers for Part B

1. (a) $-\vec{a} + \vec{b}11$ [or: $\vec{b}11 - \vec{a}$] (b) $\vec{b}$ (c) $\vec{0}$ (d) $(\vec{a} - \vec{b})14$
2. (a) $\vec{a} = \vec{b}2$ (b) $\vec{a} = -\vec{b}5 + \vec{c}3 - \vec{d}$ (c) $\vec{a} = \vec{b}$ (d) [Impossible.]

   ['Simplify' is, as always, rather vague. For some purposes, a simpler answer for Exercise 1(a) is '$\vec{a} \cdot -1 + \vec{b}11$' and, in like circumstances, a more appropriate answer for Exercise 2(b) is '$\vec{a} = \vec{b} \cdot -5 + \vec{c}3 + \vec{d} \cdot -1$' ["What multiples of $\vec{b}$, $\vec{c}$, and $\vec{d}$ can one add to obtain $\vec{a}$?"]. Suggest to students that the skill which is needed is that of seeing at a glance the results of various transformations of a term into equivalent terms. For example, on seeing any one of the following terms:

$$\vec{a} - \vec{b}c,\ \vec{a} + -(\vec{b}c),\ \vec{a} + -\vec{b}c,\ \vec{a} + \vec{b} \cdot -c,\ \vec{a}1 + \vec{b} \cdot -c, \text{ etc.}$$

   one should be "automatically" aware that it can be replaced by any of the others. Which one, if any, one chooses to replace it by will be determined, of course, by the context in which it occurs. "Simplifying is transforming to achieve some goal."]

Part C

Prove each of the following.

1. $(\vec{a}b) \cdot /b = \vec{a}$ $[b \neq 0]$
2. $\vec{a}c = \vec{b}c \Longrightarrow \vec{a} = \vec{b}$ $[c \neq 0]$ [*Hint*: Note that the converse of this theorem is a valid sentence.]
3. $\vec{c}c = \vec{0} \Longrightarrow \vec{c} = \vec{0}$ $[c \neq 0]$

5.03 $\vec{0}$-products and Cancellation Principles

In working Exercise 1 of Part C you probably recognized that the theorem you proved is analogous to the real number-theorem:

$$(ab) \cdot /b = a \ [b \neq 0]$$

and that it may be proved just as this real number-theorem is proved in steps (2) – (6) on page 148. [The only difference is that you need two of our new postulates in place of 5_1(b) and 5_2(b). Which two?]

After doing Exercise 1, the easiest way to prove the cancellation principle in Exercise 2 is to begin with the valid sentence:

$$\vec{a}c = \vec{b}c \Longrightarrow (\vec{a}c) \cdot /c = (\vec{b}c) \cdot /c$$

[How would you show that this equality principle is a valid sentence?]

The cancellation principle of Exercise 2 is analogous to one of two cancellation principles for multiplication of real numbers:

(a) $ac = bc \Longrightarrow a = b$ $[c \neq 0]$ (b) $ca = cb \Longrightarrow a = b$ $[c \neq 0]$

This suggests that there may also be a cancellation principle for multiplication of translations by real numbers which is analogous to (b). There is, and we state both cancellation principles in:

> Theorem 5-4
> (a) $\vec{a}c = \vec{b}c \Longrightarrow \vec{a} = \vec{b}$ $[c \neq 0]$
> (b) $\vec{c}a = \vec{c}b \Longrightarrow a = b$ $[\vec{c} \neq \vec{0}]$

Of course, we still have to show that Theorem 5-4(b) *is* a theorem. Either of the two cancellation principles for multiplication of real numbers can be derived from the other and one of our postulates for real numbers. [Which postulate?] However, the relation between parts (a) and (b) of Theorem 5-4 is not this simple. [Explain.] Also, we cannot prove part (b) of Theorem 5-4 in the way in which you proved

Answers for Part C

1. By 4_{a_1}, $(\vec{a}b)\cdot /b = \vec{a}(b \cdot /b)$ and, for $b \neq 0$, $b \cdot /b = 1$. So, since $\vec{a}1 = \vec{a}$ $[4_5]$, it follows that, for $b \neq 0$, $(\vec{a}b)\cdot /b = \vec{a}$. [Remind students, if necessary, that the statement in the exercise is an abbreviation for '$b \neq 0 \Longrightarrow (\vec{a}b)\cdot /b = \vec{a}$'. To arrive explicitly at this unabbreviated result, replace the first 'for' in the preceding proof by 'assuming that' and the later ', for $b \neq 0$,' by 'if $b \neq 0$ then'. A column proof is indicated in the first sentence of the following section 5.03.]

2.

(1)	$\vec{a}c = \vec{b}c \Longrightarrow (\vec{a}c)\cdot /c = (\vec{b}c)\cdot /c$		[valid]
(2)	$(\vec{a}c)\cdot /c = \vec{a}$	$[c \neq 0]$	[Exercise 1]
(3)	$(\vec{b}c)\cdot /c = \vec{b}$	$[c \neq 0]$	[Exercise 1]
(4)	$\vec{a}c = \vec{b}c \Longrightarrow \vec{a} = \vec{b}$	$[c \neq 0]$	[(2), (3), (1)]

[Compare with answer on TC 150 for Exercise 1 of Part B on page 150. As a paragraph, the proof given above might go as follows:

> By an equality principle for multiplication, if $\vec{a}c = \vec{b}c$ then $(\vec{a}c)\cdot /c = (\vec{b}c)\cdot /c$. But, by Exercise 1, $(\vec{a}c)\cdot /c = \vec{a}$ and $(\vec{b}c)\cdot /c = \vec{b}$. Hence, if $\vec{a}c = \vec{b}c$ then $\vec{a} = \vec{b}$.

The proof of the equality principle '$\vec{a} = \vec{b} \Longrightarrow \vec{a}a = \vec{b}a$' of which (1) is an instance goes as follows:

> Suppose that $\vec{a} = \vec{b}$. Since $\vec{a}a = \vec{a}a$ it follows that $\vec{a}a = \vec{b}a$. Hence, if $\vec{a} = \vec{b}$ then $\vec{a}a = \vec{b}a$.

The rules of logic involved are, as in the case of all equality principles, the introduction and replacement rules for equations and the deduction rule.]

3. [A derivation of this result from that of Exercise 2 [Theorem 5-4(a)] and Theorem 5-1(b) is given on page 187.]

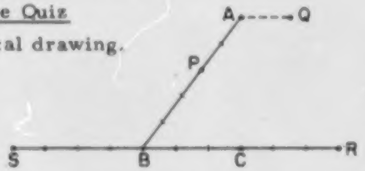
*

Sample Quiz

1. Draw three points A, B, and C which are not all on the same line:
   (a) Locate the point P such that (A to P):(P to B) is 2/3.
   (b) Locate the point Q such that $Q = A + (C - B)\frac{1}{2}$.
   (c) Locate the point R such that (B to R):(C to R) is 2.
   (d) Locate the point S such that $S = B + (B - C)\frac{4}{3}$.
2. Prove that $(A - \vec{a}c) + \vec{b}c = A + (\vec{b} - \vec{a})c$.

Answers for Sample Quiz

1. Here is a typical drawing.



2. $(A - \vec{a}c) + \vec{b}c = (A + -\vec{a}c) + \vec{b}c = A + (-\vec{a}c + \vec{b}c) = A + (-\vec{a} + \vec{b})c$
$= A + (\vec{b} + -\vec{a})c = A + (\vec{b} - \vec{a})c$.

The postulates which substitute for the APM and the PM1 are 4_8 and 4_5. Note that the former is more than an associative principle. In addition to changing the ''association'' of ''factors'', in applying it one changes from one kind of multiplication to another. Postulate 4_8 may, if you wish be called the quasi-associative principle [for multiplication of translations by real numbers]. Similarly, since two kinds of addition are involved, 4_6 is a quasi-distributive principle. On the other hand, 4_7 says that multiplication of translations by real numbers is distributive over addition of translations and, so, is properly a distributive principle.

For the proof of the validity of the second sentence displayed in this section, see the discussion on TC 186(1) of the answer for Exercise 2 of Part C.

The postulate needed in deriving one of the two cancellation principles for multiplication of real numbers from the other is, of course, the CPM [i.e.: 5_4(b)]. Note that, contrary to some students' initial reaction, it would be of no help in proving Theorem 5-4(b) to introduce ''multiplication of real numbers by translations'' by adopting the definition '$a\vec{a} = \vec{a}a$'. Doing so would make possible a restatement of Theorem 5-4(b), but this would be of no help in proving the theorem in question. As this example indicates, it makes no real difference whether or not one chooses to allow translations to be ''multiplied on the right'' by real numbers as well as ''on the left''. The operation which we chose to call 'multiplication of a translation by a real number' is an important one, but how one chooses to indicate it symbolically — and what one chooses to call it — is of importance only in so far as the choices made have some mnemonic value. The use of the word 'multiplication' and of product-notation is suggested by the fact that the operation in question does have the effect of multiplying the magnitude of the translation by the absolute value of the real number, and by the similarity of 4_5 - 4_8 to familiar statements about multiplication of real numbers. The decision to use '$\vec{a}a$' rather than '$a\vec{a}$' stems from the feeling that it is more reasonable for multipliers to be written on the right, as divisors are, and the feeling that it is the translation which is multiplied by the real number rather than vice versa. Both feelings are, admittedly, not much more than prejudices.

To prove Theorem 5-4(b) as Theorem 5-4(a) was proved would require multiplication and reciprocating operations on $\mathcal{T}$ which are not available. [In the second part of this course there will be introduced a real-valued multiplication of translations such that

$$\vec{a} \cdot \vec{b} \in \mathcal{R},\ \vec{a} \cdot (\vec{b}c) = (\vec{a} \cdot \vec{b})c, \text{ and } \vec{a} \cdot \vec{a} > 0\ [\vec{a} \neq 0].$$

Using this we could derive Theorem 5-4(b) from the corresponding cancellation principle (b) for multiplication of real numbers and the theorem that no positive number is 0.]

The ''third'' way of proving the cancellation principle (b) for multiplication of real numbers is, briefly, as follows:

> From (a) and '$0c = 0$', derive:
>
> $$ac = 0 \Longrightarrow a = 0\ [c \neq 0]$$
>
> By contraposition [and our convention as to the meaning of restrictions], this is equivalent to:
>
> $$c \neq 0 \Longrightarrow [a \neq 0 \Longrightarrow ac \neq 0]$$

> and, as is easily shown — as well as being intuitively obvious, this is equivalent to:
>
> $$a \neq 0 \Longrightarrow [c \neq 0 \Longrightarrow ac \neq 0]$$
>
> and, so, to:
>
> $$ac = 0 \Longrightarrow c = 0\ [a \neq 0]$$
>
> This last has the relevant instance:
>
> $$c(a - b) = 0 \Longrightarrow a - b = 0\ [c \neq 0]$$
>
> From this and the appropriate distributive principle [the LDPMS], follows:
>
> $$ca - cb = 0 \Longrightarrow a - b = 0\ [c \neq 0]$$
>
> The cancellation principle (b) follows from this and two instances of '$a - b = 0 \Longleftrightarrow a = b$'.

This procedure is easily modified to derive Theorem 5-4(b) from Theorem 5-4(a), Theorem 5-1(b), Theorem 5-3(a), an instance of '$\vec{a} - \vec{b} = \vec{0} \Longleftrightarrow \vec{a} = \vec{b}$', and '$a - b = 0 \Longleftrightarrow a = b$'. The last two rather trivial but useful theorems are proved in the same way. We prove the second:

> Suppose that $a = b$. It follows that $a + -b = b + -b$ and so, [by 5_7(a) and 5_3(a)], $a - b = 0$. Hence, if $a = b$ then $a - b = 0$.
>
> Suppose that $a - b = 0$. It follows that $(a - b) + b = 0 + b$ and so [by 5_7(a), 5_1(a), 5_4(a), 5_3(a), and 5_2(a)], $a = b$. Hence, if $a - b = 0$ then $a = b$.

In proving the corresponding theorem concerning $\mathcal{T}$, Definition 3-1(b) plays the role of 5_7(a), and 4_1 - 4_4 take the place of 5_1(a) - 5_4(a), respectively.

part (a). [Why not?] Fortunately, there is a third way of proving the cancellation principle (b) for multiplication of real numbers, and this third way can be used to prove Theorem 5-4(b).

One key to the proof of Theorem 5-4(b) is the theorem proved in Exercise 3 of Part C:

$$(*) \qquad \vec{c}c = \vec{0} \Longrightarrow \vec{c} = \vec{0}\ [c \neq 0]$$

This theorem is an easy consequence of Theorem 5-4(a) and Theorem 5-1(b):

$$\begin{array}{lcl} (1) & \vec{c}c = \vec{0}c \Longrightarrow \vec{c} = \vec{0}\ [c \neq 0] & \text{[Theorem 5-4(a)]} \\ (2) & \vec{0}c = \vec{0} & \text{[Theorem 5-1(b)]} \\ (3) & \vec{c}c = \vec{0} \Longrightarrow \vec{c} = \vec{0}\ [c \neq 0] & \text{[(2), (1)]} \end{array}$$

Conversely, we can derive Theorem 5-4(a) from (*) and Theorem 5-3(b) [and the fairly trivial theorem '$\vec{a} - \vec{b} = \vec{0} \Longrightarrow \vec{a} = \vec{b}$']:

$$\begin{array}{lcl} (1') & (\vec{a} - \vec{b})c = \vec{0} \Longrightarrow \vec{a} - \vec{b} = \vec{0}\ [c \neq 0] & \text{[(*)]} \\ (2') & (\vec{a} - \vec{b})c = \vec{a}c - \vec{b}c & \text{[Theorem 5-3(b)]} \\ (3') & \vec{a}c - \vec{b}c = \vec{0} \Longrightarrow \vec{a} - \vec{b} = \vec{0}\ [c \neq 0] & \text{[(2'), (1')]} \end{array}$$

Clearly, Theorem 5-4(a) follows from (3') and two instances of the "fairly trivial theorem".

The second of these two arguments suggests that we could obtain a proof of Theorem 5-4(b) by using another possible theorem:

$$(**) \qquad \vec{c}c = \vec{0} \Longrightarrow c = 0\ [\vec{c} \neq \vec{0}]$$

and Theorem 5-3(a). [The first step would be to substitute '$a - b$' for 'c' in (**).] This suggests that we try to prove (**). If you think hard, you may see that (*) and (**) seem to say the same thing. In fact, as it turns out, (*) and (**) are logically equivalent.

To show the equivalence of (*) and (**) we may begin by noting that they are equivalent, respectively, to their contrapositives:

$$\begin{array}{ll} (*') & \vec{c} \neq \vec{0} \Longrightarrow \vec{c}c \neq \vec{0}\ [c \neq 0] \\ (**') & c \neq 0 \Longrightarrow \vec{c}c \neq \vec{0}\ [\vec{c} \neq \vec{0}] \end{array}$$

Next, we recall what the restrictions mean. By our convention about this, (*') and (**') are merely abbreviations for:

$$\begin{array}{ll} (*'') & c \neq 0 \longrightarrow [\vec{c} \neq \vec{0} \longrightarrow \vec{c}c \neq \vec{0}] \\ (**'') & \vec{c} \neq \vec{0} \longrightarrow [c \neq 0 \longrightarrow \vec{c}c \neq \vec{0}] \end{array}$$

The derivation of Theorem 5-4(b) from (☆☆), Theorem 5-3(a) and two "trivial theorems" is described in the preceding commentary.

That the restricted conditional sentences (☆') is not strictly the contrapositive of (☆) [nor (☆☆') of (☆☆)] has been pointed out on TC 157(1). Nevertheless, that:

$$(☆') \qquad c \neq 0 \Longrightarrow [\vec{c} \neq \vec{0} \Longrightarrow \vec{c}c \neq \vec{0}]$$

and:

$$(☆) \qquad c \neq 0 \Longrightarrow [\vec{c}c = \vec{0} \Longrightarrow \vec{c} = \vec{0}]$$

are equivalent follows, by the replacement rule for biconditional sentences, from the logical equivalence of '$\vec{c}c = \vec{0} \Longrightarrow \vec{c} = \vec{0}$' and its (proper) contrapositive — that is, from the fact that the sentence:

$$[\vec{c}c = \vec{0} \Longrightarrow \vec{c} = \vec{0}] \Longleftrightarrow [\vec{c} \neq \vec{0} \Longrightarrow \vec{c}c \neq \vec{0}]$$

is valid.

TC 188 (1)

Parts A - E, if used as one assignment, make a rather lengthy one. If you are pressed for time and feel that one day is all the time available for these exercises then we again recommended assigning exercises to teams. In this case you might consider making the derivation of each inference form a team project and have each student do the exercises involving theorems about translations and real numbers.

Answers for Part A

1. First way:

$$\cfrac{(p \text{ and } q) \overset{*}{\Longleftrightarrow} (q \text{ and } p) \qquad \cfrac{p \Longrightarrow [q \Longrightarrow r]}{(p \text{ and } q) \Longrightarrow r}}{\cfrac{(q \text{ and } p) \Longrightarrow r}{q \Longrightarrow [p \Longrightarrow r]}} \qquad \left[\begin{array}{c} \cfrac{\cfrac{\overset{*}{p \text{ and } q}}{q} \qquad \cfrac{\overset{*}{p \text{ and } q}}{p}}{\cfrac{q \text{ and } p}{(p \text{ and } q) \Longrightarrow (q \text{ and } p)}\,{}^{*}} \\ \text{etc.} \end{array}\right]$$

['*' indicates a valid sentence.]

Second way:

$$\cfrac{\cfrac{\cfrac{\overset{\dagger}{q} \qquad \cfrac{\overset{*}{p} \qquad p \Longrightarrow [q \Longrightarrow r]}{q \Longrightarrow r}}{r}}{p \Longrightarrow r}\,{}^{*}}{q \Longrightarrow [p \Longrightarrow r]}\,{}^{\dagger}$$

Now it should not be difficult to see that (*'') and (**'') do say the same thing – and, in fact, it is easy to show that any inference of the form:

$$\dfrac{p \longrightarrow [q \longrightarrow r]}{q \longrightarrow [p \longrightarrow r]}$$

is valid. [See Part A, below.]

Exercises

Part A

1. Show that any inference of the form:

$$\dfrac{p \longrightarrow [q \longrightarrow r]}{q \longrightarrow [p \longrightarrow r]}$$

is valid. [*Hint*: One way is to use importation and then exportation (page 101), noting in between that any sentence of the form '(p and q) ⟷ (q and p)' is valid. Another way is to use modus ponens twice – first to strip off the 'p' from 'p ⟶ [q ⟶ r]' and then the 'q' – and then use the deduction rule twice – first to obtain 'p ⟶ r' and then 'q ⟶ [p ⟶ r]'.]

2. Sentence (**) follows from (*) because the inference:

$$\dfrac{c \neq 0 \longrightarrow [\overrightarrow{cc} = \vec{0} \longrightarrow \vec{c} = \vec{0}]}{\vec{c} \neq \vec{0} \longrightarrow [\overrightarrow{cc} = \vec{0} \longrightarrow c = 0]}$$

is valid. Use the fact that a conditional sentence and its contrapositive are interchangeable, together with the result of Exercise 1, to show that any inference of the form:

$$\dfrac{\text{not } p \longrightarrow [q \longrightarrow r]}{\text{not } r \longrightarrow [q \longrightarrow p]}$$

is valid. [*Hint*: In the premiss, replace 'q ⟶ r' by its contrapositive and then apply Exercise 1.]

3. Complete the following derivation of Theorem 5–4(b) from (*) and previously noted theorems.

(1) $\overrightarrow{cc} = \vec{0} \longrightarrow \vec{c} = 0$ [c ≠ 0] [(*)]
(2) $\overrightarrow{cc} = \vec{0} \longrightarrow c = 0$ [$\vec{c} \neq \vec{0}$] [(1)] } [by Ex. 2]
(3) $\vec{c}(a - b) = \vec{0} \longrightarrow a - b = 0$ [$\vec{c} \neq \vec{0}$] [(2)]

Part B

One important theorem about multiplication of real numbers is the "0-product theorem":

$$ab = 0 \longrightarrow (a = 0 \text{ or } b = 0)$$

TC 188 (2)

2.

$$\dfrac{[q \Longrightarrow p] \overset{*}{\Longleftrightarrow} [\text{not } p \Longrightarrow \text{not } q] \qquad \dfrac{\dfrac{[q \Longrightarrow r] \overset{*}{\Longleftrightarrow} [\text{not } r \Longrightarrow \text{not } q] \qquad \text{not } p \Longrightarrow [q \Longrightarrow r]}{\text{not } p \Longrightarrow [\text{not } r \Longrightarrow \text{not } q]}}{\text{not } r \Longrightarrow [\text{not } p \Longrightarrow \text{not } q]}\,[\text{Ex. } 1]}{\text{not } r \Longrightarrow [q \Longrightarrow p]}$$

['*'s indicate valid sentences.]

Another scheme, based on the symmetric rules for contraposition:

$$\dfrac{\dfrac{[\text{not } p \Longrightarrow r] \overset{*}{\Longleftrightarrow} [\text{not } r \Longrightarrow p] \qquad \dfrac{\text{not } p \Longrightarrow [q \Longrightarrow r]}{q \Longrightarrow [\text{not } p \Longrightarrow r]}\,[\text{Ex. } 1]}{q \Longrightarrow [\text{not } r \Longrightarrow p]}}{\text{not } r \Longrightarrow [q \Longrightarrow p]}\,[\text{Ex. } 1]$$

3. (4) $\vec{c}(a - b) = \vec{c}a - \vec{c}b$ [Theorem 5-3(a)]
(5) $\vec{c}a - \vec{c}b = \vec{0} \Longrightarrow a - b = 0$ [$\vec{c} \neq 0$] [(4), (3)]
(6) $\vec{c}a - \vec{c}b = \vec{0} \Longleftrightarrow \vec{c}a = \vec{c}b$ [theorem]
(7) $a - b = 0 \Longleftrightarrow a = b$ [theorem]
(8) $\vec{c}a = \vec{c}b \Longrightarrow a = b$ [$\vec{c} \neq 0$] [(8), (7), (6)]

[For proofs of the two theorems for steps (6) and (7), see TC 186(3).]

TC 189 (1)

Answers for Part B

[The direct proof of the 0-product theorem is not too messy, but it is somewhat less intuitive than the indirect proof suggested in Exercise 3. The direct proof derives the theorem from the analogue of (☆) by a dilemma based on an example of the law of the excluded middle:

$$\dfrac{\dfrac{b = 0 \text{ or } b \neq 0 \qquad \dfrac{\dfrac{b \overset{*}{=} 0}{a = 0 \text{ or } b = 0}}{b = 0 \Longrightarrow (a = 0 \text{ or } b = 0)}{}^{*} \qquad \dfrac{\dfrac{\dfrac{ab \overset{\dagger}{=} 0 \qquad \dfrac{b \overset{*}{\neq} 0 \qquad ab = 0 \Longrightarrow a = 0\ [b \neq 0]}{ab = 0 \Longrightarrow a = 0}}{a = 0}}{a = 0 \text{ or } b = 0}}{b \neq 0 \Longrightarrow (a = 0 \text{ [illegible]} = 0)}{}^{*}}{a = 0 \text{ or } b = 0}}{ab = 0 \Longrightarrow (a = 0 \text{ or } b = 0)}{}^{\dagger}$$

The derivation can be shortened slightly by using the complex dilemma mentioned in the hint for Exercise 2 of Part D.]

1. not (a = 0 or b = 0) ⟹ ab ≠ 0

This *can* be proved by deriving its consequent from its antecedent and previously proved theorems about real numbers, and then using the deduction rule. Such a proof is, however, somewhat messy. [This is usually the case when the consequent of the conditional sentence to be proved is an alternation sentence.]

Now that we have learned that a conditional sentence is a consequence of its contrapositive we have another way of proving a conditional sentence. For example, instead of trying to prove the 0-product theorem directly, we might begin by trying to prove its contrapositive.

1. State the contrapositive of the 0-product theorem.
2. Use what you have learned about 'and', 'or', and 'not' to show that the contrapositive of the 0-product theorem is logically equivalent to a theorem you proved in Chapter 4. [For help, see page 157.]
3. Prove the "$\vec{0}$-product theorem":

Theorem 5–5 $\vec{a}a = \vec{0} \Longrightarrow (a = 0 \text{ or } \vec{a} = \vec{0})$

[*Hint*: To *discover* a proof, use rules of logic to transform the contrapositive of Theorem 5–5 into a previously proved theorem. Then, see if you can derive Theorem 5–5 from the previously proved theorem by retracing your steps.]

4. In Exercise 3 you probably showed that Theorem 5–5 is a consequence of the instance:

(*$_1$) $\vec{a}a = \vec{0} \Longrightarrow \vec{a} = \vec{0}\ [a \neq 0]$

of the theorem (*) on page 187. And, in discovering your proof you probably showed that this instance is a consequence of Theorem 5–5. Using the same procedure you can show that any inference of either of the forms:

$$\frac{\text{not } q \Longrightarrow [p \Longrightarrow r]}{p \Longrightarrow (q \text{ or } r)} \qquad \frac{p \Longrightarrow (q \text{ or } r)}{\text{not } q \Longrightarrow [p \Longrightarrow r]}$$

is valid. Do so.

Part C

1. The second kind of inference in Exercise 4 of Part B [which shows that (*$_1$) is a consequence of Theorem 5–5] can be justified very easily by using one of the rules for denying an alternative. Do so, by constructing an appropriate tree-diagram.
2. An argument similar to that described in Exercise 1 shows that the instance:

(**$_1$) $\vec{a}a = \vec{0} \Longrightarrow a = 0\ [\vec{a} \neq \vec{0}]$

of (**) on page 187 is a consequence of Theorem 5–5. Give this argument.

2. By Exercise 6(a) on page 163 the sentence of Exercise 1 is equivalent to:

$$(a \neq 0 \text{ and } b \neq 0) \Longrightarrow ab \neq 0$$

The latter is (2) on page 157.

3. By (☆) on page 187, $a \neq 0 \Longrightarrow [\vec{a}a = \vec{0} \Longrightarrow \vec{a} = \vec{0}]$. Hence, $a \neq 0 \Longrightarrow [\vec{a} \neq 0 \Longrightarrow \vec{a}a \neq 0]$. So, $(a \neq 0 \text{ and } \vec{a} \neq \vec{0}) \Longrightarrow \vec{a}a \neq 0$ or, equivalently, $\text{not } (a = 0 \text{ or } \vec{a} = \vec{0}) \Longrightarrow \vec{a}a \neq 0$. Hence, $\vec{a}a = 0 \Longrightarrow (a = 0 \text{ or } \vec{a} = \vec{0})$.

[The logical rules involved are the replacement rule for biconditional sentences and importation. The biconditional premisses for the first and third applications of the replacement rule come from the equivalence of a conditional sentence with its contrapositive. That for the second application of this rule comes from the second of DeMorgan's Laws on page 171.]

4.

$$\dfrac{\dfrac{\text{not } (q \text{ or } r) \overset{*}{\Longleftrightarrow} (\text{not } q \text{ and not } r) \qquad \dfrac{\dfrac{[p \Longrightarrow r] \Longleftrightarrow [\text{not } r \Longrightarrow \text{not } p] \qquad \text{not } q \Longrightarrow [p \Longrightarrow r]}{\text{not } q \Longrightarrow [\text{not } r \Longrightarrow \text{not } p]}}{(\text{not } q \text{ and not } r) \Longrightarrow \text{not } p}}{\text{not } (q \text{ or } p) \Longrightarrow \text{not } p}}{p \Longrightarrow (q \text{ or } r)}$$

['*'s indicate valid sentences.]

$$\dfrac{[p \Longrightarrow r] \overset{*}{\Longleftrightarrow} [\text{not } r \Longrightarrow \text{not } p] \qquad \dfrac{\dfrac{\text{not } (q \text{ or } r) \overset{*}{\Longleftrightarrow} (\text{not } q \text{ and not } r) \qquad \dfrac{p \Longrightarrow (q \text{ or } r)}{\text{not } (q \text{ or } r) \Longrightarrow \text{not } p}}{(\text{not } q \text{ and not } r) \Longrightarrow \text{not } p}}{\text{not } q \Longrightarrow [\text{not } r \Longrightarrow \text{not } p]}}{\text{not } q \Longrightarrow [p \Longrightarrow r]}$$

Answers for Part C

1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{*}{p} \qquad p \Longrightarrow (q \text{ or } r)}{q \text{ or } r} \qquad \overset{\dagger}{\text{not } q}}{r}}{p \Longrightarrow r}\,*}{\text{not } q \Longrightarrow [p \Longrightarrow r]}\,\dagger$$

2. By Theorem 5-5, if $\vec{a}a = \vec{0}$ then $(a = 0 \text{ or } \vec{a} = 0)$. Suppose that $\vec{a}a = \vec{0}$ and that $\vec{a} \neq 0$. It follows [by modus ponens and denial of an alternative] that $a = 0$. Hence, for $\vec{a} \neq \vec{0}$, if $\vec{a}a = \vec{0}$ then $a = 0$.

[Notice the minor difference between this argument and that which is schematized in Exercise 1. In comparison with Exercise 1, the second assumption in Exercise 2 is "not r" rather than "not q". The effect is to validate inferences of the form '$[p \Longrightarrow (q \text{ or } r)]/[\text{not } r \Longrightarrow [p \Longrightarrow q]]$'.]

Part D

1. The first of the two kinds of inference in Exercise 4 of Part B can be made to seem intuitively reasonable in the following way:

> Suppose that if not q then [if p then r], and suppose that p. It follows that if not q then r. But, q or not q. So, q or r. Hence, if p then (q or r).

Translate the first two sentences of this argument into a tree-diagram:

$$\dfrac{p \qquad \dfrac{\text{not } q \longrightarrow [p \longrightarrow r]}{\cdots}\ [\text{Part A, Exercise 1}]}{\cdots}$$

2. The claim made in the second two sentences is that any inference of the form:

$$\dfrac{q \text{ or not } q \qquad \text{not } q \longrightarrow r}{q \text{ or } r}$$

is valid. Show that this is the case. [*Hint*: You may find it helpful to begin by showing that any inference of the form:

$$\dfrac{p \text{ or } q \qquad p \longrightarrow r \qquad q \longrightarrow s}{r \text{ or } s}$$

is valid.]

3. Complete the tree-diagram you began in Exercise 1 to justify inferences of the first kind in Exercise 4 of Part B.

Part E

From the preceding exercises it follows that there are four ways of saying what is said by a restricted conditional sentence. For example, we have seen that any two of the following sentences are logically equivalent:

$$\vec{a}a = \vec{0} \longrightarrow \vec{a} = \vec{0} \quad [a \neq 0]$$
$$\vec{a}a = \vec{0} \longrightarrow a = 0 \quad [\vec{a} \neq \vec{0}]$$
$$\vec{a}a = \vec{0} \longrightarrow (\vec{a} = \vec{0} \text{ or } a = 0)$$
$$(\vec{a} \neq \vec{0} \text{ and } a \neq 0) \longrightarrow \vec{a}a \neq \vec{0}$$

In general, any two corresponding sentences of the following forms are logically equivalent:

$$p \longrightarrow q\ [\text{not } r]$$
$$p \longrightarrow r\ [\text{not } q]$$
$$p \longrightarrow (q \text{ or } r)$$
$$(\text{not } q \text{ and not } r) \longrightarrow \text{not } p$$

Answers for Part D

The complex constructive dilemma of the hint for Exercise 2 is validated as follows:

$$\dfrac{p \text{ or } q \qquad \dfrac{\dfrac{\dfrac{\overset{*}{p} \qquad p \Longrightarrow r}{r}}{r \text{ or } s}}{p \Longrightarrow (r \text{ or } s)}{}^{*} \qquad \dfrac{\dfrac{\dfrac{\overset{*}{q} \qquad q \Longrightarrow s}{s}}{r \text{ or } s}}{q \Longrightarrow (r \text{ or } s)}{}^{*}}{r \text{ or } s}$$

The answers for Exercises 1, 2, and 3 are combined in:

$$\dfrac{q \text{ or not } q \qquad q \overset{*}{\Longrightarrow} q \qquad \dfrac{p \qquad \dfrac{\text{not } q \Longrightarrow [p \Longrightarrow r]}{p \Longrightarrow [\text{not } q \Longrightarrow r]}}{\text{not } q \Longrightarrow r}}{q \text{ or } r}$$

TC 191 (1)

Answers for Part E

1. $ab = 0 \Longrightarrow (a = 0 \text{ or } b = 0)$, $ab = 0 \Longrightarrow a = 0\ [b \neq 0]$, $ab = 0 \Longrightarrow b = 0\ [a \neq 0]$
2. $(a \neq b \text{ and } \vec{c} \neq 0) \Longrightarrow \vec{c}a \neq \vec{c}b$, $\vec{c}a = \vec{c}b \Longrightarrow a = b\ [\vec{c} \neq \vec{0}]$, $\vec{c}a = \vec{c}b \Longrightarrow \vec{c} = \vec{0}\ [a \neq b]$
3. $A + \vec{a}c = A + \vec{b}c \Longrightarrow c = 0\ [\vec{a} \neq \vec{b}]$, $A + \vec{a}c = A + \vec{b}c \Longrightarrow (\vec{a} = \vec{b} \text{ or } c = 0)$, $(\vec{a} \neq \vec{b} \text{ and } c \neq 0) \Longrightarrow A + \vec{a}c \neq A + \vec{b}c$
4. $\vec{a}a = \vec{b} \Longrightarrow a = 0\ [\vec{a} \neq \vec{b}\cdot/a]$, $\vec{a}a = \vec{b} \Longrightarrow (\vec{a} = \vec{b}\cdot/a \text{ or } a = 0)$, $(\vec{a} \neq \vec{b}\cdot/a \text{ and } a \neq 0) \Longrightarrow \vec{a}a \neq \vec{b}$

For each of the following sentences, write three other sentences equivalent to it.

1. $(a \neq 0 \text{ and } b \neq 0) \Longrightarrow ab \neq 0$
2. $\vec{c}a = \vec{c}b \Longrightarrow (a = b \text{ or } \vec{c} = \vec{0})$
3. $A + \vec{a}c = A + \vec{b}c \Longrightarrow \vec{a} = \vec{b}\ [c \neq 0]$
4. $\vec{a}a = \vec{b} \Longrightarrow \vec{a} = \vec{b} \cdot /a\ [a \neq 0]$

5.04 Vector Spaces

In Chapter 3 we noted that the set $\mathcal{T}$ of translations of $\mathcal{E}$ is a commutative group with respect to the operation of composition of functions. Using '+' for composition, '$\vec{0}$' for $i_{\mathcal{E}}$, and '−' for inversion of functions, we expressed this in Postulate 4‴:

4_0. (a) $\vec{a} + \vec{b} \in \mathcal{T}$ (b) $\vec{0} \in \mathcal{T}$ (c) $-\vec{a} \in \mathcal{T}$
4_1. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
4_2. $\vec{a} + \vec{0} = \vec{a}$
4_3. $\vec{a} + -\vec{a} = \vec{0}$
4_4. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

In the present chapter we have seen that the real numbers "operate" on translations in such a way that

4_0. (d) $\vec{a} \cdot b \in \mathcal{T}$
4_5. $\vec{a} \cdot 1 = \vec{a}$,
4_6. $\vec{a} \cdot (b + c) = \vec{a} \cdot b + \vec{a} \cdot c$,
4_7. $(\vec{a} + \vec{b}) \cdot c = \vec{a} \cdot c + \vec{b} \cdot c$,
and 4_8. $(\vec{a} \cdot b) \cdot c = \vec{a} \cdot (bc)$.

A commutative group on which the real numbers operate according to rules like 4_0(d) and $4_5 - 4_8$ is called *a vector space over the real numbers* [or, for short, 'a real vector space']. So, we can summarize $4_0 - 4_8$ in:

> **Postulate 4″** $\mathcal{T}$, under function composition, is a vector space over $\mathcal{R}$.

Vector spaces come up often in mathematics and in its applications to physics. [Some of these applications are described in the next section.] When one is considering operations on a set with respect to which the set is a vector space it is customary to call the members of the set *vectors*. So, in this course, we shall sometimes speak of translations as *vectors*.

If you imagine erasing each '→' in $4_0 - 4_8$ and replacing each '$\mathcal{T}$' by an '$\mathcal{R}$', you should see that—without knowing it—you have been acquainted for some time with one vector space. Clearly, one might properly think of the set of real numbers as a vector space and refer to real numbers as vectors. One usually doesn't, because he can say

[Here are some items which will give the students some practice in recognizing theorems about points and translations, as well as some practice in using Postulates $4_5 - 4_8$.]

The following are sentences about points and translations. If a given sentence is a theorem, write 'T' in the space provided. If it is not a theorem, write 'N' in the space.

___ 1. $[D + (D - C) \cdot -2] - C = C - D$
___ 2. $A - (B + \vec{c}3) - \vec{d}2 \Longleftrightarrow B - A = \vec{c}3 - \vec{d}2$
___ 3. $(A - C) + C = A$
___ 4. $B - [A - (A - B)3] = (B - A) \cdot -2$
___ 5. $\vec{c}2 - \vec{d}3 = B - [(B - \vec{c}2) + \vec{d}3]$
___ 6. $(A - \vec{b}4) - \vec{c}2 = A \Longleftrightarrow \vec{c} = \vec{b}2$
___ 7. $(B - A)2 - \vec{a}3 = \vec{a} \cdot -3 \Longleftrightarrow B = A$
___ 8. $\vec{b}3 + \vec{c} = \vec{f}3 \Longleftrightarrow \vec{c} = (\vec{f} - \vec{b})3$
___ 9. $\vec{a}2 = -\vec{b}\frac{1}{2} \Longleftrightarrow (C + \vec{b}) + \vec{a}4 = C$
___ 10. $(A + \vec{b}2) - (C + \vec{d}2) = A - [C - (\vec{b} + \vec{d})2]$

Answers. 1. T 2. N 3. N 4. T 5. T
6. N 7. T 8. T 9. T 10. N

We are building a very particular kind of algebraic structure in our Postulate 4. The following brief remarks are to help you to see where this particular structure lies among the large variety of algebraic structures. None of these remarks are intended for presentation to the students at this time.

If the arrows are removed from $4_0 - 4_8$ and '$\mathcal{T}$' is replaced by '$\mathcal{R}$' then the resulting sentences assert that $\mathcal{R}$ is a ring (with identity). ['ring', like 'group' and 'field', refers to a type of algebraic structure.] Let's refer to these sentences as the ring postulates. As they stand [with arrows, etc.], 4_0(a) - (c) and $4_1 - 4_4$ assert that $\mathcal{T}$ is a commutative group; and 4_0(d) and $4_5 - 4_8$ assert that the members of $\mathcal{R}$ are operators on this group. Because these operators constitute a ring, $\mathcal{T}$ is said to be a module over $\mathcal{R}$. In short, a module is a commutative group together with a ring of operators. Setting aside the particular interpretations we have given to the letter '$\mathcal{T}$' and '$\mathcal{R}$', the theory of modules is the theory which has as postulates the ring postulates together with postulates $4_0 - 4_8$. By definition, a vector space is a module whose operators constitute a field. So, again setting aside the interpretations we have given to '$\mathcal{T}$' and '$\mathcal{R}$', the theory of vector spaces is the theory which has as postulates $5_0 - 5_7$ [omitting references to order] together with $4_0 - 4_8$.

As pointed out on TC 177, the integers can be introduced as operators on any commutative group. Since, as is easily seen, addition and multiplication of integers satisfy the ring postulates, any commutative group can be considered as a module over the integers. Since the integers do not constitute a field, such a module is an example of one which is not a vector space.

Since the rational numbers constitute a field, $\mathcal{T}$ with [only] rational numbers as operators is a vector space — and is quite a different one from the vector space $\mathcal{T}$ over $\mathcal{R}$ with which we are dealing.

more about the algebra of real numbers than is included in the postulates for a vector space.

In Chapter 3 we discovered a rather close analogy between our algebra of points and translations based on Postulates 1 – 3 and 4‴ [and Definition 3-1] and the algebra of addition and subtraction of real numbers. Briefly, for each sentence of our algebra there is an analogous sentence about real numbers, and a sentence of our algebra follows from Postulates 1 – 3, 4‴, and Definition 3-1 if and only if the analogous sentence about real numbers follows from 5_0(a) – (c), 5_1 – 5_4, and 5_7 on page 170. Now that we have enlarged our algebra of points and translations, our sentences still have real number analogues but, with more postulates, more of our sentences are theorems. From what you have seen on removing the arrows from 4_0 – 4_x it is still the case that the real number analogue of any of these new theorems is bound to be a theorem about real numbers. The reverse situation is too complicated now to be of much use. All that is worth saying in this connection is that if the real number analogue of a given sentence of our extended algebra is a theorem then the sentence *may* be a theorem of our algebra, and various proofs of the analogous sentence about real numbers *may* suggest a proof of the given sentence.

Exercises

Part A

Given any translation $\vec{d}$, we shall have many occasions to speak of and to think about all of the multiples of $\vec{d}$ by real numbers. The set of all such multiples is $\{\vec{x}\colon \exists_x\, \vec{x} = \vec{d}x\}$. For convenience we adopt the following definition:

> Definition 5-1 $[\vec{d}]$ is the set of all real multiples of $\vec{d}$. That is,
>
> $$[\vec{d}] = \{\vec{x}\colon \exists_x\, \vec{x} = \vec{d}x\}.$$

1. Show that each of the following belongs to the set $[\vec{d}]$.

$$\vec{d},\ \vec{0},\ \vec{d} - \vec{d}2,\ \vec{d}\sqrt{2} + \vec{d}3$$

2. What postulate tells you that $[\vec{d}] \subseteq \mathcal{T}$?
3. Is composition a binary operation on $[\vec{d}]$? That is, is it the case that $\vec{a} + \vec{b} \in [\vec{d}]$ for all $\vec{a}$ and $\vec{b}$ in $[\vec{d}]$? Justify your answer.
4. Why does $\vec{0} \in [\vec{d}]$?

TC 191 (3)

Any field is a vector space over itself. In terms of the notion of dimensionality introduced in a later chapter, such a vector space is 1-dimensional. A field is also a vector space over any of its subfields and, as such, has dimension greater than 1. For example, the real numbers constitute an infinite dimensional vector space over the rational numbers. [One way to see that this is so is to note that there are infinitely many prime numbers and that any linear combination of square roots of distinct primes with rational multipliers is zero only if all of the multipliers are zero (i.e., for distinct primes p_1, p_2, ..., $r_1\sqrt{p_1} + r_2\sqrt{p_2} + \ldots = 0$ only if $r_1 = 0$, $r_2 = 0$, ...).] As another — and simpler — example of an infinite dimensional vector space, consider the set of all polynomial functions. These functions form a commutative group under the usual definition of addition of real-valued functions [$f + g = \{(x, y)\colon\ y = f(x) + g(x)\}$]. They also admit the real numbers as operators [$f \cdot a = \{(x, y)\colon\ y = f(x) \cdot a\}$]. In fact, since the set of all polynomial functions of degree at most n is closed with respect to addition and to multiplication by real numbers [as defined above], this set of functions is a vector space over $\mathcal{R}$ and, as it turns out, has dimension $n + 1$. Vector spaces whose members are functions occur very frequently in mathematics and are of great importance. Our vector space $\mathcal{T}$ is, of course, an example.

As will become clear when we take up dimension, any two vector spaces over the same field which have the same finite dimension are very much alike. More precisely, any two such spaces are isomorphic with respect to their respective group operations of addition and their respective operations of multiplication by field elements. [The same is true of vector spaces over a given field which have the same infinite dimension.] For example, any 3-dimensional vector space over $\mathcal{R}$ is isomorphic to the space of all triples of real numbers with addition defined by '$(x, y, z) + (u, v, w) = (x + u, y + v, z + w)$' and multiplication by '$(x, y, z) \cdot a = (xa + ya + za)$'. This vector space is often chosen as the "typical example" of a 3-dimensional vector space. Aside from this case with which it can be described, this is a rather unfortunate choice on pedagogical grounds. For some important concepts which are readily grasped when considering, say, $\mathcal{T}$, are somewhat difficult to bring into focus when one is viewing only the ordered-triple example. This, too, will become evident when we study dimension.

TC 192 (1)

The exercises of Parts A and B serve to concentrate students' attention on what it means to be a vector space and, also, to introduce concepts and notation which are fundamental to the remainder of the course. The set $[\vec{d}]$ of all real number multiples of a translation consists of $\vec{0}$ and all translations which, intuitively, have the same direction as $\vec{d}$. [This is a discovery we hope students will make in Exercise 8 of Part A.] In view of this we shall, later, refer to $[\vec{d}]$ as the direction of $\vec{d}$. [For example, $\vec{c}$ is a translation "in the direction of $\vec{d}$" if and only if $\vec{c} \in [\vec{d}]$.] The sense of a translation will be defined in a similar manner, but with the multiplication restricted to be positive. [So, it will be the case that two translations "have the same sense" — i.e., belong to the same sense-class — if and only if either [and, so, each] is a positive multiple of the other.] In terms of these notions one can define the direction of a line as the direction of any proper translation which maps the line onto itself, and the sense of a ray as the sense of any proper translation which maps the ray into itself. The set $[\vec{d}, \vec{e}]$ of all real linear combinations of $\vec{d}$ and $\vec{e}$ forms the basis for a similar definition of the "direction" — or, as we shall say, the bidirection — of a plane.

The exercises deal, implicitly, with the notion of a subspace of a vector space and are analogous to earlier exercises dealing with subgroups and subfields. You may wish to recall to students a discovery they made in Chapter 3. Knowing that $\mathcal{R}$ is a commutative group with respect to addition, they were able to show that the set I of integers is also a commutative group with respect to addition merely by noting that integers are real numbers, that $0 \in I$, and that I is closed with respect to addition and oppositing. This is enough because each of the group postulates [the APA, PA0, IPO, and CPA] is a universal statement about real numbers which refers only to 0, addition, and oppositing. [A slightly more sophisticated procedure, which you may wish to suggest, is to show merely that I is not empty and that it is closed with respect to subtraction. Since, given any real number a, $0 = a - a$ it follows that if there is an integer a, and the set of integers is closed with respect to subtraction, then 0 is an integer. Since $-a = 0 - a$ it follows from the assumed properties of I that I is closed with respect to oppositing. Since $a + b = a - -b$ it follows that I is closed with respect to addition.] For the same reason, it was possible to show that the rational numbers form a field by using the fact that $\mathcal{R}$ is a field, that rational numbers are real numbers, that 0 and 1 are rational, and that the set of rational numbers is closed with respect to addition, multiplication, oppositing, and reciprocating of nonzero numbers.

Answers for Part A

1. $\vec{d} = \vec{d}1$; $\vec{0} = \vec{d}0$; $\vec{d} - \vec{d}2 = \vec{d} \cdot -1$; $\vec{d}\sqrt{2} + \vec{d}3 = \vec{d}(\sqrt{2} + 3)$ [And, of course, 1, 0, -1, and $\sqrt{2} + 3$ are real numbers.]
2. 4_0(d)
3. Yes.; this follows from 4_6. If $\vec{a} = \vec{d}a$ and $\vec{b} = \vec{d}b$ then $\vec{a} + \vec{b} = \vec{d}a + \vec{d}b = \vec{d}(a + b)$. [The preceding is as much as you should expect from your students, especially in view of the fact that we have, as yet, given no rules for existential quantifiers. You may wish to point out that implicit use is made of the fact that the equality principle '$(\vec{a} = \vec{c}$ and $\vec{b} = \vec{d}) \Longrightarrow \vec{a} + \vec{b} = \vec{c} + \vec{d}$' is a valid sentence.]
4. $\vec{0} \in [\vec{d}]$ because there is a number x [namely 0] such that $\vec{0} = \vec{d}x$.

Answers for Part A [cont.]

5. Yes.; this follows from Theorem 5-2(a). If $\vec{a} = \vec{d}a$ then $-\vec{a} = -(\vec{d}a) = \vec{d} \cdot -a$.
6. Yes. Since, by 4_0(a)-(c) and $4_1 - 4_4$, $\mathcal{T}$ is a commutative group with respect to composition and $[\vec{d}]$ is a subset of $\mathcal{T}$ which contains $\vec{0}$ and is closed with respect to composition and inversing, $[\vec{d}]$ is also a group with respect to composition. For, since $4_1 - 4_4$ are universal statements about composition, inversing and $\vec{0}$ for all members of $\mathcal{T}$ the corresponding statements must also hold for $[\vec{d}]$.
7. All that remains is to show that if $\vec{a} \in [\vec{d}]$ and $b \in \mathcal{R}$ then $\vec{a}b \in [\vec{d}]$.
8. Yes. If $\vec{a} = \vec{d}a$ then $\vec{a}b = (\vec{d}a)b = \vec{d}(ab)$, by 4_8.
9. Each member of $[\vec{d}]$ other than $\vec{0}$ has the same direction as $\vec{d}$ does; each translation which does not belong to $[\vec{d}]$ has a different direction than $\vec{d}$ does.

Answers for Part B

1. If $\vec{a} = \vec{d}a$ then, also, $\vec{a} = \vec{d}a + \vec{e}0$. So, each member of $[\vec{d}]$ belongs to $[\vec{d}, \vec{e}]$. [Similarly, $[\vec{e}] \subseteq [\vec{d}, \vec{e}]$ and, so, $[\vec{d}] \cup [e] \subseteq [\vec{d}, \vec{e}]$. There are, however, many translations in $[\vec{d}, \vec{e}]$ which are neither in $[\vec{d}]$ nor in $[\vec{e}]$ — except in the cases in which $\vec{d} = \vec{0}$, $\vec{e} = \vec{0}$, or $[\vec{d}] = [\vec{e}]$.]
2. Yes. Since $\mathcal{T}$ is a vector space over $\mathcal{R}$ and $[\vec{d}, \vec{e}] \subseteq \mathcal{T}$, all that needs to be shown is that $\vec{0} \in [\vec{d}, \vec{e}]$ and that $[\vec{d}, \vec{e}]$ is closed with respect to addition, oppositing, and multiplication by members of $\mathcal{R}$. That $\vec{0} \in [\vec{d}, \vec{e}]$ follows from the fact that $\vec{0} \in [\vec{d}] \subseteq [\vec{d}, \vec{e}]$. [Alternatively, $\vec{0} = \vec{0} + \vec{0} = \vec{d}0 + \vec{e}0 \in [\vec{d}, \vec{e}]$.] The three closure properties are shown as follows:

$$(\vec{d}a_1 + \vec{e}a_2) + (\vec{d}b_1 + \vec{e}b_2) = (\vec{d}a_1 + \vec{d}b_1) + (\vec{e}a_2 + \vec{e}b_2) = \vec{d}(a_1 + b_1) + \vec{e}(a_2 + b_2)$$

$$-(\vec{d}a_1 + \vec{e}a_2) = -(\vec{d}a_1) + -(\vec{d}a_2) = \vec{d} \cdot -a_1 + \vec{d} \cdot -a_2$$

$$(\vec{d}a_1 + \vec{e}a_2)a = (\vec{d}a_1)a + (\vec{e}a_2)a = \vec{d}(a_1 a) + \vec{e}(a_2 a)$$

The purpose of this section is to point out, by discussing examples like those in the Introduction, that the notion of vector plays a role in subjects other than geometry. The ''other vector space'' of the title refers to the space [properly 'spaces'] of ''measure vectors'' for which we try to give students a feeling in the discussion which precedes the exercises on page 195. To avoid devoting too much time to the somewhat extraneous subject of nongeometric applications of vector spaces, a precise description of the spaces of measure vectors is relegated to the optional section 5.06. As far as the remainder of the course is concerned the present section 5.05 may also be considered as optional, and both sections might be omitted. It is even possible to omit section 5.05 and to take up section 5.06 merely for the purpose of showing students that there are vector spaces other than $\mathcal{T}$ and its subspaces.

It is worth stressing the point that, although there is a meaning of the word 'vector' according to which directed trips, velocities, and forces are correctly referred to as vectors, it is a different meaning from that which we have given to this word in section 5.04. According to the latter, an object may be spoken of as a vector in any context in

5. Is it the case that $-\vec{a} \in [\vec{d}]$ for every $\vec{a}$ in $[\vec{d}]$? That is, is $[\vec{d}]$ closed with respect to inversing? Justify your answer.

6. Is $[\vec{d}]$ a commutative group with respect to composition? Explain.

7. Your explanation in answer to Exercise 5 should have shown that, because $\mathcal{T}$ is a commutative group with respect to composition, any subset of $\mathcal{T}$ which contains $\vec{0}$ and is closed with respect to composition and inversing is, also, a commutative group with respect to composition. What more would you need know about such a subset in order to be sure that it is a vector space over $\mathcal{R}$?

8. Is $[\vec{d}]$ a vector space over $\mathcal{R}$ [with respect to the operations in $\mathcal{T}$]?

9. Intuitively, what do you think is true about all the translations – except, perhaps, $\vec{0}$ – which belong to $[\vec{d}]$? About any translation which does not belong to $[\vec{d}]$?

Part B

> Definition 5-2 $[\vec{d}, \vec{e}]$ is the set of all *real linear combinations* of $\vec{d}$ and $\vec{e}$. That is,
>
> $$[\vec{d}, \vec{e}] = \{\vec{x} : \exists_x \exists_y \, \vec{x} = \vec{d}x + \vec{e}y\}$$

1. Show that $[\vec{d}] \subseteq [\vec{d}, \vec{e}]$.

2. Is $[\vec{d}, \vec{e}]$ a vector space over $\mathcal{R}$?

5.05 Another Vector Space

Throughout the rest of the course the only vector spaces we shall be interested in are the vector space $\mathcal{T}$ and its *subspaces* like those you studied in Parts A and B of the preceding exercises. But, to give you some idea of other uses of the notions of vector we shall illustrate three of them in this section. Much of what follows depends on the intuitive notions you already have about how physical objects behave and, to some extent, on common geometrical notions which will be developed more completely later in this course. Most of these notions have already been used in pages 1-5 of the Introduction.

Most kinds of quantity which have, like translations, both sense and magnitude can be treated as vectors – that is, as members of a vector space. All that is necessary for this is that such quantities can be "added", and that they can be "multiplied" by real numbers, in such a way that postulates like 4_a - 4_s are satisfied.

which it is being considered as a member of a vector space. Since, as is pointed out in the text, there is no reasonable definition of 'addition' as a binary operation on the set of directed trips, such trips are unlikely to be considered, in any context, as members of a vector space. As to forces, forces acting at a given point may be thought of as members of a vector space and, so, be referred to as vectors in our sense of the word. However, even when forces act at different points they are still spoken of as vectors even though oppositely sensed forces of the same magnitude acting at different points of a body do not "cancel" one another.

In the second paragraph the word 'quantity' refers to "denominate [vector] measures", such as 20 miles at a heading of 10°. Such "quantities" of directed trips, for example, can always be added, although the trips themselves can be added only if they fit onto one another properly.

In line with the distinctions made in this course 'sensed trips' would be more appropriate than 'directed trips'. The latter, however, sound less queer than the former and so is perhaps preferable for this peripheral section.

Sample Quiz

1. We have postulates that tell us that T is a commutative group under function composition, and that $\vec{a} \cdot b \in T$. Give the four additional postulates we need to say that the group T of translations admits the real numbers as operators.

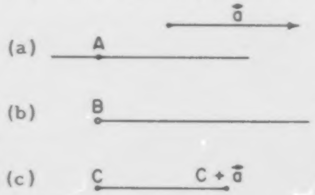
2. Suppose that $\vec{a} \neq \vec{0}$. Draw pictures of each of the following sets of points.
   (a) All points Q such that $Q = A + \vec{a}q$, for some q.
   (b) All points R such that $R = B + \vec{a}r$, for some $r > 0$.
   (c) All points S such that $S = C + \vec{a}s$ and $0 \leq s \leq 1$.

3. Given that $\vec{a} \neq \vec{0}$, determine the values of 'b' that satisfy the following.
   (a) $\vec{a} \cdot 3b + \vec{a} \cdot -2b + \vec{a}16 = \vec{0}$
   (b) $\vec{a} \cdot b^2 = \vec{a}(3b - 1) + \vec{a}(1 - b^2)$

Answers for Sample Quiz

1. $\vec{a} \cdot 1 = \vec{a}$; $\vec{a}(b + c) = \vec{a}b + \vec{a}c$; $(\vec{a} + \vec{b})c = \vec{a}c + \vec{b}c$; $(\vec{a}b)c = \vec{a} \cdot (bc)$

2. Here are typical pictures:



3. (a) -16 $[\vec{a} \cdot 3b + \vec{a} \cdot -2b + \vec{a}16 = \vec{0} \Longrightarrow \vec{a}(b + 16) = \vec{0}]$
   (b) $0, \frac{3}{2}$ $[\vec{a} \cdot b^2 = \vec{a}(3b - 1) + \vec{a}(1 - b^2) \Longrightarrow \vec{a}(2b^2 - 3b) = \vec{0}]$

Directed Trips

Suppose that Bill starts at a point R and takes a trip of 4 miles at a heading of 225° to a point S. This trip may be illustrated as in Fig. 5-2.

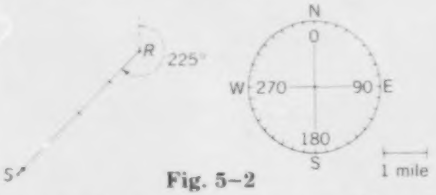


Fig. 5-2

Given that Bill now makes a return trip from S to R, this return trip is one of ___ miles at a heading of _____. [Complete this sentence.]

Make a drawing to show a trip of 3 miles northeasterly from a point A. [Be sure to show the direction of north and to indicate the scale of your drawing.]

Before saying more about trips like Bill's let's consider the simpler case of trips all of which are taken along a straight road which runs, say, east and west.

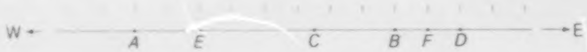


Fig. 5-3

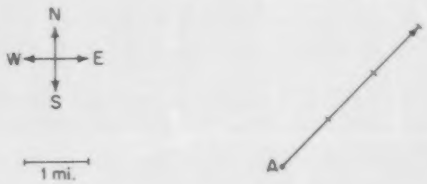
[For convenience, a sequence of milestones is indicated on the north side of the road. And we have labelled some points along the road. Note that, for example, the distance between A and E is 2 miles, and the distance between B and C is 2.5 miles.] In describing what kind of a trip it is that one makes in going from E, say, to C one might say that this is an easterly trip of 3.5 miles. What kind of trip is one from C to E?

As you know, another way of describing directed trips along a road is to use real numbers. If we take the unit of distance as understood, and agree that the eastward sense, say, is to be considered to be "positive", we can say that the measure of the directed trip from E to C is $^+3.5$ and that the measure of the trip from C to E is $^-3.5$. [Or, as we have been doing, instead of '$^+3.5$' we can say '3.5' and, instead of '$^-3.5$', we can say '-3.5'.]

Using real numbers to measure directed trips is an advantage when one must deal with two or more successive trips. For example, suppose that Bill starts from his home beside the road and makes two trips, one after the other. Suppose that the measure of the first trip is 3 and that of the second is -4.5. Without looking at the figure—which, anyway, doesn't show Bill's home—you should be able to compute the measure of the directed trip from Bill's home to the point where his second trip ends. Can you do this? What is the measure of the third trip Bill must make to reach home? What is the direction of this trip?

Fill the blanks as follows: 4; 45°

The students should have drawings something like this:



You might want to have them make this drawing as a seat-work exercise. In that case, you can walk around the room and give individual help where needed.

Answer to question: a westerly trip of 3.5 miles.

You may want to ask similar questions of the class in order to check their understanding of the situation.

If, starting at his home, Bill makes a trip of measure 3 followed by one of measure -4.5 then the measure of the directed trip from his home to his final position is -1.5. The measure of the trip from there to Bill's home is 1.5. Since we have chosen the eastward sense as positive, Bill must travel east to reach his home.

In preparation for the exercises of Part A it may be well to bring out the meaning of 'heading' by questions such as 'What is the heading of a trip from center of the compass disk [for Part A] through the second graduation clockwise from north? Through the fourth graduation counterclockwise from west?' [Answers: 30°, 210°]. Also, you may wish to remind students of the Pythagorean theorem. [Exercise 1 deals with a 3-4-5 triangle.]

Let's collect some ideas from the preceding discussion. First, directed trips along a road can be measured by real numbers. Second, sometimes – but not always – one trip can be thought of as being "added onto" another. For example, you can "add" the trip from *B* to *E* onto the trip from *C* to *B* and consider the "resultant" trip from *C* to *E*. But, it doesn't make much sense to add the trip from *B* to *D* onto the trip from *A* to *C*. Third, measures of directed trips can always be added and, when it makes sense to "add" two trips then the sum of their measures is the measure of the resultant of the two trips.

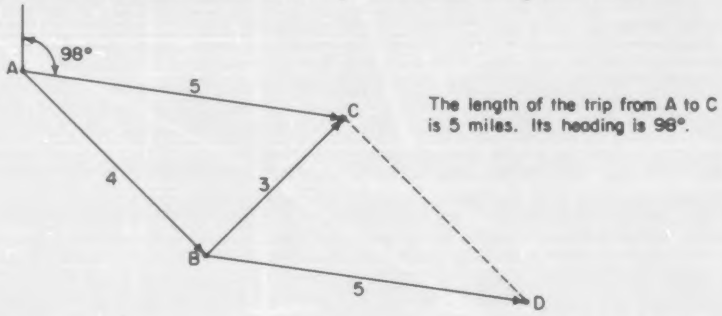
Now, let's see what all this has to do with vectors. In the first place, although a directed trip along a road can be represented by an arrow from the point where the trip begins to the point where it ends, different arrows of the same length and sense will represent different trips. Also, trips can be added only if they fit together properly. So, there seems to be no way of thinking of directed trips as members of a vector space. On the other hand, you have seen in the preceding section that the real numbers constitute a vector space. If we like, we can think of an arrow drawn on the picture of our road as representing – not a trip, but – the real number which is the measure of the trip from the place where the arrow begins to its point. For example, the arrow from *A* to *E* and the arrow from *B* to *D* would both represent the real number 2. Since real numbers form a vector space with respect to addition, you can "compute" sums of real numbers just as you do sums of translations. For example, to "compute" in this way 2 + –5, draw any arrow [along the road] which represents 2 and, next, draw the arrow which represents –5 from the point of the first arrow. Then, the arrow which begins where the first does and ends at the point of the second arrow represents 2 + –5.

Now, let's see what happens when we consider trips in other directions beside east and west. Again, any such trip can be represented by an arrow; but different arrows – even if they have the same length and the same sense – will represent different trips. And, it makes sense to "add" two trips only if the second starts where the first ends. In spite of this, trips which are represented by arrows which have the same length and the same sense are "similar". Just as real numbers can be used to measure directed trips along a road, there are "numbers" which can be used to measure directed trips in all directions. These numbers, which we shall call 'measure vectors', form a vector space and can be represented by arrows. In fact, the arrow which represents a directed trip also represents the measure vector of this trip. The difference is that the different arrows which represent different but "similar" trips all represent the same measure vector. When one directed trip can be "added onto" another, the measure vector of the resultant trip can be computed just as you compute the resultant of two translations.

In order to make it easier to check the results in the exercises for Part A [and other parts in this section, you might duplicate work sheets on which a circular compass has been printed. The ease with which such papers are checked more than makes up for the effort made to prepare such work sheets. This is also an excellent place to bring in a parallel ruler as a tool for making fast and accurate scale drawings.

Answers for Part A

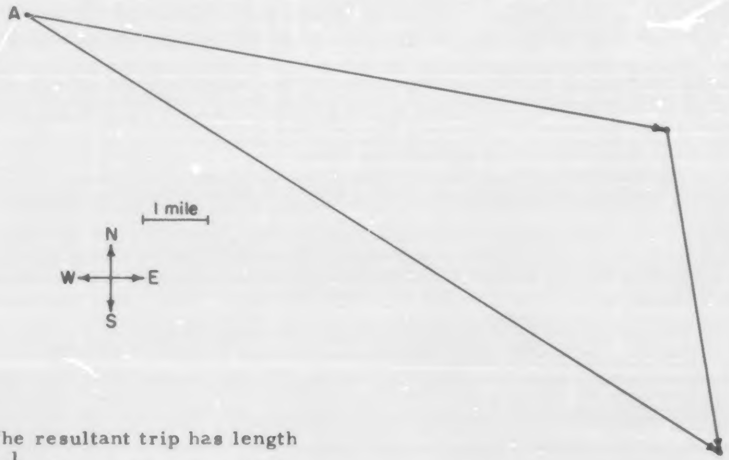
1, 2. Your students should have a picture something like this:



The length of the trip from A to C is 5 miles. Its heading is 98°.

3. Among the correct answers are these: A to B and C to D; B to A and D to C; C to A and D to B

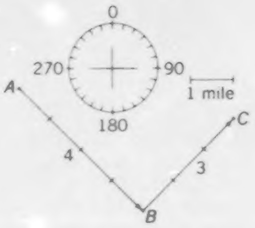
4. Your students should have drawings something like this:



The resultant trip has length $12\frac{1}{2}$ miles and heading 123°.

Exercises

Part A



Suppose that Jack starts at a point *A* and walks 4 miles at a heading of 135° to a point *B*. Then Jack walks 3 miles at a heading of 45° to a point *C*.

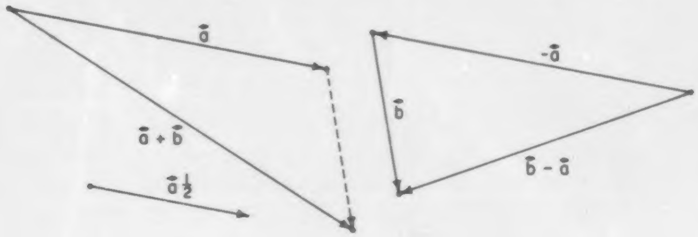
1. Copy the figure, using a scale of $\frac{1}{2}$ inch for 1 mile. Draw an arrow to represent the measure of the trip from *A* to *C*. What is the length of this trip? What is its heading?
2. On your drawing for Exercise 1, locate the point *D* such that the trips from *A* to *C* and from *B* to *D* have the same measure vector.
3. What other two directed trips between points of your figure have the same measure vector?
4. Draw another figure, showing compass directions and scale [$\frac{1}{2}$ inch to 1 mile], and mark a point *A*. Bill starts at a point *P* which is 300 miles from *A*. He makes a 10 mile trip from *P* at a heading of 100° and follows this with a trip of 5 miles at a heading of 170°. On your paper, draw arrows which represent the measure vectors of these trips and of the resultant trip. What are the [approximate] length and heading of the resultant trip?
5. On the same sheet of paper you used for Exercise 4, draw arrows to represent the measure vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ of Bill's two trips; but, this time use a scale of $\frac{1}{4}$ inch to 1 mile. Then, draw an arrow to represent [to the same scale] the measure vector $\overrightarrow{a} + \overrightarrow{b}$ of the resultant trip.
6. Continuing with Exercise 5, draw arrows to represent the measure vectors $-\overrightarrow{a}$, $\overrightarrow{a}\frac{1}{2}$, and $\overrightarrow{b} - \overrightarrow{a}$. What are the lengths and headings of trips which have these measures?

Part B

In applying mathematics to physical situations one must always make assumptions, and usually these assumptions are only approximately satisfied. [Recall, for example, the interpretation on page 27 of points as "absolutely precise locations".] In talking about directed trips we have assumed that, for instance, there are trips exactly 10 miles long, precisely at a heading of 30°. More generally, we have assumed that a directed trip has a single, precise, measure vector. This is, of course, not true of "real" trips. It is possible, however, to assign approximate measure vectors to trips and, by using the algebra of vector spaces as though these were the exact measures of the trips, reach sufficiently accurate conclusions as to, say, resultants of successive trips.

There is another assumption which we make when assigning measure vectors in the way we have done to trips on the surface of the earth.

5, 6. Your students should have drawings something like this:


$\overrightarrow{a}$
$\overrightarrow{a} + \overrightarrow{b}$
$\overrightarrow{a}\frac{1}{2}$
$-\overrightarrow{a}$
$\overrightarrow{b}$
$\overrightarrow{b} - \overrightarrow{a}$

The lengths and headings of trips which have these measures are:

$-\overrightarrow{a}$: 10 miles; 280°
$\overrightarrow{a}\frac{1}{2}$: 5 miles; 100°
$\overrightarrow{b} - \overrightarrow{a}$: $9\frac{1}{2}$ miles; 252°

1. Suppose that Bill starts at Dallas, Texas and makes three successive trips. The length of each trip is 500 miles. The first trip is north, the second is east, and the third is south. In what direction must Bill travel to return to Dallas? How far must he go?
2. When you answered Exercise 1, you may have thought that Bill would be 500 miles east of Dallas at the end of his third trip. If you did, then you should look at – or imagine – a globe. At the end of his third trip, is Bill more or less than 500 miles from Dallas?
3. Suppose that Bill travels 300 miles west, then 400 miles north, and, finally, 500 miles in the direction of his starting point. Will Bill be back where he started at the end of his third trip?
4. **(a)** What assumption are we making about the earth when we assign measure vectors to directed trips on the earth's surface?
   (b) Is this assumption correct?
   (c) If your answer to (b) is 'No.', is it still possible to use this assumption to make useful predictions?

Part C

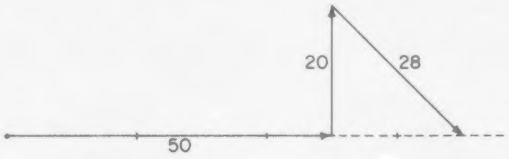
1. A ship leaves Charleston, South Carolina and sails due east. After sailing 50 miles east, the ship sails due north for 20 miles to evade a local storm. Then, the ship sails southeast until it reaches the line of its original easterly course.
   (a) Make a scale diagram showing the path of the ship to the point where it gets back on course. Use a scale of 1 inch to represent 20 miles.
   (b) Estimate the distance the ship traveled from the beginning of its trip to the point where it gets back on course.
   (c) How far from Charleston is the ship when it gets back on course?
2. A ship starts at Boston, Massachusetts, sails 80 miles due east, and then sails due south. The pilot falls asleep during the trip and loses track of how far south the ship sailed. By means of radio contacts, he is able to determine that the ship is exactly 100 miles from Boston.
   (a) Make a scale diagram [$\frac{1}{2}$ inch for 20 miles] representing the path of the ship.
   (b) How far to the south had the ship sailed when it was 100 miles from Boston?
   (c) Suppose that the ship now turns and sails toward a point D which is 160 miles due east from Boston. How many miles must the ship travel to get to D?
   (d) How many miles has the ship sailed from the time it left Boston until its arrival at D?
3. A ship leaves a point A and sails west for 100 miles to a point B. From B the ship makes a trip of measure $\vec{a}$ directly to a point C.
   (a) If the resultant trip from A to C is a trip of 0 miles, draw an arrow to represent the measure vector $\vec{a}$. [Is there more than one possibility?]

Answers for Part B

1. Bill must travel due west about 560 miles in order to return to Dallas. [Notice that he is not 500 miles from Dallas. Given that Bill makes the three trips as specified, he will go north to a point near Marysville, Kansas, then east to a point near Terre Haute, Indiana, and then south to a point near Montgomery, Alabama.]
   Students who have trouble seeing this are most easily convinced by having them plot the successive trips on a large globe. Also, by using a globe it is easier to see why the resultant trip is not as one would expect in a vector space.
2. More.
3. No.; he must go a little further to reach his starting point.
4. (a) We assume that the surface of the earth is flat.
   (b) No.
   (c) Yes. For many purposes, and for not too large distances, the predictions will be useful. And, with a little practice, one can make fairly good approximations to the errors inherent in this assumption. On the other hand, for an accurate survey of a moderately large region, one must use different assumptions.

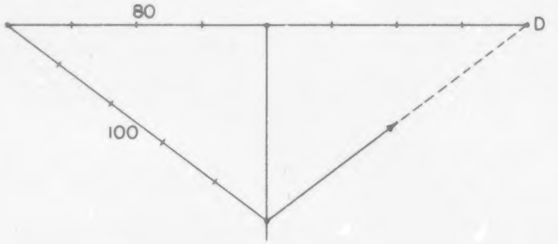
Answers for Part C

1. (a) Your students should have pictures something like this:



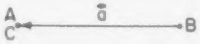
   (b) About 98 miles
   (c) 70 mile.

2. (a)



   (b) 60 miles [See drawing in (a).]
   (c) 240 miles

3. (a) [No.]



(b) If the magnitude of $\vec{a}$ is 60 miles and the resultant trip from A to C is 40 miles long, draw an arrow to represent the measure vector $\vec{a}$. [Is there more than one possibility?]

(c) If the magnitude of $\vec{a}$ is 60 miles and the trip from A to C is 120 miles long, draw an arrow to represent the measure vector $\vec{a}$. [Is there more than one possibility?]

(d) If the magnitude of $\vec{a}$ is 60 miles and the resultant trip from A to C is 40 miles long, draw an arrow to represent the measure vector $\vec{a}$. [Is there more than one possibility?]

(e) Suppose that the magnitude of $\vec{a}$ is 60 miles and that the resultant trip from A to C is d miles long. Explain, by referring to a diagram,

  (i) why it is impossible for d to be less than 40,
  (ii) why it is impossible for d to be greater than 160,
  (iii) why it is possible for d to be any number between, and including, 40 and 160,
  (iv) why $\vec{a}$ may be any one of two possible measure vectors if $40 < d < 160$.

Velocity

Read again the discussion of a problem about velocity on pages 3–5. Before considering the solution to this problem, let us consider some simpler problems. But first we should make precise the distinction between *speed* and *velocity*. The speed of an object only refers to how fast the object is moving. [For example, we may say that the speed of an automobile is 50 miles per hour.] The velocity of an object refers not only to how fast the object is moving [that is, its speed], but also to the sense of its path of movement. [For example, we may say that the velocity of an automobile is 50 miles per hour *to the northwest.*]

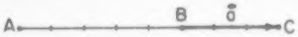
We may represent the velocity of an object by drawing an arrow whose length represents [with reference to a suitable scale] the speed, and whose sense represents the sense of the path of movement. For example, using a scale of $\frac{1}{2}$ inch to represent 1 mile per hour, the velocity of a boat sailing 5 miles per hour to the east is represented by the arrow:

Fig. 5–4

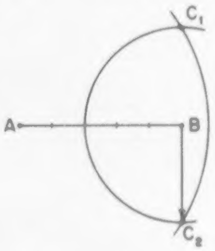
Exercises

Part A

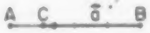
1. Did you ever try walking up or down a moving escalator? If you walk up on an "up" escalator, the speed with which you climb will be (greater than/less than) ______________ your corresponding

(b)


[No.]

(c)


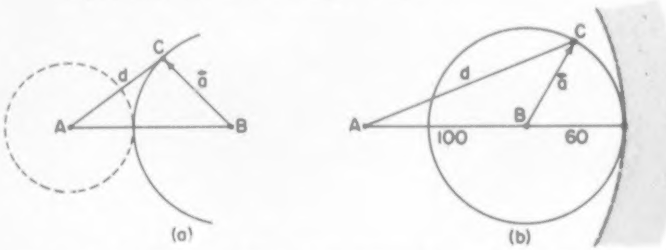
[Yes.]

(d)


[No.]

(e) All of these explanations are based on intuitive notions about circles and triangles.

  (i) All of the points which are less than 40 miles from A are "outside" of the circle with radius 60 [miles] and center B. [See figure (a), below.] So, no point on this circle can be less than 40 miles from A. Since C is some point of this circle and d is the distance from A to C, it is impossible for d to be less than 40.



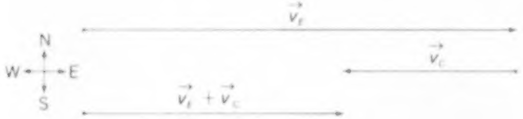
  (ii) All of the points which are more than 160 miles from A are "outside" the circle with radius 160 [miles] and center A. [See figure (b), above.] This circle intersects the circle with center B and radius 60 [miles] in the point on the ray $\overleftrightarrow{AB}$ which is 160 miles from A. So, C is "inside" the circle with center A and radius 160 [miles], which means that it is impossible for d [the distance from A to C] to be greater than 160.

  (iii) As C "moves" along the circle with center B and radius 60 [miles] from the point which is 160 miles from A to the point which is 40 miles from A, d "shrinks" in a systematic, or continuous, fashion from 160 to 40. Intuitively, any value between these extremes can be obtained by a suitable location for the "moving point" C.

  (iv) This follows from the fact that the circle with center B and radius 60 [miles] is symmetric about the line $\overleftrightarrow{AB}$.

speed on ordinary stairs. If you walk down an "up" escalator, the speed with which you climb will be (increased/decreased) ____________.

2. An airplane can fly 250 miles per hour if there is no wind. If the plane is flying into the wind, its speed will be (increased/decreased) ____________. If it is flying with the wind, its speed will be (increased/decreased) ____________. If the pilot keeps the plane pointed north and a wind is blowing in a westerly direction the plane (will/will not) ____________ travel due north.
3. Suppose that when a ship's engines are operating the ship can move with a speed of 10 miles per hour in still water.
   (a) If the ship is traveling against a current, its speed will be (increased/decreased) ____________. If the ship is headed east with its engines operating, against a current which is moving west with a speed of 4 miles per hour, the ship's resultant velocity will be ____ miles per hour to the east. The resultant velocity can be found graphically by making the following drawing:



Scale: $\frac{1}{2}$ inch = 1 mile per hour

$\vec{v}_E$ = velocity of the ship due to the engines

$\vec{v}_C$ = velocity of the ship due to the current

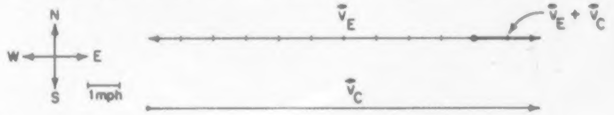
$\vec{v}_E + \vec{v}_C$ = resultant velocity of the ship

   (b) If the ship is headed west, with its engines operating, against a 12 mile per hour easterly current, the ship's resultant velocity will be ____ miles per hour to the ________. Make a scale drawing to show how you can find the resultant velocity graphically.
   (c) If the ship is headed north with its engines operating and a current is moving in the same direction with a speed of 5 miles per hour, the ship's resultant velocity will be ____ miles per hour to the ________. Make a scale drawing to show how you can find the resultant velocity graphically.
   (d) Suppose that the ship is headed north with its engines operating and across a current moving east at a speed of 5 miles per hour. How far north will the ship be after one hour? How far east will the ship be after one hour? How far from the starting point will the ship be after one hour? Make a scale drawing to show how you can find [approximately] the path of the ship and the distance it travels in one hour. [Make your scale drawing using a scale of $\frac{1}{2}$ inch to represent 1 mile.]

Answers for Part A

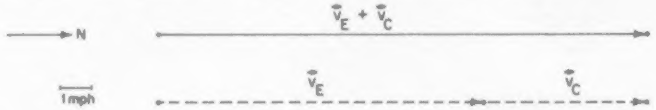
1. greater than; decreased
2. decreased; increased; will not
3. (a) decreased; 6
3. (b) 2; east.

The students should have drawings something like this:



(c) 15; north.

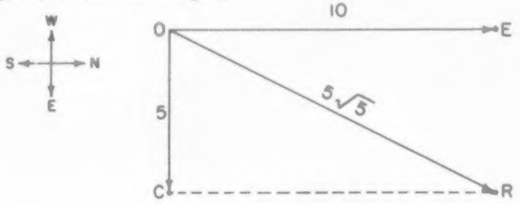
The students should have drawings something like this:



(d) 10 miles; 5 miles; the ship's path is one at a heading of [about] 27° and it travels about 11.2 miles in one hour.

[Note that, by the Pythagorean theorem, the length of $\overline{OC}$ (see the picture below) is $5\sqrt{5}$.]

The students should have drawings something like this [only twice as large]:

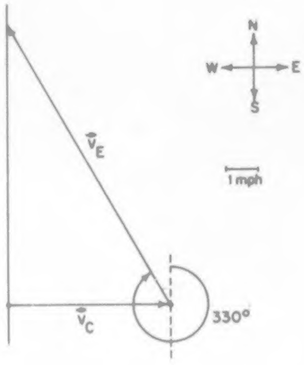


TC 200 (1)

3. (e) The picture drawn for part (d) is appropriate for this exercise. Label the arrows from O to E, C, and R, respectively, '$\vec{v}_E$', '$\vec{v}_C$', and '$\vec{v}_E + \vec{v}_C$'.

(f) The students should have drawings something like this one on the right.

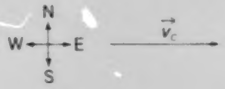
The captain must steer his ship at a heading of [about] 330° [or, 30° west of north] in order that the ship's resultant velocity is one which is due north.



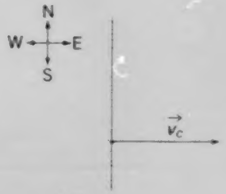
(e) Consider the same situation described in Exercise 3(c) above. Using a scale of ½ inch to represent 1 mile per hour, make a drawing to show:

$\vec{v}_E$ = velocity of the ship due to the engines
$\vec{v}_C$ = velocity of the ship due to the current
$\vec{v}_E + \vec{v}_C$ = resultant velocity of the ship

(f) Suppose that the captain wants to sail his ship due north, but there is a current of 5 miles per hour to the east. Make a drawing to find [approximately] at what heading the captain must steer his ship in order that the resultant velocity of the ship should be due north.
Hint: Let us use a scale of ¼ inch to represent 1 mile per hour. $\vec{v}_C$, the velocity of the ship due to the current, is represented by the arrow at the right:



$\vec{v}_E$, the velocity of the ship due to the engines, must be represented by an arrow 2½ inches long—our job is to determine the sense of that arrow. The resultant velocity $\vec{v}_E + \vec{v}_C$ must be represented by an arrow pointing north. So, we start with $\vec{v}_C$ and a line in the north-south direction:



Now set a pair of compasses so that you can draw an arc of radius 2½ inches. [Make a drawing on your paper and solve the problem.]

4. Make a drawing that will help you solve the problem posed on page 4.

Part B

Let V be the set of all velocities that a given ship can attain. Some elements of V are:

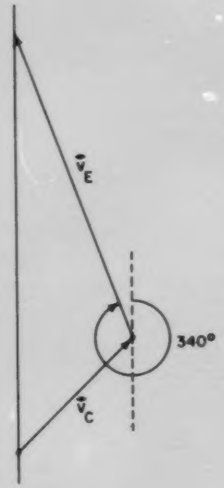
$\vec{v}_1$ = 7 miles per hour west
$\vec{v}_2$ = 4.5 miles per hour northeast

The elements of V can be represented by arrows [with reference to a chosen scale].

1. Choose a suitable scale, and draw arrows to represent the velocities $\vec{v}_1$ and $\vec{v}_2$.

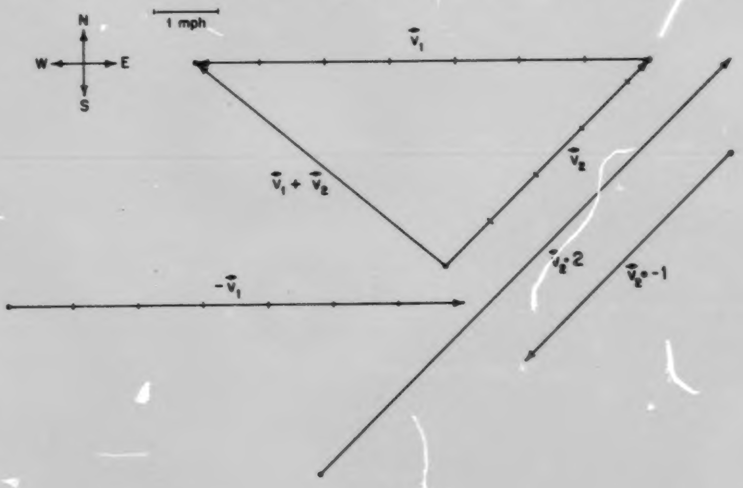
4. In order to solve the problem posed on page 4, we make use of the diagram at the right.

The captain must steer his ship at a heading of [about] 340° [or, 20° west of north] in order that the ship's resultant velocity is one which is due north.



Answers for Part B

1, 2, 3. The students should have drawings something like this.



2. Find $\vec{v}_1 + \vec{v}_2$ [the resultant of $\vec{v}_1$ and $\vec{v}_2$].
3. Draw an arrow to represent each of the following velocities [using a natural interpretation of the symbols].
   (a) $-\vec{v}_1$ **(b)** $\vec{v}_2 2$ **(c)** $\vec{v}_2 \cdot -1$

Part C

1. A boy is swimming east across a lake. In still water, he swims with a speed of 4 mph. The velocity of the current in the lake is 2 mph to the north.
   (a) What is the boy's resultant velocity under the conditions described above?
   (b) Suppose that the lake is 8 miles wide, [east-west], 16 miles long, and generally rectangular in shape. How long will it take him to swim across the lake? How far north of his starting point will he be when he reaches the other side of the lake?
   (c) Suppose that he wants to reach the other side of the lake at a point due east of his starting point. In which direction should he begin to swim in order to do this?
2. With no wind blowing, a certain airplane can fly at a speed of 250 mph. Determine the velocity of this plane if it is flying west and the wind acts on the plane with a velocity of 50 mph to the south.
3. A plane leaves New York and is to be 400 miles due west in one hour. However, there is a wind blowing out of the northwest that is strong enough to move the plane 80 mph. Make a diagram [¼ inch represents 20 mph] and estimate the speed and heading at which the plane must fly to arrive at its destination on time.

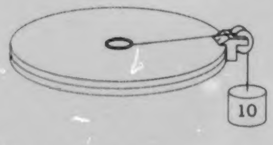
Force

Read again the description of a force table in the Introduction, pages 1 and 2. Before considering the experiments described there, we shall consider some simpler ones.

Exercises

Part A

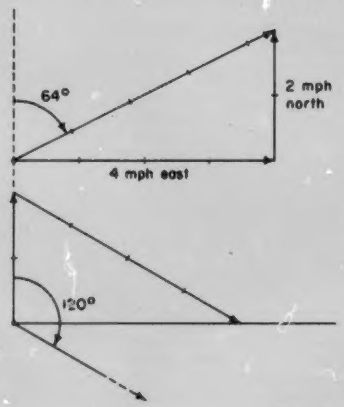
1. Consider an experiment in which just one string is tied to the ring. Suppose that a 10-gram weight is attached to the end of the string, and the pin is pulled out of the table.



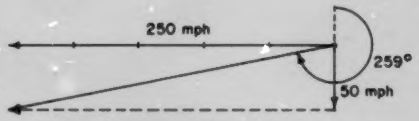
Will the ring move? If so, draw an arrow to describe its path of movement.

Answers for Part C

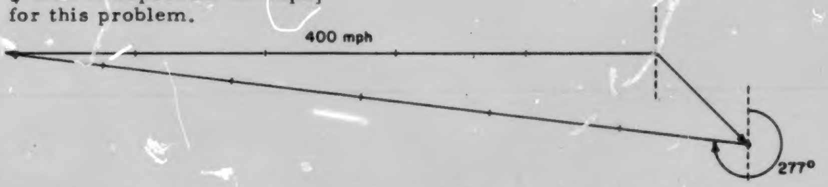
1. (a) 4.5 mph at a heading of 64°
   (b) about 2 hours; about 4 miles
   (c) at a heading of [about] 120°



2. About 255 miles per hour at a heading of [about] 259°.

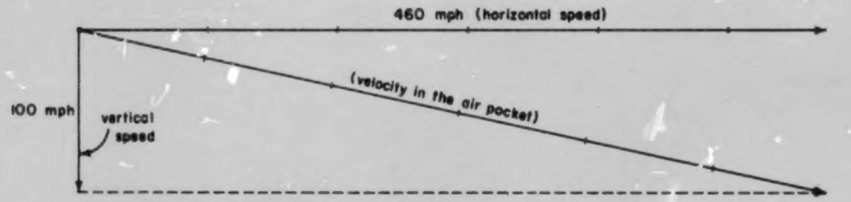


3. Here is a scale diagram [using ¼ inch to represent 20 mph] for this problem.



The plane must fly about 460 miles per hour at a heading of 277°.

4. About 470 mph.

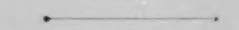


Answers for Part A

1. Yes. The ring will be pulled toward the pulley.

*

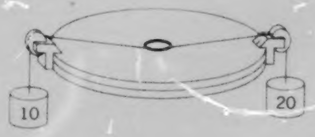
The measure of the force on the ring is a vector and can be represented by an arrow. Using a scale of $\frac{1}{4}$ inch to a 5-gram force, and representing the sense from the center of the table to the pulley by left-to-right, the measure vector of the force on the ring is represented by:



or by any other horizontal, right-pointing, arrow 1 inch long.

*

2. Suppose that, as shown in the figure at the right, two strings are attached to the ring. A 10-gram weight is attached to the string over the left-hand pulley, and a 20-gram weight is attached to the other string.

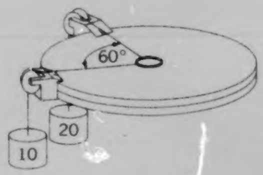


(a) If the pin which holds the ring at the center of the table is removed, will the ring move? If so, describe the motion.

(b) Draw a *force diagram* consisting of two arrows from a common origin, which represents, to scale, the measure vectors of the two forces acting on the ring.

(c) What single *resultant* force acting on the ring is equivalent to the two given forces?

3. Suppose that two strings are tied to the ring as shown in the figure at the right. A 10-gram weight is attached to one string, a 20-gram weight is attached to the second string, and the pin is removed.



(a) Will the ring move? Draw a plan of the force table as seen from above and, on it, draw an arrow pointing the way you think the ring will begin to move.

(b) Draw a force diagram representing the measure vectors of the two forces which act on the ring when it is at the center of the table.

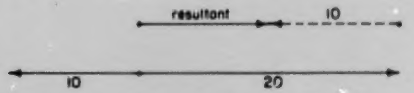
(c) Make a guess as to how to find the resultant of these two forces, and use your guess to construct an arrow which represents the measure vector of this resultant force. What is the approximate magnitude of the resultant?

(d) Supposing that your guess in part (c) is correct, can you use it to check your answer for part (a)?

Answers for Part A [cont.]

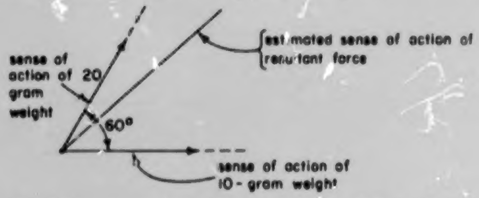
2. (a) Yes. The ring will move toward the pulley of the 20-gram weight.

(b)



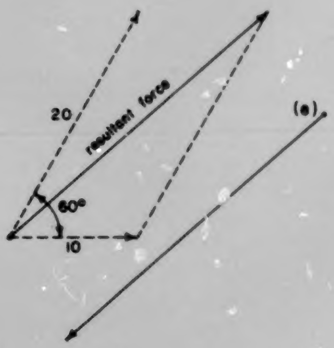
(c) 10 grams toward the pulley of the 20-gram weight.

3. (a) Yes.


sense of action of 20 gram weight
60°
estimated sense of action of resultant force
sense of action of 10-gram weight

The students should have enough intuition about this system to feel that the 20-gram weight will "contribute" more to the resultant than the 10-gram weight, so that the resultant force will "lean" toward the sense of action of the heavier weight.

(b), (c), (e)



"Add" the forces graphically as we pictured the addition of translations.

The resultant force is one of about 25.3 grams and making an angle of about 36° with the sense of action of the 10-gram weight.

(d) Yes. Simply compare the sketch made in (a) with the computed resultant force in (c).

(e) On your force diagram, draw an arrow which represents the measure vector of the third force which must be applied to the ring to keep it at the center of the table.

*

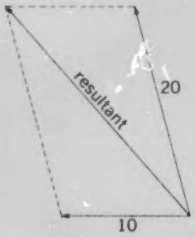


Fig. 5–5

The assumption that the quantity [or: "amount"] of a force can [like the quantity of a velocity or of a directed trip] be described by a measure vector amounts to assuming that the quantity of the resultant of two forces can be computed by the same graphical procedure we used to compute the resultant of two translations. This assumption can be justified experimentally. [Your answer for part (e) of Exercise 3 may suggest force table experiments which would help to justify this assumption.]

*

* 4. (a) Consider the same set-up as in Exercise 3. Notice that when the ring has moved away from the center of the table the angle between the 10-gram and 20-gram forces will have changed. Guess where the ring will be when it has moved two-thirds of the way to the edge of the table. Mark this point on your picture for Exercise 3(a). Then, draw a force diagram and compute the measure vector of the resultant force on the ring.

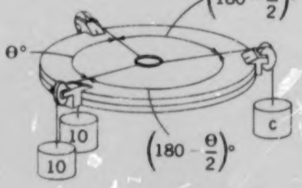
(b) Do you think that the ring will move in a straight line?

Part B

1. (a) Use a force diagram to find the answers to the questions on page 2.

(b) What should be the weight of *C* on page 2 [*A* and *B* are 20-gram weights] to keep the ring from moving?

2. Suppose that three strings are tied to the force table ring, as shown in the figure at the right. Complete the table so that, for corresponding values of θ and c, the ring will stay at the center of the table.


$\left(180-\frac{\theta}{2}\right)^\circ$
θ°
10
10
c
$\left(180-\frac{\theta}{2}\right)^\circ$

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
θ [degrees]		10	60		150	175	
c [grams]	20			10			0

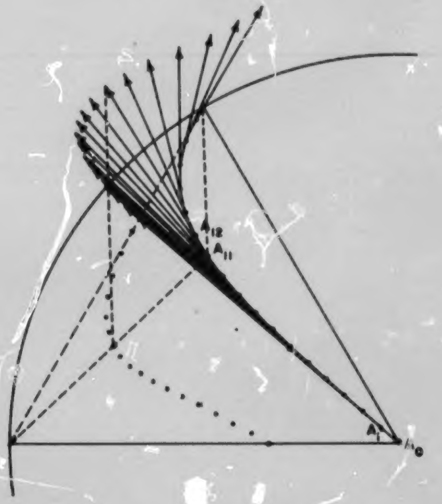
Answers for Part A [cont.]

3. (e) [See drawing made in (c).] The required force is the opposite of the resultant of the given forces.

4. [The figure below shows an approximation to the path of the ring. The arrows indicate the forces operating at various points of the path. The procedure for drawing such an approximation may interest your students. One begins with the ring at A_0 and assumes that the force is the same for a short portion of the path — to A_1, say. At this point one recomputes the force, graphically and proceeds as before. For example, a sequence of eleven such steps brings us to A_{11}. The dashed lines show the directions of the two forces. The construction dot in the longer dashed line indicates the position of the arrow-head for the 10-gram force. An arrow twice as long from this point and parallel to the shorter dashed line ends at the head of the arrow for the resultant force at A_{11}. A_{12} is then chosen a short distance out along the arrow representing this resultant, and the process is repeated. Better results can be obtained by choosing shorter distances between successive positions.

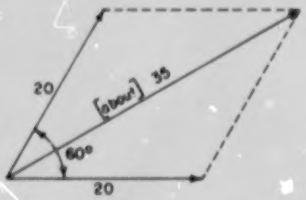
It is interesting to note that the curvature of the path is very slight for nearly half its length, but increases rapidly from there on. The path is tangent to the indicated chord when the ring reaches the edge of the table.

Note that the assumptions made will be more nearly satisfied if the strings attached to the ring run through small, smooth-edged, holes at the edge of the table rather than over pulleys.]



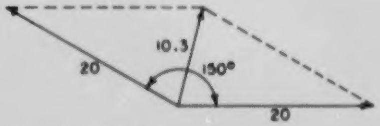
Answers for Part B

1. (a) Here is a force diagram picturing the situation described on page 2.


20
[about] 35
60°
20

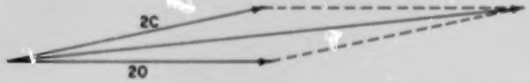
40 grams at C is too much. About 35 grams is needed at C to keep the ring at the center of the table.

(b) Here is a force diagram picturing the situation described on page 2.


20
10.3
150°
20

The weight at C should be about 10.3 grams.

(c) Here is a force diagram picturing the situation described on page 2.

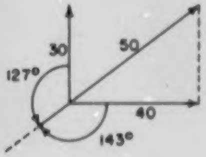


About 39.8 grams is needed at C to keep the ring at the center of the table. This is more than is needed at C in the situation described in Position 1 on page 1.

2. (a) 0 (b) 19.9 (c) 17.5 (d) 120 (e) 5.2 (f) 1.7 (g) 180

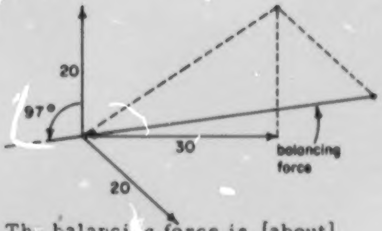
Answers for Part B

3. (a)


30
50
127°
40
143°

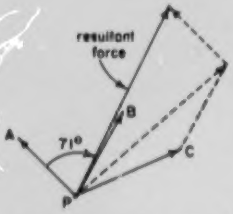
The balancing force is 50 at an angle of 143° clockwise from the sense of the force of measure 40.

(b)


20
97°
30
20
balancing force

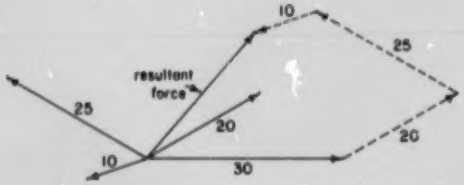
The balancing force is [about] 45 at an angle of 97° counterclockwise from the vertical force of measure 20.

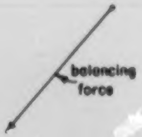
4. Here is a force diagram picturing the situation.


resultant force
B
A
71°
C
P

The resultant force is [about] 130 pounds at an angle of [about] 71° clockwise from the sense of A's force.

5. (a) Here is a force diagram picturing this situation.

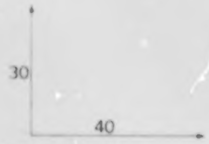

10
25
resultant force
25
20
20
10
30


balancing force

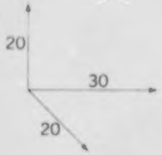
The resultant force is about 24 pounds at an angle of [about] 50° counterclockwise from the 30 pound force.

3. Find the "balancing force" when the forces applied to the ring are as shown in these force diagrams:

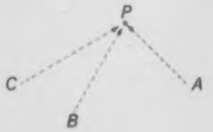
(a)



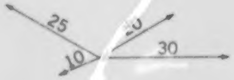
(b)



4. Three men [A, B, and C] are pushing on a piano [P] in the directions indicated in the diagram at the right. A is pushing with a force of 50 pounds, B is pushing with a force of 60 pounds, and C is pushing with a force of 70 pounds. What is the resultant force exerted on the piano by the three men?



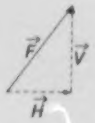
5. Here is a scale diagram of four forces acting at a point. [$\frac{1}{2}$ inch represents 10 pounds.]

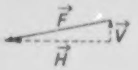


   (a) Draw an arrow to represent the resultant force of this system and use it to estimate this resultant force.
   (b) Draw an arrow to represent the force needed to put the given system in equilibrium.

• Part C

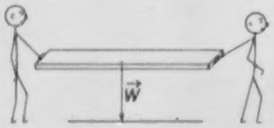
It is often convenient to think of a force $\vec{F}$ as being the resultant of two forces $\vec{H}$ and $\vec{V}$ in the horizontal and vertical directions, respectively. $\vec{H}$ is the *horizontal component* of $\vec{F}$ and $\vec{V}$ is the *vertical*



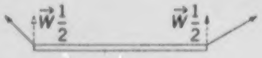


component of $\vec{F}$. If the direction of $\vec{F}$ is horizontal then the vertical component of $\vec{F}$ is $\vec{0}$. The weight of an object is one force which acts on the object. What is the horizontal component of such a force?

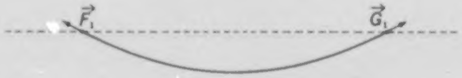
1. (a) Draw a force diagram to show [the measure vector of] a force whose horizontal component is a force of 4 pounds and whose vertical component is a force of 3 pounds.
   (b) What is the magnitude of a force such as that described in part (a)?
   (c) Is there enough information given in part (a) to describe completely the measure vector of the force? Explain.
2. Consider a board which has a handle at each end for convenience



in carrying it. Two boys pick up the board so that it is level, and start carrying it. Since the board is level, each boy supports half of its weight. You can judge, by the directions of the boys' arms,

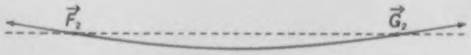


the directions in which they are pulling on the board. Which boy is pulling harder? How do you know? Why should he pull harder?
3. Here is a picture of a rope which is being held so that both ends are at the same height above the ground. The rope is supported by the



forces $\vec{F}_1$ and $\vec{G}_1$ at its ends. As in the case of the board, the vertical component of each of these two forces is half the weight of the rope. Since the rope is merely being held in position by $\vec{F}_1$ and $\vec{G}_1$, the horizontal components of $\vec{F}_1$ and $\vec{G}_1$ must have the same magnitude.

Here is a picture of the *same* rope after it has been pulled tighter.



   (a) What is the magnitude of the vertical component of $\vec{F}_2$? of $\vec{G}_2$?
   (b) If the rope were to be pulled so tight as to be straight, what would be the vertical components of the forces on the ends of the rope?
   (c) Can a rope be pulled straight?

5.06 Measure Vectors

The existence of a kind of number which can be used to measure such things as directed trips, velocities, and forces was mentioned in the preceding section. For completeness we include here a discussion of these measure vectors.

To come directly to the point, there are different spaces of measure vectors, one for each dimension. The space of 1-dimensional measure vectors is just the vector space of real numbers. [Recall that $\mathscr{R}$ is a vector space over $\mathscr{R}$]. As you know, these 1-dimensional measure vectors can be used as measures of directed trips along a line, as well as velocities or forces in a given direction. To do so, it is necessary, first, to choose one of the two senses along the line, or in the given direction, as positive, and to choose a unit of length [foot, mile], or of speed [foot per second, mile per hour], or of magnitude of force [pound, gram]. After you have made these choices, each directed trip on the line, or each velocity or force in the given direction, has a unique real number as its measure. [With different choices, the same trips, velocities, or forces would have different measures, and it is easy to see how changing one's choices affects the assignment of measures.]

To measure directed trips, velocities, or forces in a given plane, one needs 2-dimensional measure vectors. These are ordered pairs of real numbers with addition of ordered pairs and multiplication of ordered pairs by real numbers defined by:

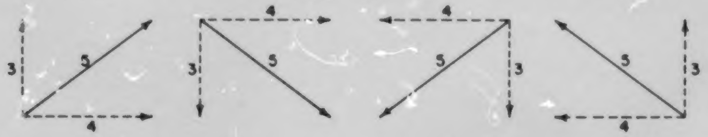
$$(a,b) + (c,d) = (a + c, b + d), \quad (a,b) \cdot c = (ac,bc)$$

It is easy to check that, with these definitions of addition and multiplication, the set of all ordered pairs of real numbers is a vector space over $\mathscr{R}$. What, for example, is the 2-dimensional measure vector $\vec{0}$? What is the opposite of (a,b)? [Note that in the definition of addition for ordered pairs, the first of the three '+'s refers to the operation which is being defined and the other two refer to ordinary addition of real numbers.]

In order to use 2-dimensional measure vectors as measures of directed trips on a given plane we must first make some choices, just as we do to use real numbers to measure trips on a line. As before, we must choose a unit of length. In addition to this we must make a choice similar to that in the 1-dimensional case of choosing a positive sense. In the 2-dimensional case we choose, *in order*, two positive senses which are in directions at right angles to one another. In a drawing, we can indicate this choice by picturing two rays and labeling them '(1)' and '(2)'. [In Fig. 5-6, we have also indicated a chosen unit

Answers for Part C

1. (a) Since the senses of the components in the horizontal and vertical directions are not specified there are four possible answers:



(b) 5 pounds

(c) No. [See remark on part (a).]

2. The boy on the right is pulling harder, as is shown by the fact that his arm is more nearly horizontal. He needs to pull harder at the beginning to get the board in motion. Once this is achieved, it is sufficient that the horizontal component of his force balance that of his partner.

3. (a) The magnitude of the vertical components of $\vec{F}_2$ and $\vec{G}_2$ is half the weight of the rope — the same as that of $\vec{F}_1$ and $\vec{G}_1$.

(b) Since the forces at the ends of a rope must be in the direction of the rope, if the rope were horizontal then the magnitudes of the vertical components would be zero.

(c) No. The forces at the ends must have nonzero vertical components and, so, can never be horizontal.

Answers to questions.

The 2-dimensional measure vector $\vec{0}$ is (0, 0). The opposite of (a, b) is (-a, -b).

of length.] We can now assign a 2-dimensional measure vector (a,b)

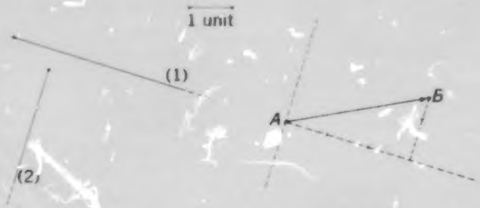


Fig. 5–6

to a directed trip. Without our going into the details of how this is done, you should be able to guess when we say that the measure of the pictured trip from A to B is $(3, -1.5)$. [Other choices of the unit of length and of the ordered pair of positive senses would result in different measures being assigned to the same trip. For example, if we had made the same choices of positive senses, but had made them in the reverse order, then the measure of the trip from A to B would have been $(-1.5, 3)$. If, instead, our second choice of a positive sense had been that of the ray opposite to the one we did choose then the measure of the trip from A to B would have been $(3, 1.5)$.]

You should have no difficulty now in seeing how 2-dimensional measure vectors are used as measures of velocities and of forces in a plane. And you should be able to guess correctly what the space of 3-dimensional measure vectors is and how it is used in measuring directed trips, velocities, or forces in space.

Exercises

1. Check that the set of all ordered pairs of real numbers is a vector space over $\mathscr{R}$ when addition and multiplication are defined as in the text.
2. The members of the space of 3-dimensional measure vectors are the ordered triples of real numbers. Define addition of such ordered triples and multiplication of ordered triples by real numbers in such a way as to obtain a vector space over $\mathscr{R}$.
3. What choices must be made before 3-dimensional measure vectors can be assigned as measures to directed trips?

Answers for Exercises.

1. 4_1: $$\begin{aligned}[(a, b) + (c, d)] + (e, f) &= (a + c,\ b + d) + (e, f)\\ &= ([a + c] + e,\ [b + d] + f)\\ &= (a + [c + e],\ b + [d + f])\\ &= (a, b) + (c + e,\ d + f)\\ &= (a, b) + [(c, d) + (e, f)]\end{aligned}$$

   4_2: $(a, b) + (0, 0) = (a + 0,\ b + 0) = (a, b)$

   4_3: $(a, b) + -(a, b) = (a, b) + (-a, -b) = (a + -a,\ b + -b) = (0, 0)$

   4_4: $(a, b) + (c, d) = (a + c,\ b + d) = (c + a,\ d + b) = (c, d) + (a, b)$

   4_5: $(a, b)\cdot 1 = (a1, b1) = (a, b)$

   4_6: $$\begin{aligned}(a, b)(c + d) &= (a(c + d),\ b(c + d))\\ &= (ac + ad,\ bc + bd)\\ &= (ac, bc) + (ad, bd) = (a, b)c + (a, b)d\end{aligned}$$

   4_7: $$\begin{aligned}[(a, b) + (c, d)]e &= (a + c,\ b + d)e\\ &= ((a + c)e,\ (b + d)e)\\ &= (ae + ce,\ be + de)\\ &= (ae, be) + (ce, de) = (a, b)e + (c, d)e\end{aligned}$$

   4_8: $$\begin{aligned}[(a, b)c]d = (ac, bc)d &= ([ac]d,\ [bc]d)\\ &= (a(cd),\ b(cd)) = (a, b)(cd)\end{aligned}$$

2. $(a, b, c) + (d, e, f) = (a + d,\ b + e,\ c + f)$

   $(a, b, c)d = (ad, bd, cd)$

   [Check of vector space postulates is similar to that in Exercise 1.]

3. Choose unit of length and, in succession, three positive senses so that each two are perpendicular.

5.07 Chapter Summary

Vocabulary Summary

operator [on a group]
vector space
vector
subspace
measure vector
speed
velocity
force
force diagram
resultant
equilibrium
component

Postulates

1. (a) $B - A \in \mathscr{T}$ (b) $A + \vec{a} \in \mathscr{E}$
2. (a) $A + (B - A) = B$ (b) $\vec{a} = (A + \vec{a}) - A$
3. $(B - A) + (C - B) = C - A$
4″. $\mathscr{T}$, under function composition, is a vector space over $\mathscr{R}$. [See page 191.]
5′. $\mathscr{R}$ is an ordered field. [See page 170.]

Definitions

[See page 141 for Definition 3–1 and Definition 3–2.]
5–1. $[\vec{d}] = \{\vec{x}: \exists_x\, \vec{x} = \vec{d}x\}$
5–2. $[\vec{d}, \vec{e}] = \{\vec{x}: \exists_x \exists_y\, \vec{x} = \vec{d}x + \vec{e}y\}$

Other Theorems

[See page 141 for Theorems 2–1 through 3–14.]
5–1. (a) $\vec{a}0 = \vec{0}$ (b) $\vec{0}a = \vec{0}$
5–2. (a) $\vec{a} \cdot -a = -(\overrightarrow{aa})$ (b) $-\vec{a} \cdot a = -(\overrightarrow{aa})$
5–3. (a) $\vec{a}(a - b) = \overrightarrow{aa} - \overrightarrow{ab}$ (b) $(\vec{a} - \vec{b})a = \overrightarrow{aa} - \overrightarrow{ba}$
5–4. (a) $\overrightarrow{ac} = \overrightarrow{bc} \Longrightarrow \vec{a} = \vec{b}\ [c \neq 0]$ (b) $\overrightarrow{ca} = \overrightarrow{cb} \Longrightarrow a = b\ [\vec{c} \neq \vec{0}]$
5–5. $\overrightarrow{aa} = \vec{0} \Longrightarrow (a = 0 \text{ or } \vec{a} = \vec{0})$

Rules of Logic

[For earlier summaries, see pages 111–113, 142, and 170–172.]
Inferences of any of the following forms are valid:

$$\frac{p \longrightarrow [q \longrightarrow r]}{q \longrightarrow [p \longrightarrow r]} \qquad \left[\text{i.e.: } \frac{q \longrightarrow r \quad [p]}{p \longrightarrow r \quad [q]}\right]$$

$$\frac{\text{not } p \longrightarrow [q \longrightarrow r]}{\text{not } r \longrightarrow [q \longrightarrow p]} \qquad \left[\text{i.e.: } \frac{q \longrightarrow r \quad [\text{not } p]}{q \longrightarrow p \quad [\text{not } r]}\right]$$

$$\frac{\text{not } q \longrightarrow [p \longrightarrow r]}{p \longrightarrow (q \text{ or } r)} \qquad \frac{p \longrightarrow (q \text{ or } r)}{\text{not } q \longrightarrow [p \longrightarrow r]}$$

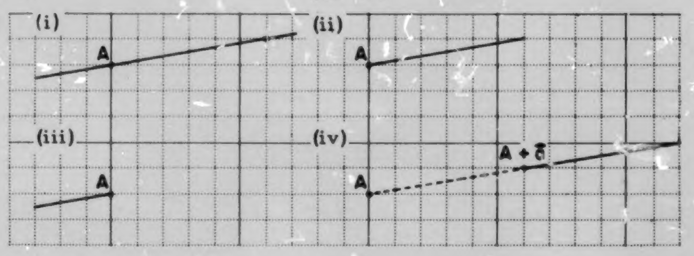
$$\frac{p \text{ or } q \quad p \longrightarrow r \quad q \longrightarrow s}{r \text{ or } s}$$

Key to Chapter Test

1. (a) $\vec{a} \cdot -3 + \vec{b} \cdot -5$ (b) $P + \vec{a}4 - \vec{b}4$
(c) $C + \vec{c}4$ (d) $(P - Q)6$

2. (a) III (b) II
(c) IV (d) VI

3. (a)

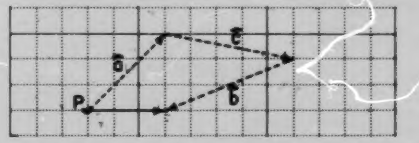


(b) (i) – (ii) and (iv)
(ii) – (ii), (iii)
(iii) – (i)
(iv) – (i) and (ii)

4. (a) True. (b) True. (c) False.
(d) True. (e) False. (f) True.

5. Since it is assumed that $\vec{b}$ and $\vec{c}$ belong to $\vec{a}$ it follows that, for some numbers b and c, $\vec{b} = \vec{a}b$ and $\vec{c} = \vec{a}c$. So, for any number a, $\vec{b}a = \vec{a} \cdot ba$. Thus, for any a, $\vec{b}a + \vec{c} = \vec{a} \cdot ba + \vec{a}c = \vec{a} \cdot (ba + c)$. So, there is a number y such that $\vec{b}a + \vec{c} = \vec{a}y$. It follows that, for any a, $\vec{b}a + \vec{c} \in [\vec{a}]$. Hence, for each x, $\vec{b}x + \vec{c} \in [\vec{a}]$.

6. (a) The drawing should look something like this:

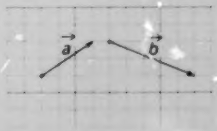


(b) No. $\vec{a} + \vec{b} + \vec{c}$ is not $\vec{0}$ [Many possible explanations possible here.]

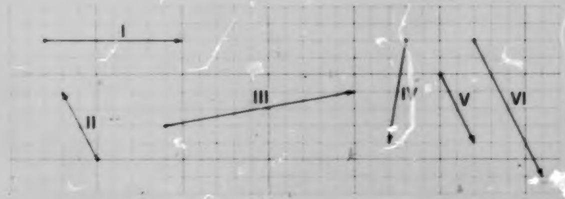
Chapter Test

1. Simplify.
   - (a) $\vec{a} \cdot -5 + \vec{b}3 + \vec{a}2 + \vec{b} \cdot -8$
   - (b) $[P + (\vec{a}4 - \vec{b}2)] - [Q - (Q + \vec{b} \cdot -2)]$
   - (c) $(C + \vec{c} \cdot -2) - [(C - \vec{c}3) - (C + \vec{c}3)]$
   - (d) $(P - Q)3 + [(Q + \vec{q}) - (P + \vec{p})] \cdot -3 + (\vec{a} - \vec{p})3$

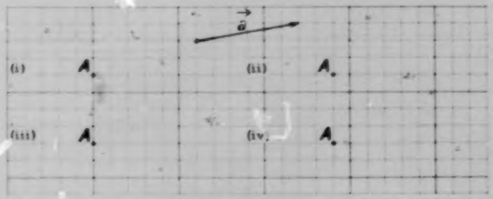
2. Consider the translations $\vec{a}$ and $\vec{b}$ described at the right.



Tell which of the arrows drawn below describes:
   - (a) $\vec{a}2 + \vec{b}$
   - (b) $\vec{a} - \vec{b}$
   - (c) $\vec{b} - \vec{a}2$
   - (d) $\vec{b}2 - \vec{a}2$

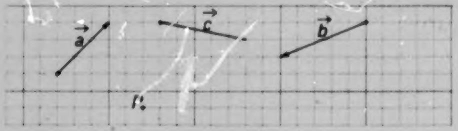


3. Consider the following sets of points:
   - (i) $\{X : \exists_x\, X = A + \vec{a}x\}$ (ii) $\{X : \exists_{x \geq 0}\, X = A + \vec{a}x\}$
   - (iii) $\{X : \exists_{x \leq 0}\, X = A + \vec{a}x\}$ (iv) $\{X : \exists_{x \geq 1}\, X = A + \vec{a}x\}$
   - (a) Suppose that $\vec{a}$ is the translation described by the arrow drawn below. Copy this drawing and draw pictures of the sets (i) – (iv).
   - (b) Answer these questions about the sets (i) – (iv).



   - (i) Which of the given sets are rays with the same sense?
   - (ii) Which of the given sets are rays with the same vertex?
   - (iii) Which of the given sets has each of the others as a subset?
   - (iv) Which of the given sets has as a subset the segment with end points $A + \vec{a}\frac{1}{3}$ and $A + \vec{a}\frac{2}{3}$?

4. Recall that $[\vec{a}] = \{\vec{x} : \exists_x\, \vec{x} = \vec{a}x\}$. Which of the following are true and which are false?
   - (a) $\vec{0} \in [\vec{a}]$
   - (b) $\vec{a}3 - \vec{a}5 \in [\vec{a}]$
   - (c) $[\vec{0}] = \varnothing$
   - (d) If $\vec{b} \in [\vec{a}]$ then $\vec{b} \cdot -4 \in [\vec{a}]$.
   - (e) If $\vec{b} \in [\vec{a}]$ then $\vec{a} \in [\vec{b}]$.
   - (f) If $(\vec{a} \in [\vec{b}]$ and $\vec{b} \in [\vec{c}])$ then $\vec{a} \in [\vec{c}]$.
5. Suppose that $\vec{b}$ and $\vec{c}$ belong to $[\vec{a}]$. Show that, for each x, $\vec{b}x + \vec{c}$ belongs to $[\vec{a}]$.
6. The following arrows describe three forces acting on a point P.
   - (a) Copy this figure and make a careful drawing of the resultant of these three forces.
   - (b) Are these forces in equilibrium? Give a reason for your answer.



Chapter Six
Linear Dependence and Independence

6.01 Linear Combinations of Vectors

According to Definition 5–2 on page 193, a vector $\vec{g}$ is a *linear combination* of vectors $\vec{a}$ and $\vec{b}$ if and only if there are real numbers x and y such that

$$\vec{g} = \vec{a}x + \vec{b}y.$$

[Then 'linear combination of $\vec{a}$ and $\vec{b}$' is short for 'sum of a multiple of $\vec{a}$ and a multiple of $\vec{b}$'.] For example,

$\vec{a}5 + \vec{b} \cdot -3$ is a linear combination of $\vec{a}$ and $\vec{b}$, and
$\vec{a}_1 \cdot \frac{7}{2} + \vec{a}_2 \cdot -\frac{2}{3}$ is a linear combination of $\vec{a}_1$ and $\vec{a}_2$.

Similarly, we shall say that

$\vec{c} \cdot -\frac{2}{3} + \vec{b} \cdot -3 + \vec{a} \cdot 5$ is a linear combination of $\vec{c}$, $\vec{b}$, and $\vec{a}$.

To make best use of the phrase 'linear combination' it turns out to be convenient to agree to refer to any multiple of a vector $\vec{a}$ as *a linear combination of* $\vec{a}$. So, for example, $\vec{a}5$ is a linear combination of $\vec{a}$ and $\vec{c} \cdot -\frac{2}{3}$ is a linear combination of $\vec{c}$.

> Definition 6–1
> $\vec{a}$ is a linear combination of $\vec{a}_1, \ldots, \vec{a}_n$
> $\Longleftrightarrow$
> there are real numbers $x_1, \ldots, x_n$ such that
> $$\vec{a} = \vec{a}_1x_1 + \ldots + \vec{a}_nx_n$$

[Be sure you understand what the '. . .'s mean in Definition 6–1. For examples, here are the first two instances of this definition:

$\vec{a}$ is a linear combination of $\vec{a}_1 \Longleftrightarrow$
there is a number x_1 such that $\vec{a} = \vec{a}_1x_1$
$\vec{a}$ is a linear combination of $\vec{a}_1$ and $\vec{a}_2 \Longleftrightarrow$
there are numbers x_1 and x_2 such that $\vec{a} = \vec{a}_1x_1 + \vec{a}_2x_2$

What is the fourth instance of the definition?]

The concepts of linear dependence and independence are of basic importance for any work at all with vector spaces as such. The only theorems about vectors which do not depend on these notions are theorems which are completely analogous to theorems from the algebra of real numbers. The reason for this is simple. First, linear independence is needed for the purpose of defining the dimension of a vector space. [Roughly, the dimension of a vector space $\mathcal{T}$ is the number of "independent directions" which vectors may have. An analogue of this is that the 3-dimensionality of Euclidean space $\mathcal{E}$ is due to the fact that one can choose in $\mathcal{E}$ only three independent directions. Of course, these may be chosen in many ways. One self-centered choice is up-down, right-left, and front-back.] Second, two vector spaces over $\mathcal{R}$ [or any other given field] are isomorphic if and only if they have the same dimension. It follows that if one lacks the means of defining dimension then the only results one can obtain are those which hold for all vector spaces. Since the real numbers constitute a vector space, such results will be strict analogues of theorems about real numbers.

For our purposes, linear dependence and independence serve not only in defining the dimension of $\mathcal{T}$ — and, with this, the dimension of $\mathcal{E}$ — but also in defining such basic notions as collinearity and coplanarity of points. It is in terms of these latter notions that we define 'line' and 'plane'; the noncollinearity of its vertices is an essential part of the notion of a triangle; the only conceptual difference between a circle and a sphere — or between a disc and a ball — is that one is a plane figure while the other is not. Such a list of illustrations of the importance for geometry of linear dependence and independence could be extended indefinitely. But these should be sufficient to convince you of the importance of the material in the present chapter.

As you will see, the definition of 'linear dependence', and the theorems depending on it, require the use of the quantifiers '∀' and '∃' — or, at least, the use of equivalent modes of expression such as 'for all' and 'there are'. Up to now we have made only rather informal use of quantifiers or of quantifying phrases and we continue this practice in the first six sections of this chapter. However, a real understanding of the meanings of 'for all' and 'there are' as these phrases are used in mathematics can best be gained by study and use of the rules of logic for dealing with '∀' and '∃'. Hence, sections 6.07 and 6.08 are devoted to the logic of quantifiers and to illustrations of its use in proving theorems concerning linear dependence and independence. [Incidentally, this is the last large hunk of logic to be treated in the text.] You may find it necessary to omit or deal lightly with some — but, certainly, by no means all — of the material in these sections. On the other hand, you may find it desirable to insert some of this material into your treatment of the earlier sections. In fact, we urge that you study sections 6.07 and 6.08 with this end in view.

The notion of a linear combination of given vectors is an important one and forms a good basis for understanding the notion of linear dependence. As mentioned in earlier commentary on Definitions 5-1 and 5-2, the former notion is an aid to defining the direction and sense of a vector and, so, to defining the direction of a line and the sense of an oriented line, as well as to defining higher dimensional analogues of these notions.

The occurrence of the parameter 'n' in the definitions and many of the theorems of this chapter requires some comment here. It will

TC 211 (2)

turn out that our vector space of translations is 3-dimensional and so, for example, any linear combination of any number of given translations is bound to be a linear combination of some three, at most, of them, and any sequence of more than three translations is bound to be linearly dependent. So, as far as our later needs are concerned, it would be sufficient to deal only with the case $n \leq 4$. Had we chosen to do so there would have been two alternatives. We could have introduced a restriction — say, '$n \leq 4$' — into each definition and theorem — a restriction which would never come into play — or we could have split up each definition into four definitions and each theorem into four theorems. Neither alternative makes sense either mathematically or pedogogically.

The use of subscripts in the second example on page 214 is merely for the purpose of calling attention to the fact that subscripts could be used to create new letters. The more sophisticated convention, by which 'f_1', say, is an alternative to the notation '$f(1)$' for the value of a function f at the argument 1, is not introduced in this book [although you will find mention of it in the commentary for section 6.04].

Be sure students understand the '...'-convention illustrated in Definition 6-1. [You might tell them that it is a pattern for a whole sequence of definitions.] They may require more examples like those following the definition, and the matter should be reviewed when discussing Definition 6-2 on page 220, Theorem 6-2 on page 221, etc. The fourth instance of the definition is:

$$\begin{gathered}\vec{a} \text{ is a linear combination of } \vec{a}_1, \vec{a}_2, \vec{a}_3, \text{ and } \vec{a}_4 \\ \Longleftrightarrow \\ \text{there are numbers } x_1, x_2, x_3, \text{ and } x_4 \text{ such that} \\ \vec{a} = \vec{a}_1 x_1 + \vec{a}_2 x_2 + \vec{a}_3 x_3 + \vec{a}_4 x_4\end{gathered}$$

Although we do not stress the point here, the notions of linear combination and linear dependence come in two slightly different forms. One can speak either of linear combinations of the members of the set $\{\vec{a}_1, \ldots, \vec{a}_n\}$ or of linear combinations of the terms of the sequence $(\vec{a}_1, \ldots, \vec{a}_n)$. As explained on page 218, an n-termed sequence is a function whose domain is $\{1, \ldots, n\}$ and can be named by listing its values — which are called its terms — in order, between parentheses. Such a list may contain repetitions and, if it does, these repetitions are significant; they indicate that the function has the same value for two or more arguments. Also, the order of listing is significant; it shows which values go with which arguments. A set, on the other hand, can be named by listing its members between braces. The order in which the members are listed is irrelevant and, although it may happen that a member is listed more than once, any such repetitions are irrelevant.

As far as linear combinations are concerned it makes little difference whether one speaks in terms of sequences or of sets. Since, however, it is necessary to distinguish between the two notions in the case of linear dependence, and since we shall be concerned mainly with linear dependence of sequences, it is better, if the question arises, to lean toward the sequence-interpretation of Definition 6-1. If the question does arise spontaneously in the minds of your students, they will probably be prepared to realize that, according to the definition, to say that $\vec{a}$ is a linear combination of $\vec{b}$ and $\vec{c}$ mean that there are numbers x and y such that $\vec{a} = \vec{b}x + \vec{c}y$, while to say that $\vec{a}$ is a linear combination of $\vec{c}$ and $\vec{b}$ is to say that there are numbers x and y such that

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$\vec{a} = \vec{c}x + \vec{b}y$. So since addition of vectors is commutative, it is a theorem that $\vec{a}$ is a linear combination of $\vec{b}$ and $\vec{c}$ if and only if $\vec{a}$ is a linear combination of $\vec{c}$ and $\vec{b}$. Since this theorem is not of itself, a logically valid sentence, the difference between the concept of being a linear combination of $\vec{b}$ and $\vec{c}$ and that of being a linear combination of $\vec{c}$ and $\vec{b}$ may not be merely ignored. Incidentally, since $\{\vec{b}, \vec{c}\} = \{\vec{c}, \vec{b}\}$ [although, for $\vec{b} \neq \vec{c}$, $(\vec{c}, \vec{b}) \neq (\vec{b}, \vec{c})$], one would have to prove the same theorem:

$$(\star) \qquad \exists_x \exists_y\ \vec{a} = \vec{b}x + \vec{c}y \Longleftrightarrow \exists_x \exists_y\ \vec{a} = \vec{c}x + \vec{b}y$$

before he would be justified in adopting the definition:

$$\begin{gathered}\vec{a} \text{ is a linear combination of the members of } \{\vec{b}, \vec{c}\} \\ \Longleftrightarrow \\ \exists_x \exists_y\ \vec{a} = \vec{b}x + \vec{c}y\end{gathered}$$

For, $(\star)$ is a consequence of this definition and the identity of $\{\vec{b}, \vec{c}\}$ and $\{\vec{c}, \vec{b}\}$; so, if $(\star)$ were not a theorem then the definition would be creative — as no definition should be [see TC 108 and TC 120].

Definition 6-1 is, clearly, closely related to Definitions 5-1 and 5-2. Indeed, this definition might be replaced by:

$$[\vec{a}_1, \ldots, \vec{a}_n] = \{\vec{x} \colon \exists_{x_1} \ldots \exists_{x_n}\ \vec{x} = \vec{a}_1 x_1 + \ldots + \vec{a}_n x_n\}$$

[Read the left side as 'brackets $\vec{a}_1$ through $\vec{a}_n$'.] If this were done then '$\vec{a} \in [\vec{a}_1, \ldots, \vec{a}_n]$' might replace the phrase '$\vec{a}$ is a linear combination of $\vec{a}_1, \ldots, \vec{a}_n$'. However, the phrase 'linear combination' is useful, and we shall have need of the bracket notation only in the cases covered by Definitions 5-1 and 5-2.

Finally, as you will notice, our practice as to omitting multiplication dots is somewhat random. If oppositing signs and subtraction signs were more readily distinguishable, multiplication dots could always be omitted. As it is, we never omit them before oppositing signs. In other contexts, the mood of the moment is likely to rule.

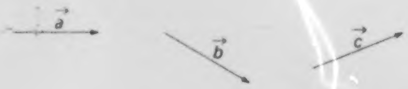
TC 212 (1)

Parts A - E involve more work than can usually be handled in one class period. One way to handle these exercises is to take Parts A and B as one homework assignment, Part C and the first two or three exercises in Part D as class exercises and demonstration, and the rest of Part D and Part E as a second homework assignment perhaps using teams for the derivations.

Exercises

Part A

1. Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be the translations described in the following diagram.



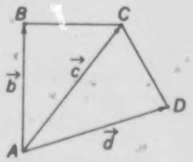
On your paper, make use of the translations just described to illustrate the following linear combinations.

(a) $\vec{a}5$, $\vec{b} \cdot -3$, $\vec{a} \cdot \frac{7}{2}$, $\vec{c} \cdot -\frac{2}{3}$
(b) $\vec{a}5 + \vec{b} \cdot -3$, $\vec{a} \cdot \frac{7}{2} + \vec{c} \cdot -\frac{2}{3}$, $\vec{c} \cdot -\frac{2}{3} + \vec{b} \cdot -3 + \vec{a}5$
(c) Four other examples [of your choice] of linear combinations of $\vec{a}$, or of $\vec{a}$ and $\vec{b}$, or of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

2. Consider any vectors $\vec{a}$ and $\vec{b}$. Answer each of the following questions and explain your answers.

(a) Is $\vec{a}7$ a linear combination of $\vec{a}$ and $\vec{b}$? [To answer this in the affirmative, of course, you must find multiples of $\vec{a}$ and $\vec{b}$ such that their sum is $\vec{a}7$. To put it another way, you must find real numbers a and b such that $\vec{a}7 = \vec{a}a + \vec{b}b$.]
(b) Is $\vec{b}$ a linear combination of $\vec{a}$ and $\vec{b}$?
(c) Is $\vec{0}$ a linear combination of $\vec{a}$ and $\vec{b}$?
(d) Is $\vec{b}$ a linear combination of $\vec{b}$? Of $\vec{a}$?

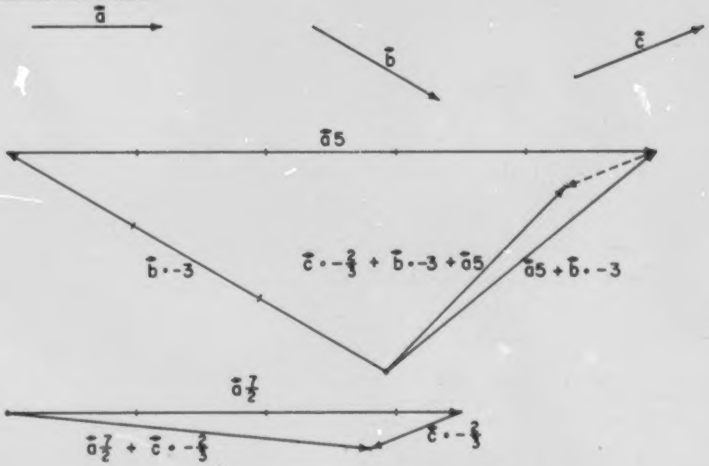
3. Given the quadrilateral $ABCD$ shown at the right. Notice, in the figure, that $\vec{b} = B - A$, $\vec{c} = C - A$ and $\vec{d} = D - A$.



(a) Express the translation $C - B$ as a linear combination of $\vec{b}$ and $\vec{c}$. As a linear combination of $\vec{b}$, $\vec{c}$, and $\vec{d}$.
(b) Express each of the translations $D - C$ and $D - B$ as a linear combination of $\vec{b}$, $\vec{c}$, and $\vec{d}$.
(c) By Postulate 3, we know that $(C - B) + (D - C) = D - B$. Use this to check your answers in parts (a) and (b). [*Hint*: Substitute the appropriate results from (a) and (b) for '$C - B$', '$D - C$', and '$D - B$' and check whether the resulting sentence is true. What can you conclude if it is? If it isn't?]
(d) We did not say that A, B, C, and D are coplanar [—that is, all belong to the same plane]. And, we did not say that they aren't coplanar. Can you tell from the figure whether or not these four points are coplanar? Does it make a difference as far as your answers for (a) and (b) are concerned?

Answers for Part A

1.


$\vec{a}$
$\vec{b}$
$\vec{c}$
$\vec{a}5$
$\vec{b} \cdot -3$
$\vec{c} \cdot -\frac{2}{3} + \vec{b} \cdot -3 + \vec{a}5$
$\vec{a}5 + \vec{b} \cdot -3$
$\vec{a}\frac{7}{2}$
$\vec{a}\frac{7}{2} + \vec{c} \cdot -\frac{2}{3}$
$\vec{c} \cdot -\frac{2}{3}$

(c) [For this one, ask several students to illustrate one of their choices at the chalkboard.]

2. (a) Yes, since $\vec{a}7 = \vec{a}7 + \vec{b}0$.
(b) Yes, since $\vec{b} = \vec{a}0 + \vec{b}1$.
(c) Yes, since $\vec{0} = \vec{a}0 + \vec{b}0$.
(d) Yes, since $\vec{b} = \vec{b}1$.; Whether $\vec{b}$ is a linear combination of $\vec{a}$ depends on special properties of these vectors. Specifically, $\vec{b}$ will be a linear combination of $\vec{a}$ if and only if $\vec{b} = \vec{0}$ or ($\vec{a} \neq \vec{0}$ and $\vec{a}$ and $\vec{b}$ have the same "direction").

3. (a) $C - B = \vec{c} - \vec{b} = \vec{b} \cdot -1 + \vec{c}1$; $C - B = \vec{b} \cdot -1 + \vec{c}1 + \vec{d}0$
(b) $D - C = \vec{b}0 + \vec{c} \cdot -1 + \vec{d}1$; $D - B = \vec{b} \cdot -1 + \vec{c}0 + \vec{d}1$
(c) Substituting in the given instance of Postulate 3 yields:

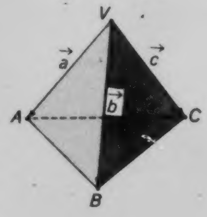
(☆) $(\vec{b} \cdot -1 + \vec{c}1 + \vec{d}0) + (\vec{b}0 + \vec{c} \cdot -1 + \vec{d}1) = \vec{b} \cdot -1 + \vec{c}0 + \vec{d}1$

Simplification of the left side by using theorems based on Postulates 4 and 5 reduces (☆) to a sentence of the form '$\vec{a} = \vec{a}$', and reversing these simplification steps shows that (☆) is true. Actually, this tells us nothing. Had (☆) turned out to be false we might have concluded that not all the results obtained in answers to parts (a) and (b) could be correct [or that something was wrong about Postulates 3, 4, or 5].

(d) No.; No. [Anticipate some problems in visualizing this situation by having a simple stick model available for manipulation. Join sticks representing $\vec{b}$, $\vec{c}$, and $\vec{d}$ at point A so that the join is flexible. Elastic thread for $C - B$ and $D - C$ works nicely. Tape, say, the sticks for $\vec{c}$ and $\vec{d}$ to the chalkboard and let students move the $\vec{b}$ stick to various positions.]

(e) If the figure is correctly drawn, $C - B$ is not a linear combination of $D - C$. [Why?] But, suppose that the figure is incorrect and that $C - B$ *is* a linear combination of $D - C$. On the basis of this assumption, show that $\vec{c}$ is a linear combination of $\vec{b}$ and $\vec{d}$. [In doing this you will find that you need to make one other assumption about the points.]

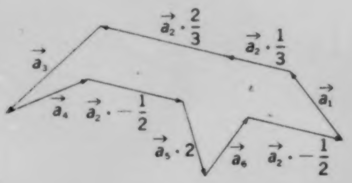
4. Consider the pyramid V-ABC, shown at the right. Notice, in the figure, that $A - V = \vec{a}$, $B - V = \vec{b}$, and $C - V = \vec{c}$.



(a) Express each of the translations $B - A$, $C - B$, and $A - C$ as linear combinations of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

(b) Use your results from (a) to verify that $(B - A) + (C - B) + (A - C) = \vec{0}$.

5. Given the translations shown in the following figure:


$\vec{a}_2 \cdot \frac{2}{3}$
$\vec{a}_2 \cdot \frac{1}{3}$
$\vec{a}_3$
$\vec{a}_1$
$\vec{a}_4$
$\vec{a}_2 \cdot -\frac{1}{2}$
$\vec{a}_5 \cdot 2$
$\vec{a}_6$
$\vec{a}_2 \cdot -\frac{1}{2}$

(a) From the figure it follows that [Complete.]

$$\vec{a}_1 \cdot \underline{\quad} + \vec{a}_2 \cdot \underline{\quad} + \vec{a}_3 \cdot \underline{\quad} + \vec{a}_4 \cdot \underline{\quad} + \vec{a}_5 \cdot \underline{\quad} + \vec{a}_6 \cdot \underline{\quad} = \vec{0}.$$

(b) Express $\vec{a}_1$ as a linear combination of $\vec{a}_2$, $\vec{a}_3$, $\vec{a}_4$, $\vec{a}_5$, and $\vec{a}_6$.

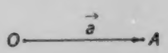
(c) Express $\vec{a}_2$ as a linear combination of $\vec{a}_1$, $\vec{a}_3$, $\vec{a}_4$, $\vec{a}_5$, and $\vec{a}_6$.

(d) Express $\vec{a}_5$ as a linear combination of $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{a}_4$, and $\vec{a}_6$.

6. Prove: $\vec{a}a + \vec{b}b = \vec{0} \Longrightarrow \vec{a} = \vec{b} \cdot -(b \cdot /a)$ $[a \neq 0]$

Part B

1. Draw a figure like this:



(a) Locate the points D, E, F, G, H, and I, where

$$D = O + \vec{a}\tfrac{1}{2}, \quad E = O + \vec{a} \cdot -\tfrac{1}{2}, \quad F = A + \vec{a} \cdot -\tfrac{1}{2},$$
$$G = F + \vec{a}2, \quad H = E + \vec{a}, \quad I = O + \vec{a} \cdot -2.$$

(e) Intuitively, if C - B were a linear combination of C - D then B, C, and D would be collinear. Assuming [despite the evidence of the figure] that C - B is a linear combination of C - D it follows, by definition, that there is a number — say, a — such that C - B = (D - C)a. It follows from parts (a) and (b) that

$$\begin{aligned} \vec{b} \cdot -1 + \vec{c}1 &= (\vec{c} \cdot -1 + \vec{d}1)a, \\ &= \vec{c} \cdot -a + \vec{d}a, \\ \vec{c}(1 + a) &= \vec{b}1 + \vec{d}a, \\ \vec{c} &= \vec{b} \cdot \frac{1}{1 + a} + \vec{d}\frac{a}{1 + a} \quad [a \neq -1] \end{aligned}$$

We note that a = −1 if and only if C - B = C - D — that is, if and only if B = D. So, assuming that C - B is a linear combination of C - D, and that B ≠ D, it follows that $\vec{c}$ is a linear combination of $\vec{b}$ and $\vec{d}$.

[Part (d) should alert students to the fact that flat drawings need not represent plane geometric figures, and illustrates the fact that many of our theorems will be independent of dimension. Part (e) is a step in a campaign which will eventually lead to the recognition that collinearity can be defined in terms of linear dependence of appropriate point differences. It is also an example of the use of "position vectors" [in this case "from A"] to characterize points on a line [the line $\overleftrightarrow{BD}$]. $C \in \overleftrightarrow{BD}$ if and only if $\vec{c}$ is a linear combination of $\vec{b}$ and $\vec{d}$.]

4. (a) $B - A = \vec{a} \cdot -1 + \vec{b}1 + \vec{c}0$; $C - B = \vec{a}0 + \vec{b} \cdot -1 + \vec{c}1$;

$A - C = \vec{a}1 + \vec{b}0 + \vec{c} \cdot -1$

(b) $$\begin{aligned} &(\vec{a} \cdot -1 + \vec{b}1 + \vec{c}0) + (\vec{a}0 + \vec{b} \cdot -1 + \vec{c}1) + (\vec{a}1 + \vec{b}0 + \vec{c} \cdot -1) \\ &= \vec{a}(-1 + 0 + 1) + \vec{b}(1 + -1 + 0) + \vec{c}(0 + 1 + -1) \\ &= \vec{a}0 + \vec{b}0 + \vec{c}0 = \vec{0} \end{aligned}$$

[In discussing this exercise, point out that the choice of notation which leads to such equations as '$B - A = \vec{b} - \vec{a}$', '$C - B = \vec{c} - \vec{b}$', and '$A - C = \vec{a} - \vec{c}$' has obvious advantages if one has much to do with the points A, B, and C. The result stated in (b) is more easily checked by noting that it is equivalent to '$(\vec{b} - \vec{a}) + (\vec{c} - \vec{b}) + (\vec{a} - \vec{c}) = \vec{0}$'. It is still more easily checked by using Postulate 3 and Theorem 3-1(b). You might check understanding of the '...'-convention on the theorem:

$$(A_2 - A_1) + (A_3 - A_2) + \ldots + (A_n - A_{n-1}) = A_n - A_1$$

[A proof of this theorem requires the use of mathematical induction; but individual instances can be established by using instances of Postulate 3.]]

Answers for Part A [cont.]

5. (a) 1, 0, 1, 1, 2, 1

   (b) $\vec{a}_1 = \vec{a}_2 0 + \vec{a}_3 \cdot -1 + \vec{a}_4 \cdot -1 + \vec{a}_5 \cdot -2 + \vec{a}_6 \cdot -1$

   (c) Insufficient information. [One might be inclined to assume from the figure that, for example, $\vec{a}_2 \cdot \frac{1}{3} + \vec{a}_5 \cdot 2 + \vec{a}_6 = \vec{0}$ and, so, that

   $$\vec{a}_2 = \vec{a}_1 0 + \vec{a}_3 0 + \vec{a}_4 0 + \vec{a}_5 \cdot -6 + \vec{a}_6 \cdot -3.$$

   However, as Exercise 3(d) has pointed out, such assumptions are unwarranted. A demonstration will help to clarify the point that $\vec{a}_2$ is not necessarily a linear combination of $\vec{a}_1$, $\vec{a}_3$, $\vec{a}_4$, $\vec{a}_5$, and $\vec{a}_6$. Copy the diagram of Exercise 5 on the board. Now make a new diagram using $\vec{a}_1$, $\vec{a}_3$, $\vec{a}_4$, $\vec{a}_5 \cdot 2$, and $\vec{a}_6$. Students will see that $\vec{a}_1 + \vec{a}_3 + \vec{a}_4 + \vec{a}_5 \cdot 2 + \vec{a}_6 = \vec{0}$ regardless of the direction of $\vec{a}_2$. In some sense, the translations $\vec{a}_1$, $\vec{a}_3$, $\vec{a}_4$, $\vec{a}_5 \cdot 2$, and $\vec{a}_6$ are "free" of, or do not depend upon $\vec{a}_2$.]

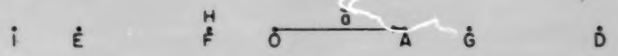
   (d) $\vec{a}_5 = \vec{a}_1 \cdot -\frac{1}{2} + \vec{a}_2 0 + \vec{a}_3 \cdot -\frac{1}{2} + \vec{a}_4 \cdot -\frac{1}{2} + \vec{a}_6 \cdot -\frac{1}{2}$

6. Suppose that $\vec{a}a + \vec{b}b = \vec{0}$. It follows that $(\vec{a}a + \vec{b}b) \cdot /a = \vec{0} \cdot /a$. Since $(\vec{a}a + \vec{b}b) \cdot /a = (\vec{a}a) \cdot /a + (\vec{b}b) \cdot /a = \vec{a}(a \cdot /a) + \vec{b}(b \cdot /a)$ and since, for $a \neq 0$, $a \cdot /a = 1$ it follows that $\vec{a}1 + \vec{b}(b \cdot /a) = \vec{0} \cdot /a$. Since $\vec{a}1 = \vec{a}$ and $\vec{0} \cdot /a = \vec{0}$ it follows that $\vec{a} + \vec{b}(b \cdot /a) = \vec{0}$. So, $\vec{a} = -\vec{b}(b \cdot /a) = \vec{b} \cdot -(b \cdot /a)$. Hence, for $\vec{a} \neq \vec{0}$, if $\vec{a}a + \vec{b}b = \vec{0}$ then $\vec{a} = \vec{b} \cdot -(b \cdot /a)$. [Remind students that the "unabbreviated form" of this theorem is:

   $$a \neq 0 \Longrightarrow [\vec{a}a + \vec{b}b = \vec{0} \Longrightarrow \vec{a} = \vec{b} \cdot -(b \cdot /a)] \quad]$$

Answers for Part B

1. (a)



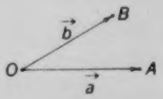
(b) Consider the vectors which map O onto the points D, E, F, G, H, and I. Is each a linear combination of $\vec{a}$? Justify your answer.

(c) Give the following ratios of sensed segments:

$(O \text{ to } A):(A \text{ to } D)$; $(A \text{ to } G):(F \text{ to } O)$; $(F \text{ to } E):(O \text{ to } A)$;
$(I \text{ to } E):(O \text{ to } D)$; $(I \text{ to } E):(E \text{ to } H)$; $(O \text{ to } F):(O \text{ to } H)$.

2. (a) In Exercise 1, is each of the points D, E, F, G, H, and I on the same line as O and A?
   (b) Suppose that $\vec{c}$ is a linear combination of $\vec{a}$, where $\vec{a}$ is the vector pictured in Exercise 1, and that $J = O + \vec{c}$. Do you think that J is on the same line as O and A?
   (c) Suppose that K is any point on the same line as the points O and A of Exercise 1. Do you think that $K - O$ is a linear combination of $\vec{a}$?

3. Draw a figure like this:



(a) Locate the points P, Q, R, S, T, and U, where

$P = O + \vec{a}2$, $\quad Q = O + \vec{b}2$,
$R = A + (\vec{a} + \vec{b})$, $\quad S = O + (\vec{a} \cdot -\frac{3}{2} + \vec{b})$,
$T = O + (\vec{a} + \vec{b} \cdot -2)$, $\quad U = O + (\vec{a} + \vec{b}) \cdot -\frac{1}{2}$.

(b) Consider the vectors which map O onto the points P, Q, R, S, T, and U. Is each of these vectors a linear combination of $\vec{a}$ and $\vec{b}$? Justify your answer.

4. (a) In Exercise 3, is each of the points P, Q, R, S, T, and U in the same plane as A, B, and O?
   (b) Suppose that $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$, where $\vec{a}$ and $\vec{b}$ are the vectors pictured in Exercise 3, and that $V = O + \vec{c}$. Do you think that V is in the same plane as A, B, and O?
   (c) Suppose that W is any point in the same plane as A, B, and O. Do you think that $W - O$ is a linear combination of $\vec{a}$ and $\vec{b}$?

5. Use a pencil [or, your finger] to represent a vector $\vec{c}$ which is not a linear combination of the vectors $\vec{a}$ and $\vec{b}$ from Exercise 3. Demonstrate how you would find the following points:

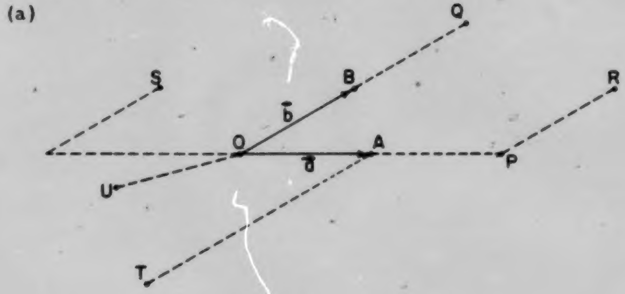
(a) $O + \vec{c}$ (b) $O + \vec{c} \cdot -5$ (c) $O + (\vec{a} + \vec{c})$
(d) $O + (\vec{a} + \vec{b} + \vec{c})$ (e) $O + (\vec{b}2 + \vec{c} \cdot -3)$ (f) $O + (\vec{a}3 + \vec{b} + \vec{c}2)$

(b) Yes.; $D - O = \vec{a}\frac{5}{2}$, $E - O = \vec{a} \cdot -\frac{3}{2}$, $F - O = \vec{a} \cdot -\frac{1}{2}$,
$G - O = \vec{a}\frac{3}{2}$, $H - O = \vec{a} \cdot -\frac{1}{2}$, $I - O = \vec{a} \cdot -2$.

(c) 2/3; 1; −1; 1/5; 1/2; 1

2. (a) Yes. (b) Yes. (c) Yes. [As yet, these answers can be justified only on intuitive grounds. When, in the next chapter, the notion of collinearity is defined, it will be possible to prove that the answers are correct. If students exhibit doubts as to the correctness of the answers, suggest intuitive arguments based, say, on the use of a tracing sheet. But, point out that such arguments are purely intuitive.]

3. (a)



(b) Yes.; $P - O = \vec{a}2 + \vec{b}0$, $Q - O = \vec{a}0 + \vec{b}2$, $R - O = \vec{a}2 + \vec{b}1$,
$S - O = \vec{a} \cdot -\frac{3}{2} + \vec{b}1$, $T - O = \vec{a}1 + \vec{b} \cdot -2$,
$U - O = \vec{a} \cdot -\frac{1}{2} + \vec{b} \cdot -\frac{1}{2}$.

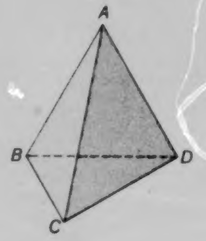
4. (a) Yes. (b) Yes. (c) Yes. [The translation $\vec{a}$, and any multiple of it, will map any plane containing the line $\overleftrightarrow{OA}$ onto itself. The translation $\vec{b}$, and any multiple of it, will map any plane containing the line $\overleftrightarrow{OB}$ onto itself. The resultant of any multiple of $\vec{a}$ followed by any multiple of $\vec{b}$ will map the plane determined by O, A, and B onto itself. Intuitively, using just the sums of a multiple of $\vec{a}$ and a multiple of $\vec{b}$ there is no way to "get out of" the plane of O, A, and B.]

5. [To the teacher.] Each student should be able to demonstrate how to find the various given points. Stick and string models may be both helpful and useful to make at this stage of the game.

6. Consider the pyramid A-BCD.

(a) Do you think that $C - B$ is a linear combination of $B - A$ and $C - A$? Of $D - B$ and $D - C$? Of $A - B$ and $D - B$? Explain your answers.

(b) Do you think that $C - A$ is a linear combination of $B - A$ and $D - A$? Of $B - A$, $D - A$, and $C - B$? Explain your answers.



*Part C Points to Ponder

1. It should be intuitively clear that for any two points there is exactly one line that contains them. Another way to express this same idea is to say that any two points are collinear. Of course, it is not necessarily the case that any three points are collinear. Try to formulate a definition of collinearity of points A, B, and C in terms of translations.

2. As stated in Exercise 1, there is exactly one line that contains two given points. Try to formulate a definition of a line containing two points, say, the points A and B.

3. Three points need not be collinear. However, it is intuitively obvious that three noncollinear points are contained in exactly one plane. Try to formulate a definition of coplanarity of points A, B, C, and D. Do it, of course, in terms of translations.

4. Try to formulate a definition of the plane containing three noncollinear points A, B, and C.

Part D

1. Recall that, by Definition 5-1,

$$[\overrightarrow{p}] = \{\overrightarrow{x}: \exists_y \overrightarrow{x} = \overrightarrow{p}y\}$$

— that is, $[\overrightarrow{p}]$ is the set of all linear combinations of $\overrightarrow{p}$. [Read '$[\overrightarrow{p}]$' as 'brackets $\overrightarrow{p}$'.]

(a) Show that each of the following vectors belongs to $[\overrightarrow{a}]$:

$$\overrightarrow{a}, -\overrightarrow{a}, \overrightarrow{0}, \overrightarrow{a} \cdot \tfrac{1}{2} + \overrightarrow{a} \cdot -3$$

(b) Show that each of the vectors in part (a) belongs to $[\overrightarrow{a}2]$.

(c) Show that each vector in $[\overrightarrow{a}]$ is in $[\overrightarrow{a}2]$. [What does this tell you about how $[\overrightarrow{a}]$ and $[\overrightarrow{a}2]$ are related?]

(d) Show that $[\overrightarrow{a}] = [\overrightarrow{a}2]$—that is, show that the set of all linear combinations of $\overrightarrow{a}$ is the same as the set of all linear combinations of $\overrightarrow{a}2$. [*Hint*: Having done part (c), what more need you do?]

---

6. (a) Yes.; C - B = (B - A)· -1 + (C - A)1.

Yes.; C - B = (D - B)1 + (D - C)· -1.

No. The word 'pyramid' [and the shading of the figure] indicates that we are not to consider the possibility of C being in the plane of A, B, and D. And, intuitively, no translation which is a linear combination of A - B and D - B can move B out of this plane.

(b) No, for the same reason given in support of the answer for the last question of part (a).

Yes.; C - A = (B - A)1 + (D - A)0 + (C - B)1.

Answers for Part C

[The answers to these questions ought to generate a lot of healthy discussion about how translations act on points. While not all students will be able to do the kind of creative thinking necessary to generate adequate definitions, trying to do so should be encouraged and, most important, no such attempts which seem to be or are "incorrect" should be denounced. We give below possible answers.]

1. If it is assumed that A, B, and C are three points, a correct answer is:

A, B, and C are collinear if and only if
C - A is a linear combination of B - A.

[This is still correct under the weaker assumption that A ≠ B.] Without any assumption concerning the number of elements in the set {A, B, C}:

A, B, and C are collinear if and only if (A = B or C - A is a linear combination of B - A).

A, B, and C are collinear if and only if one of B - A and C - A is a linear combination of the other.

2. For A ≠ B, the line $\overleftrightarrow{AB}$ is {X: X - A is a linear combination of B - A}.

3. If it is assumed that A, B, and C are noncollinear, a correct answer is:

A, B, C, and D are coplanar if and only if
D - A is a linear combination of B - A and C - A.

Without any assumption concerning A, B, C, and D:

A, B, C, and D are coplanar if and only if (A, B, and C are collinear or D - A is a linear combination of B - A and C - A).

A, B, C, and D are coplanar if and only if one of B - A, C - A, and D - A is a linear combination of the other.

4. For A, B, and C noncollinear, the plane containing A, B, and C is

{X: X - A is a linear combination of B - A and C - A}.

[We stress, again, that these are only some possible answers, and that there is no point, at this time, in striving to force "correct" answers from all students. "Points to Ponder" should be taken literally.]

Answers for Part D

1. (a) $\vec{a} = \vec{a}1$; $-\vec{a} = \vec{a}\cdot -1$; $\vec{0} = \vec{a}0$; $\vec{a}\cdot\frac{1}{2} + \vec{a}\cdot -3 = \vec{a}(\frac{1}{2} + -3) = \vec{a}\cdot -\frac{5}{2}$

[In connection with these answers you might foreshadow the introduction rule for '∃'. This is (2) on page 240 and is stated formally, along with the elimination rule for '∀', in the box on page 244. Its application to the exercises of part (a) comes about in the following way. To show that $\vec{b} \in [\vec{a}]$ we must, by definition, show that there is a real number x such that $\vec{b} = \vec{a}x$. The most satisfactory way to show that there is such a number is to exhibit one. This is what is done in the answers for the exercises. However, a complete solution of the first of these exercises would run as follows:

> By Postulate 4_5, $\vec{a} = \vec{a}1$. So [by the introduction rule for '∃'], $\exists_x\ \vec{a} = \vec{a}x$. Hence, by Definition 5-1, $\vec{a} \in [\vec{a}]$.

There is no point in requiring, in part (a), that students give answers in this extended form. It should not be difficult, however, to get them to see the necessity of this if 'Show' were changed to 'Prove'. Doing so now will help prepare them for later exercises.]

(b) $\vec{a} = \vec{a}1 = \vec{a}(2\cdot\frac{1}{2}) = (\vec{a}2)\frac{1}{2}$ [or: $\vec{a} = \vec{a}1 = \vec{a}(2\cdot /2) = (\vec{a}2)\cdot /2$];
$-\vec{a} = \vec{a}\cdot -1 = (\vec{a}\cdot -1)1 = (\vec{a}\cdot -1)(2\cdot /2) = (\vec{a}2)(-1\cdot /2)$ [$= (\vec{a}2)\cdot -\frac{1}{2}$]
$\vec{0} = (\vec{a}2)0$; $\vec{a}\cdot\frac{1}{2} + \vec{a}\cdot -3 = (\vec{a}2)\cdot -\frac{5}{4}$

[The general procedure is illustrated in the answer for the second of the three exercises of part (b). It depends on the easily proved theorem:

$$(\vec{a}b)(ac) = (\vec{a}a)(bc)$$

This theorem is used in the answer, below, for part (c).]

(c) Suppose that $\vec{b} \in [\vec{a}]$. It follows, by definition, that $\exists_x\ \vec{b} = \vec{a}x$. Suppose that $\vec{b} = \vec{a}b$. It follows by 4_5 that $\vec{b} = (\vec{a}b)1$. Since $2 \neq 0$, $2\cdot /2 = 1$ and it follows that $\vec{b} = (\vec{a}b)(2\cdot /2) = (\vec{a}2)(b\cdot /2)$. So, $\exists_x\ \vec{b} = (\vec{a}2)x$ and, by definition, $\vec{b} \in [\vec{a}2]$. Hence, if $\vec{b} = \vec{a}b$ then $\vec{b} \in [\vec{a}2]$. Since $\exists_x\ \vec{b} = \vec{a}x$ it follows that $\vec{b} \in [\vec{a}2]$. Hence, if $\vec{b} \in [\vec{a}]$ then $\vec{b} \in [\vec{a}2]$. [This tells us that $[\vec{a}] \subseteq [\vec{a}2]$.]

[This argument illustrates the use of the elimination rule for '∃' which is stated formally on page [illegible]9.
In its application here it tells us that if we know that, for any b,

(1) $\vec{b} = \vec{a}b \Longrightarrow \vec{b} \in [\vec{a}2]$

and also know that

(2) $\exists_x\ \vec{b} = \vec{a}x$

then we may conclude that $\vec{b} \in [\vec{a}2]$. Note that this conclusion does not follow from (1) alone, since (1) might hold even if the equation '$\vec{b} = \vec{a}b$' had no solutions. In this case (1) would give no warrant for concluding that $\vec{b} \in [\vec{a}2]$.

Students should, of course, not be expected to give as complete an argument as that given above. Nevertheless, if, in class discussion, you can bring them to appreciate this argument, and the roles played in it by the two rules for '∃',

then the pay-off should be large. For one thing, much of the discussion in sections 6.07 and 6.08 will have been previewed and should cause students little difficulty when they come to it.]

(d) Having shown in part (c) that $[\vec{a}] \subseteq [\vec{a}2]$, all that remains to be done in order to show that $[\vec{a}] = [\vec{a}2]$ is to show that $[\vec{a}2] \subseteq [\vec{a}]$. That is, it remains only to be shown that if $\vec{b} \in [\vec{a}2]$ then $\vec{b} \in [\vec{a}]$.

Suppose that $\vec{b} \in [\vec{a}2]$. By definition it follows that there is a real number — say, b — such that $\vec{b} = (\vec{a}2)b$. It follows, by 4_8, that $\vec{b} = \vec{a}(2b)$. So, by definition, $\vec{b} \in [\vec{a}]$. Hence, if $\vec{b} \in [\vec{a}2]$ then $\vec{b} \in [\vec{a}]$.

[This argument has been chosen to illustrate a less complete form which might reasonably be expected of students who either do not know the use of the existential quantifier or who, having learned thoroughly how to use it, have advanced to the point of ignoring it. (Recall our contention that, by-and-large, only students who have thoroughly understood formal rules of logic are able to argue informally with both ease and safety.) By comparing the given argument with that for part (c) you should have no trouble in rewriting it so as to bring out the use of the rules for '∃'. We suggest you do so before reading further. The changes required are to replace the fourth sentence by 'By definition it follows that $\exists_x\ \vec{b} = (\vec{a}2)x$. Suppose that $\vec{b} = (\vec{a}2)b$.' and the sixth by 'So, $\exists_x\ \vec{b} = \vec{a}x$ and, by definition, $\vec{b} \in [\vec{a}]$. Hence, if $\vec{b} = (\vec{a}2)b$ then $\vec{b} \in [\vec{a}]$. Since $\exists_x\ \vec{b} = (\vec{a}2)x$ it follows that $\vec{b} \in [\vec{a}]$.'.]

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(e) One.; Infinitely many.

(f) It tells you that $\vec{a} = \vec{0}$.

2. (a) False. Would be true if followed by the restriction '[$c \neq 0$]'.

(b) True. ['$\exists_x\ \vec{a}c = \vec{a}x$' is a consequence of its valid instance '$\vec{a}c = \vec{a}c$'.]

(c) True. [By Definitions 5-1 and 6-1 this is a restatement of (b).]

(d) False. [Restatement of (a).]

(e) True.

(f) False. Would be true if followed by the restriction '[$c \neq 0$]'.

3. (a) Intuitively, this is the line through A whose direction is that of $\vec{a}$; alternatively, it is the line through A and $A + \vec{a}$.

(b) {A}

(e) How many translations are in $[\vec{a}]$ if $\vec{a} = \vec{0}$? If $\vec{a} \neq \vec{0}$?

(f) Suppose that $[\vec{a}]$ contains exactly one vector. What does this tell you about $\vec{a}$?

2. True or false? Justify each answer; and if you can "add onto" a false sentence to make a true one, do so.

(a) $\vec{a} \in [\overrightarrow{ac}]$
(b) $\overrightarrow{ac}$ is a linear combination of $\vec{a}$
(c) $\overrightarrow{ac} \in [\vec{a}]$
(d) $\vec{a}$ is a linear combination of $\overrightarrow{ac}$
(e) $[\overrightarrow{ac}] \subseteq [\vec{a}]$
(f) $[\overrightarrow{ac}] = [\vec{a}]$

3. Describe the set of points $\{X: \exists_x (X = A + \vec{x} \text{ and } \vec{x} \in [\vec{a}])\}$

(a) if $\vec{a} \neq \vec{0}$.
(b) if $\vec{a} = \vec{0}$.

4. Prove each of the following:

(a) $-\vec{a} \in [\vec{a}]$
(b) $\vec{a} \in [-\vec{a}]$
(c) $\vec{b} \in [\vec{a}] \longrightarrow [\vec{b}] \subseteq [\vec{a}]$
(d) $[\overrightarrow{aa}] \subseteq [\vec{a}]$
(e) $(\vec{a} \in [\vec{b}] \text{ and } \vec{b} \in [\vec{a}]) \longrightarrow [\vec{a}] = [\vec{b}]$
(f) $[-\vec{a}] = [\vec{a}]$
(g) $(\vec{b} \neq \vec{0} \text{ and } b \in [\vec{a}]) \longrightarrow [\vec{a}] = [\vec{b}]$
[*Hint*: Prove: $\vec{b} \in [\vec{a}] \longrightarrow \vec{a} \in [\vec{b}]$ $[\vec{b} \neq \vec{0}]$]
(h) $[\overrightarrow{aa}] = [\vec{a}]$ $[a \neq 0]$

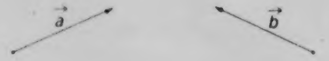
Part E

1. (a) Suppose that $\vec{a}$ is any vector. Can you find numbers a, b, and c such that

$$(\vec{a}2)a + (\vec{a}3)b + (\vec{a} \cdot -4)c = \vec{0}?$$

(b) Find numbers a, b, and c *which are not all zero* and which satisfy the equation in part (a).

2. Consider the vectors $\vec{a}$ and $\vec{b}$ indicated in the following figure:



(a) Are there real numbers a and b such that $\vec{a}a + \vec{b}b = \vec{0}$? Justify your answer.

(b) Do you think that there are real numbers a and b, *not both zero*, such that $\vec{a}a + \vec{b}b = \vec{0}$? Explain your answer. [*Hint*: Does Exercise 6 of Part A seem relevant?]

3. Suppose that P, R, and S are any points in $\mathscr{E}$.

(a) Show that $S - P$ is a linear combination of $R - P$ and $S - R$.

(b) *Use your result from part* (a) to find numbers a, b, and c such that

$$(S - P)a + (R - P)b + (S - R)c = \vec{0}.$$

(c) Use your result from part (b) to show that $S - R$ is a linear combination of $S - P$ and $R - P$.

4. Suppose that $\vec{a}$, $\vec{b}$, and $\vec{c}$ are any vectors.

(a) Show that $\vec{0}$ is a linear combination of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

4. (a) Since $-\vec{a} = \vec{a}\cdot -1$ it follows that $\exists_x -\vec{a} = \vec{a}x$. Hence, by definition, $-\vec{a} \in [\vec{a}]$.

(b) By part (a), $--\vec{a} \in [-\vec{a}]$. Since $--\vec{a} = \vec{a}$ it follows that $\vec{a} \in [-\vec{a}]$. [Another argument: Since $\vec{a} = -\vec{a}\cdot -1$ it follows that $\exists_x \vec{a} = -\vec{a}x$. Hence, $\vec{a} \in [-\vec{a}]$.]

(c) Suppose that $\vec{b} = \vec{a}b$ and that $\vec{c} = \vec{b}c$. It follows that $\vec{c} = (\vec{a}b)c = \vec{a}(bc)$ and so, by definition, that $\vec{c} \in [\vec{a}]$. Hence, if $\vec{c} \in [\vec{b}]$ then $\vec{c} \in [\vec{a}]$ — that is, $[\vec{b}] \subseteq [\vec{a}]$. Hence, if $\vec{b} \in [\vec{a}]$ then $[\vec{b}] \subseteq [\vec{a}]$.

[This argument, like that given for Exercise 1(d), is perfectly satisfactory on an intuitive level. Nevertheless, students will understand it more thoroughly, and will have more success with similar arguments, if it is put in the more formal mode illustrated by the answer for Exercise 1(c) (assuming, of course, that they have already seen the latter). Here is how it goes:

> Suppose that $\vec{b} \in [\vec{a}]$. It follows that $\exists_x \vec{b} = \vec{a}x$. Suppose that $\vec{b} = \vec{a}b$.
>
> Suppose, now, that $\vec{c} \in [\vec{b}]$. It follows that $\exists_x \vec{c} = \vec{b}x$. Suppose that $\vec{c} = \vec{b}c$. Since $\vec{b} = \vec{a}b$ it follows that $\vec{c} = (\vec{a}b)c = \vec{a}(bc)$. So, $\exists_x \vec{c} = \vec{a}x$ and, by definition, $\vec{c} \in [\vec{a}]$. Hence, if $\vec{c} = \vec{b}c$ then $\vec{c} \in [\vec{a}]$. So, since $\exists_x \vec{c} = \vec{b}x$ it follows that $\vec{c} \in [\vec{a}]$. Hence, if $\vec{c} \in [\vec{b}]$ then $\vec{c} \in [\vec{a}]$ — that is, $[\vec{b}] \subseteq [\vec{a}]$.
>
> Hence, if $\vec{b} = \vec{a}b$ then $[\vec{b}] \subseteq [\vec{a}]$. So, since $\exists_x \vec{b} = \vec{a}x$ it follows that $[\vec{b}] \subseteq [\vec{a}]$. Hence, if [illegible] $\in [\vec{a}]$ then $[\vec{b}] \subseteq [\vec{a}]$.

Note how each of the three assumptions is, in turn, picked up and discharged.]

(d) [The argument is the same as that given as part of the answer for Exercise 1(d) with '2' replaced throughout by 'a'.]

(e) [This follows at once from the result in part (c).]

(f) [This follows at once from the results in parts (a), (b), and (e).]

(g) Suppose that $\vec{b} \in [\vec{a}]$. It follows, by definition, that there is a number — say, b — such that $\vec{b} = \vec{a}b$. Since $\vec{a}0 = \vec{0}$ it follows for $\vec{b} \neq \vec{0}$, that $b \neq 0$. Since $b \neq 0$, $b\cdot/b = 1$ and, since $\vec{b} = \vec{a}b$, it follows that $\vec{b}\cdot/b = (\vec{a}b)\cdot/b = \vec{a}(b\cdot/b) = \vec{a}1 = \vec{a}$. So, by definition, $\vec{a} \in [\vec{b}]$. Hence, for $\vec{b} \neq \vec{0}$, if $\vec{b} \in [\vec{a}]$ then $\vec{a} \in [\vec{b}]$.

Suppose, now, that $\vec{b} \neq \vec{0}$ and $\vec{b} \in [\vec{a}]$. It follows [as has just been proved] that $\vec{a} \in [\vec{b}]$. Since, also, $\vec{b} \in [\vec{a}]$ it follows by the result in part (e) that $[\vec{a}] = [\vec{b}]$. Hence, if $\vec{b} \neq \vec{0}$ and $\vec{b} \in [\vec{a}]$ then $[\vec{a}] = [\vec{b}]$.

(h) Since, by part (d), $[\vec{a}a] \subseteq [\vec{a}]$ it is sufficient to prove that, for $a \neq 0$, $[\vec{a}] \subseteq [\vec{a}a]$.

Suppose that $\vec{b} \in [\vec{a}]$. By definition there is a number — say, b — such that $\vec{b} = \vec{a}b$. Now, for $a \neq 0$, $a \cdot /a = 1$. So it follows that $\vec{b} = \vec{a}b = (\vec{a}b)1 = (\vec{a}b)(a \cdot /a) = (\vec{a}a)(b \cdot /a)$. So, by definition, $\vec{b} \in [\vec{a}a]$. Hence, for $a \neq 0$, if $\vec{b} \in [\vec{a}]$ then $\vec{b} \in [\vec{a}a]$ — that is, $[\vec{a}] \subseteq [\vec{a}a]$.

Answers for Part E

[These are exploration exercises for the notion of linear dependence. For example, solving Exercise 1(b) amounts to proving that $(\vec{a}2, \vec{a}3, \vec{a} \cdot -4)$ is a linearly dependent sequence.]

1. (a) This is trivial. Take $a = b = c = 0$.

   (b) Since $(\vec{a}2)a + (\vec{a}3)b + (\vec{a} \cdot -4)c = \vec{a}(2a + 3b - 4c)$ [and since $\vec{a}0 = \vec{0}$] it is sufficient to find numbers a, b, and c which are not all zero and are such that $2a + 3b - 4c = 0$. There are infinitely many solutions. Perhaps the simplest is obtained by choosing $a = 2$, $b = 0$, and $c = 1$.

2. (a) Yes.; $\vec{a}0 + \vec{b}0 = \vec{0}$

   (b) No. If there were such numbers then one of $\vec{a}$ and $\vec{b}$ would be a multiple of the other.

3. (a) By Postulate 3, $S - P = (R - P) + (S - R) = (R - P)1 + (S - R)1$. Hence, $\exists_x \exists_y\ S - P = (R - P)x + (S - R)y$; and, by definition, $S - P$ is a linear combination of $R - P$ and $S - R$.

   (b) $(S - P)1 + (R - P) \cdot -1 + (S - R) \cdot -1 = \vec{0}$

   (c) $S - R = -[(S - R) \cdot -1] = (S - P)1 + (R - P) \cdot -1$

4. (a) $\vec{0} = \vec{a}0 + \vec{b}0 + \vec{c}0$

(b) Suppose that $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$. Then, by definition, there are numbers — say, a and b — such that $\vec{c} = \vec{a}a + \vec{b}b$. It follows that $\vec{a}a + \vec{b}b + \vec{c} \cdot -1 = \vec{0}$. Since $-1 \neq 0$, not all of a, b, and -1 are zero.

(c) Suppose that $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$ and that $c \neq 0$. Since $c \neq 0$, $c \cdot /c = 1$ and it follows that $(\vec{a}a + \vec{b}b + \vec{c}c) \cdot /c = \vec{a}(a \cdot /c) + \vec{b}(b \cdot /c) + \vec{c}$. Since $\vec{0} \cdot /c = \vec{0}$ it also follows that $\vec{a}(a \cdot /c) + \vec{b}(b \cdot /c) + \vec{c} = \vec{0}$. Hence,

$$\vec{c} = -[\vec{a}(a \cdot /c) + \vec{b}(b \cdot /c)] = \vec{a} \cdot -(a \cdot /c) + \vec{b} \cdot -(b \cdot /c)$$

[Consequently, $\exists_x \exists_y\ \vec{c} = \vec{a}x + \vec{b}y$ and, by definition, $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$.]

5. This follows from the result obtained in Exercise 4(b) together with the commutativity and associativity of addition of vectors. In detail:

   If $\vec{a}$ is a linear combination of $\vec{b}$ and $\vec{c}$ then [by Exercise 4(b)] there are numbers — say, a, b, and c — which are not all zero and are such that $\vec{b}b + \vec{c}c + \vec{a}a = \vec{0}$. Since $\vec{b}b + \vec{c}c + \vec{a}a = \vec{a}a + (\vec{b}b + \vec{c}c) = \vec{a}a + \vec{b}b + \vec{c}c$ it follows that if $\vec{a}$ is a linear combination of $\vec{b}$ and $\vec{c}$ then there are numbers — say, a, b, and c — such that $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$.

   Similarly, if $\vec{b}$ is a linear combination of $\vec{a}$ and $\vec{c}$ then [by Exercise 4(b)] there are numbers — say, a, b, and c — which are not all zero and are such that $\vec{a}a + \vec{c}c + \vec{b}b = \vec{0}$. Since $\vec{a}a + \vec{c}c + \vec{b}b = \vec{a}a + (\vec{c}c + \vec{b}b) = \vec{a}a + (\vec{b}b + \vec{c}c) = \vec{a}a + \vec{b}b + \vec{c}c$ it follows that if $\vec{a}$ is a linear combination of $\vec{b}$ and $\vec{c}$ then there are numbers — say, a, b, and c — which are not all zero and are such that $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$.

   Finally, if $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$ then [by Exercise 4(b)] there are numbers — say, a, b, and c — which are not all zero and are such that $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$.

   Consequently [by a dilemma argument] if one of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is a linear combination of the others then . . . .

6. Suppose that a, b, and c are not all zero. Then, $a \neq 0$ or $b \neq 0$ or $c \neq 0$. Suppose that $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$. By Exercise 4(c) it follows that if $c \neq 0$ then $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$ and, so, that one of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is a linear combination of the others. Since $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$ it follows that $\vec{a}a + \vec{c}c + \vec{b}b = \vec{0}$ and that $\vec{b}b + \vec{c}c + \vec{a}a = \vec{0}$. So, by repetitions of the preceding argument it follows that if $b \neq 0$ then one of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is a linear combination of the others, and that if $a \neq 0$ then one of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is a linear combination of the others. Since $a \neq 0$ or $b \neq 0$ or $c \neq 0$ it follows that one of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is a linear combination of the others. Hence, if a, b, and c are not all zero then . . . .

7. Suppose that $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$ are vectors such that at least one of them is a linear combination of the others. Suppose that $\vec{a}_i$ is a linear combination of the vectors $\vec{a}_j$ for $1 \leq j \leq k$ and $j \neq i$. It follows, by definition, that there are numbers — say, a_j — such that $\vec{a}_i$ is the sum of the vectors $\vec{a}_j a_j$. Let $a_i = -1$. It then follows that $\vec{a}_1 a_1 + \vec{a}_2 a_2 + \ldots + \vec{a}_k a_k = \vec{0}$, and, since $a_i = -1 \neq 0$, not all the numbers $a_1, \ldots, a_k$ are zero. Hence, if $\vec{a}_i$ is a linear combination of the others then there are numbers $a_1, \ldots, a_k$, not all zero, such that $\vec{a}_1 a_1 + \ldots + \vec{a}_k a_k = \vec{0}$. Since, by hypothesis,

(b) Assuming that $\overrightarrow{c}$ is a linear combination of $\overrightarrow{a}$ and $\overrightarrow{b}$, show how to find numbers a, b, and c *which are not all zero* such that

$$\overrightarrow{a}a + \overrightarrow{b}b + \overrightarrow{c}c = \overrightarrow{0}.$$

(c) Assuming that a, b, and c are numbers such that $c \neq 0$ and

$$\overrightarrow{a}a + \overrightarrow{b}b + \overrightarrow{c}c = \overrightarrow{0},$$

show that $\overrightarrow{c}$ is a linear combination of $\overrightarrow{a}$ and $\overrightarrow{b}$.

5. Suppose that $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ are vectors such that at least one of them is a linear combination of the others. Prove that there are numbers a, b, and c which are not all zero and which are such that

$$\overrightarrow{a}a + \overrightarrow{b}b + \overrightarrow{c}c = \overrightarrow{0}.$$

6. Suppose that $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ are vectors, and a, b, and c are real numbers which are not all zero, such that

$$\overrightarrow{a}a + \overrightarrow{b}b + \overrightarrow{c}c = \overrightarrow{0}.$$

Prove that at least one of the vectors $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ is a linear combination of the others.

7. Suppose that $\overrightarrow{a_1}, \overrightarrow{a_2}, \ldots, \overrightarrow{a_k}$ are vectors such that at least one of them is a linear combination of the others. Prove that there are numbers $a_1, a_2, \ldots, a_k$, not all zero, such that

$$\overrightarrow{a_1}a_1 + \overrightarrow{a_2}a_2 + \ldots + \overrightarrow{a_k}a_k = \overrightarrow{0}.$$

8. Suppose that $\overrightarrow{a_1}, \overrightarrow{a_2}, \ldots, \overrightarrow{a_k}$ are vectors and $a_1, a_2, \ldots, a_k$ are real numbers, not all zero, such that $\overrightarrow{a_1}a_1 + \overrightarrow{a_2}a_2 + \ldots + \overrightarrow{a_k}a_k = \overrightarrow{0}$. Prove that at least one of the given vectors is a linear combination of the others.

6.02 Sequences of Vectors

A *sequence* of vectors is a function whose domain is a set of consecutive positive integers, beginning with 1, and whose values are vectors. For example, the function

$$\{(1, \overrightarrow{a}), (2, \overrightarrow{a3}), (3, \overrightarrow{b}), (4, \overrightarrow{0}), (5, \overrightarrow{a})\}$$

is a sequence of vectors. The successive values of a sequence are called its terms – the *first term* of our example is $\overrightarrow{a}$, its *second term* is $\overrightarrow{a3}$, etc. Although the first term and the fifth term of this sequence happen to be equal, the sequence is called *a 5-termed sequence*. Specifically, for any positive integer n, *an n-termed sequence* is a function whose argu-

at least one of $\vec{a}_1, \ldots, \vec{a}_k$ is a linear combination of the others, there are numbers $a_1, \ldots, a_k$, not all zero, such that $\vec{a}_1a_1 + \ldots + \vec{a}_ka_k = \vec{0}$. [This is more than is to be expected of most students. As written, however, the argument suggests that there is an '∃' hidden in the undergrowth. There is. In fact, the assumption is that $\exists_i$ ($1 \leq i \leq k$ and $\vec{a}_i$ is a linear combination of the vectors $\vec{a}_1, \ldots, \vec{a}_k$ other than $\vec{a}_i$). So, the natural procedure is to show that, for any i [such that $1 \leq i \leq k$], if $\vec{a}_i$ is a linear combination of the vectors $\vec{a}_1, \ldots, \vec{a}_k$ other than $\vec{a}_i$ then $\exists_{x_1} \ldots \exists_{x_k}\ \vec{a}_1x_1 + \ldots + \vec{a}_kx_k = \vec{0}$. The argument, as given, is as close as we care to come to establishing this. Similar remarks apply to the answer for Exercise 8.]

8. Suppose that $\vec{a}_1, \ldots, \vec{a}_k$ are vectors and $a_1, \ldots, a_k$ are real numbers, not all zero, such that $\vec{a}_1a_1 + \ldots + \vec{a}_ka_k = \vec{0}$. Suppose that $a_i \neq 0$. It follows that $a_i \cdot /a_i = 1$ and, so, that $\vec{a}_i = \vec{a}_i1$ is the sum of the vectors $\vec{a}_j \cdot -(a_j \cdot /a_i)$ for $j \neq i$. So, by definition, $\vec{a}_i$ is a linear combination of the vectors $\vec{a}_1, \ldots, \vec{a}_k$ other than $\vec{a}_i$. Hence, if $a_i \neq 0$ then $\vec{a}_i$ is a linear combination of the other vectors. Since, by assumption, at least one of the numbers a_i is not zero it follows that at least one of the vectors $\vec{a}_1, \ldots, \vec{a}_k$ is a linear combination of the others.

It would be possible to avoid discussing the notion of a sequence and to discuss linear dependence in section 6.03 — much as we have discussed linear combinations in section 6.01 — without being specific as to whether this notion is meant to apply to sequences, or to sets, of vectors. The ambiguity which would be introduced by this procedure would, we feel, have increasingly serious pedogogical consequences as students get further into the course. So, it seems best to spend a short time now on the notion of a sequence rather than to attempt to correct misconceptions later.

As introduced here, 'sequence' is short for 'finite sequence'. Since we have no need in this connection for infinite sequences — that is, for functions whose domain is the set of all positive integers — we omit the word 'finite'. For some remarks on the difference in meaning between '$(\vec{a}_1, \ldots, \vec{a}_n)$' and '$\{\vec{a}_1, \ldots, \vec{a}_n\}$', see TC 211(2) and the answers given for the exercises.

ments are the positive integers less than or equal to n. Such a sequence may have any number of values up to and including n. If an n-termed sequence has n values then it is called *a sequence of distinct terms.*

A sequence is usually referred to by listing its successive terms between parentheses. For example, the 5-termed sequence given above may be referred to as

$$(\vec{a}, \vec{a}3, \vec{b}, \vec{0}, \vec{a}).$$

Since it is the order in which the terms are listed that tells you which value goes with which argument, this ordering is important.

Exercises

Part A

1. Let f be the sequence $(\vec{b}1, \vec{b}0, \vec{b})$.
   (a) What is Df? What is Rf?
   (b) Complete:
   f is a ___-termed sequence and Rf has ___ members.
   (c) What is $f(2)$? $f(1)$? $f(3)$? $f(5)$? $f(1/2)$?
   (d) $(\vec{b}1, \vec{b}0, \vec{b})$ and $\{\vec{b}1, \vec{b}0, \vec{b}\}$ are both sets. How do they differ from one another?
   (e) Just one of the following sentences is true. Which one?

$$(\vec{b}, \vec{0}, \vec{b}) = (\vec{0}, \vec{b}) \qquad \{\vec{b}, \vec{0}, \vec{b}\} = \{\vec{0}, \vec{b}\}$$

2. Consider the function g, where $g = \{(1, \vec{a}2), (3, \vec{a}1), (2, \vec{a}6), (4, \vec{a}3)\}$. Complete:
   (a) The function g is a ___-termed sequence because ________.
   (b) A simple name for the sequence g is:

   (________)

   (c) The second term of g is $g($___$)$ and, so, is ___.
   (d) g is a sequence of distinct terms if and only if ___ $\neq$ ___.
3. Consider the sequence f of Exercise 1.
   (a) Can you find real numbers a_1, a_2, and a_3 such that

$$(*) \qquad (\vec{b}1)a_1 + (\vec{b}0)a_2 + \vec{b}a_3 = \vec{0}?$$

   [*Hint*: Knowing that $\vec{a}0 = \vec{0}$ should help.]
   (b) Can you find real numbers a_1, a_2, and a_3 such that *not all of them are zero* and such that (*) is satisfied?
4. Consider the sequence g of Exercise 2. Can you find real numbers a_1, a_2, a_3, and a_4, which are *not all zero*, such that

$$(\vec{a}2)a_1 + (\vec{a}6)a_2 + (\vec{a}1)a_3 + (\vec{a}3)a_4 = \vec{0}?$$

Answers for Part A

1. (a) $\{1, 2, 3\}$; $\{\vec{b}, \vec{0}\}$ [An alternative second answer is '$\{\vec{b}1, \vec{b}0, \vec{b}\}$'. If this is suggested, point out that the answer previously given is equivalent and simpler.]
   (b) 3; 2 (c) $\vec{0}$; $\vec{b}$; $\vec{b}$; '$f(5)$' and '$f(1/2)$' are nonsense.
   (d) The first is a set of ordered pairs [whose second components are vectors]; the second is a set of vectors [whose members are the second components of the members of the first set].
   (e) The second is true.
2. (a) 4; $\mathcal{D}g = \{1, 2, 3, 4\}$ (b) $(\vec{a}2, \vec{a}6, \vec{a}1, \vec{a}3)$
   (c) 2; $\vec{a}6$ (d) $\vec{a}$; $\vec{0}$
3. (a) Simplest to choose $a_1 = a_2 = a_3 = 0$.
   (b) Simplest to choose $a_1 = a_3 = 0$ and $a_2 \neq 0$ — say, $a_2 = 1$.
4. Since $(\vec{a}2)a_1 + (\vec{a}6)a_2 + (\vec{a}1)a_3 + (\vec{a}3)a_4 = \vec{a}(2a_1 + 6a_2 + a_3 + 3a_4)$ and since $\vec{a}0 = \vec{0}$, it is sufficient to find numbers a_1, a_2, a_3, and a_4, not all zero, and such that $2a_1 + 6a_2 + a_3 + 3a_4 = 0$. A simple choice [from among the infinitely many available] is $a_1 = 3$, $a_2 = -1$, $a_3 = a_4 = 0$.

* * *

For completeness it should be noted that the parenthesis notation for sequences introduces a slight ambiguity because parentheses are already in use for referring to ordered pairs. Thus, '$(\vec{a}, \vec{b})$' may, now, refer to the set of two ordered pairs whose first components are 1 and 2 and whose second components are $\vec{a}$ and $\vec{b}$, respectively; or it may, as in the past, refer to the single ordered pair whose first component is $\vec{a}$ and whose second component is $\vec{b}$.

This ambiguity is extremely unlikely to create any confusion and will probably not be noticed by your students. The most common formal method for avoiding ambiguity is to use '<' and '>', rather than '(' and ')', in forming names for ordered pairs. Then, $<\vec{a}, \vec{b}>$ is an ordered pair and $(\vec{a}, \vec{b})$ is a 2-termed sequence.

*

Part B

1. Consider a 1-termed sequence $(\vec{a})$.
   - **(a)** Is there a real number a such that $\vec{a}a = \vec{0}$?
   - **(b)** For what value(s) of $\vec{a}$ is there a nonzero real number a such that $\vec{a}a = \vec{0}$?

2. Use two pencils to indicate vectors $\vec{a}$ and $\vec{b}$, where the following conditions hold:
   - **(a)** There is a number x such that $\vec{a} = \vec{b}x$.
   - **(b)** There is a number y such that $\vec{b} = \vec{a}y$.
   - **(c)** There is no number x such that $\vec{a} = \vec{b}x$.
   - **(d)** There is a number x such that $\vec{a} = \vec{b}x$ *and* there is no number y such that $\vec{b} = \vec{a}y$. [For this, one of your pencils should be very short.]

3. Picture vectors $\vec{a}$ and $\vec{b}$ for which the following conditions hold:
   - **(a)** There are numbers x and y, not both zero, such that
   $$\vec{a}x + \vec{b}y = \vec{0}.$$
   - **(b)** It is impossible to find numbers x and y such that (i) not both x and y are zero and (ii) $\vec{a}x + \vec{b}y = \vec{0}$.

4. **(a)** Use three pencils to indicate vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ for which there are numbers x, y, and z such that $\vec{a}x + \vec{b}y + \vec{c}z = \vec{0}$.
   - **(b)** Use three pencils to indicate vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ for which there are numbers x, y, and z which are not all zero and are such that $\vec{a}x + \vec{b}y + \vec{c}z = \vec{0}$.
   - **(c)** Use three pencils to indicate vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ for which it is impossible to find numbers x, y, and z, not all zero, such that $\vec{a}x + \vec{b}y + \vec{c}z = \vec{0}$.
   - **(d)** Given the conditions in part (b), is the vector $\vec{a}$ a linear combination of $\vec{b}$ and $\vec{c}$? Is $\vec{b}$ a linear combination of $\vec{a}$ and $\vec{c}$? Is $\vec{c}$ a linear combination of $\vec{a}$ and $\vec{b}$? Must some one of the three vectors be a linear combination of the others? Explain each of your answers.

6.03 Linearly Dependent Sequences

As we have seen in the exercises just completed, it is possible to have a sequence of vectors $(\vec{a}_1, \ldots, \vec{a}_n)$ for which there are numbers, say $a_1, \ldots, a_n$, that satisfy both of these conditions:

(i) not all of $a_1, \ldots, a_n$ are zero
(ii) $\vec{a}_1 a_1 + \ldots + \vec{a}_n a_n = \vec{0}$

Such sequences are called *linearly dependent* sequences. We formalize this notion in our next definition.

Answers for Part B [To be done in class.]

1. (a) Yes. [0] (b) $\vec{0}$ is the only such value of '$\vec{a}$'.

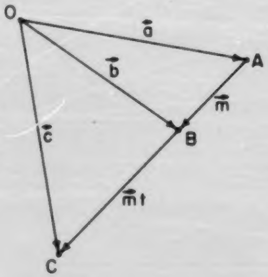
2. (a), (b) [The pencils should be held parallel to one another. Bring out the fact that the equation '$\vec{a} = \vec{b}a$' will have a unique solution — positive if the pencils point the same way, negative if they point opposite ways.]
   - (c) [The pencils should be held so as not to be parallel.]
   - (d) [The pencil representing $\vec{a}$ would have to be of zero length (and the other would have to be a ''real'' pencil).]

3. (a) [Pencils should be held parallel.]
   - (b) [Pencils should be nonparallel.]

4. (a) [Since $\vec{a}0 + \vec{b}0 + \vec{c}0 = \vec{0}$, pencils may be held in any position.]
   - (b) [Pencils should be held parallel to some plane — say, all horizontal. Point out that it is not necessary that all should be in one plane.]
   - (c) [Pencils should not all be parallel to any plane — for example, two horizontal and not parallel, the third not horizontal.]
   - (d) [Since two of the pencils may be parallel and the third be in a different direction, the answer to each of the first three questions is 'Not necessarily.'. The answer to the fourth question is 'Yes.'.]

*

Sample Quiz



The diagram at the right illustrates translations $\vec{a}$, $\vec{b}$, $\vec{c}$ from O to the points A, B, and C, respectively. Also, from the diagram we see that $B - A = \vec{m}$ and $C - B = \vec{m}t$.

1. Express $\vec{m}$ as a linear combination of $\vec{a}$ and $\vec{b}$.
2. Express $\vec{m}t$ as a linear combination of $\vec{b}$ and $\vec{c}$.
3. Express $\vec{c}$ as a linear combination of $\vec{a}$ and $\vec{b}$.

Answers for Sample Quiz

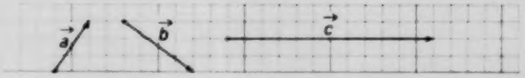
1. $\vec{b} - \vec{a}$ [or: $\vec{a} \cdot -1 + \vec{b}1$]
2. $\vec{c} - \vec{b}$ [or: $\vec{b} \cdot -1 + \vec{c}1$]
3. $\vec{a} \cdot -t + \vec{b}(1 + t)$ [From 1 and 2, we know that $\vec{m} = \vec{b} - \vec{a}$ and $\vec{m}t = \vec{c} - \vec{b}$. So, $(\vec{b} - \vec{a})t = \vec{c} - \vec{b}$. So, $\vec{c} = \vec{b} + (\vec{b} - \vec{a})t = \vec{a} \cdot -t + \vec{b}(1 + t)$.]

> **Definition 6-2**
> $(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)$ is linearly dependent
>
> $\Longleftrightarrow$
>
> there are numbers $x_1, x_2, \ldots, x_n$ such that
> (i) at least one of the numbers is not zero
> and (ii) $\vec{a}_1x_1 + \vec{a}_2x_2 \ldots + \vec{a}_nx_n = \vec{0}$.

Exercises

Part A

1. Tell which of the following sequences are linearly dependent sequences and which are not. Explain each of your answers.
   - **(a)** $(\vec{b}6, \vec{b}0, \vec{b}4)$ [Remember, to show that this sequence is linearly dependent you must find numbers a_1, a_2, and a_3, at least one of which is not zero, such that $(\vec{b}6)a_1 + (\vec{b}0)a_2 + (\vec{b}4)a_3 = \vec{0}$.]
   - **(b)** $(\vec{a}2, \vec{a})$
   - **(c)** $(\vec{a}2, \vec{a}, -\vec{a}, \vec{b})$
   - **(d)** $(\vec{a}, \vec{b}, \vec{0})$
   - **(e)** $(\vec{a}, \vec{b}, \vec{a} - \vec{b}3)$
   - **(f)** $(\vec{a}, \vec{b}, \vec{c})$, where $\vec{a}$, $\vec{b}$, and $\vec{c}$ are, respectively, translations toward the east, toward the south, and upward.
   - **(g)** $(\vec{a}, \vec{c})$, where $\vec{a}$ and $\vec{c}$ are as in part (f).
2. Here are arrows describing vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ such that $(\vec{a}, \vec{b}, \vec{c})$ is linearly dependent. Draw a similar diagram on your paper. [Graph paper should make this an easy job.]



   - **(a)** Estimate values for 'a', 'b', and 'c' such that at least one of a, b, and c is not zero and $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$ and illustrate your answer with an appropriate figure.
   - **(b)** Make use of the figure you drew for part (a) to obtain an appropriate figure to illustrate *other* values for 'a', 'b', and 'c' such that at least one of a, b, and c is not zero and $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$.
3. Consider the vectors given in Exercise 2.
   - **(a)** Use a pencil to indicate a vector $\vec{d}$ such that $(\vec{a}, \vec{b}, \vec{d})$ is *not* linearly dependent. Give a short, but convincing, argument for your particular selection of $\vec{d}$.
   - **(b)** Given your selection for $\vec{d}$ in part (a), would you say that $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly dependent or not? Explain your answer.

Part B

Consider each of the following sentences. If you believe that it is a theorem, try to prove it. If you believe that it is not true, look for a counter-example.

In discussing Definition 6-2 you might begin by bringing out the fact that, given any sequence $(\vec{a}_1, \ldots, \vec{a}_n)$, it is always possible to find numbers $a_1, \ldots, a_n$ such that

$$\text{(ii)} \qquad \vec{a}_1a_1 + \ldots + \vec{a}_na_n = \vec{0}$$

but that the possibility of finding such numbers which are not all zero exists if and only if at least one term of the sequence is a linear combination of the other terms. For, if $\vec{a}_i$ is such a term then one can find numbers satisfying (ii) and such that $a_i = -1$; while if one can find numbers satisfying (ii) and such that $a_i \neq 0$ then one can solve (ii) for '$\vec{a}_i$'. [Actually, this alternative formulation of linear dependence works only for $n \geq 2$ — a fact which some student should be eager to point out.] This discussion will foreshadow Theorem 6-2 on page 221, and the ensuing discussion of the case $n = 1$ will bring out Theorem 6-1.

Some time during the discussion of the definition it should be noticed that condition (i) is stated in two ways:

$$\text{not all of } a_1, \ldots, a_n \text{ are zero}$$

and:

$$\text{at least one of the numbers is not zero}$$

Although students are not likely to have any difficulty in seeing that these are ways of saying the same thing it should be of interest to note that they may be written, respectively, as:

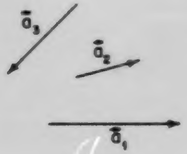
$$\text{not } (a_1 = 0 \text{ and } \ldots a_n = 0) \qquad (a_1 \neq 0 \text{ or } \ldots a_n \neq 0)$$

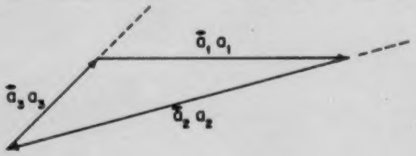
and that, for any given value of 'n', their equivalence could be shown by repeated use of the validity of sentences of the form:

$$\text{not } (p \text{ and } q) \Longleftrightarrow (\text{not } p \text{ or not } q) \qquad [\text{page } 171]$$

In further discussion of the definition it may be helpful to resort again to pencils. [In doing so you will, of course, be restricting considerations to non-$\vec{0}$ vectors.] A linearly dependent sequence $(\vec{a}_1, \vec{a}_2)$ is illustrated by two parallel pencils. Note that if the pencils are of the same length then the two terms of the sequence are the same vector or opposite vectors. A linearly dependent sequence $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is illustrated by three pencils parallel to a given plane — say, by three horizontal pencils. If no two of the pencils are parallel then each term of the sequence is a linear combination of the other two. This is also the case if all three are parallel. In case just two pencils are parallel then each of the corresponding terms is a linear combination of the other and, hence, of the other two terms of the sequence. In this case the term corresponding to the third pencil is not a linear combination of the other terms.

The preceding remarks can be clarified by drawings on the chalkboard. Draw three arrows to represent non-$\vec{0}$ vectors $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$





TC 220 (2)

in different directions. Then draw an arrow representing some nonzero multiple $\vec{a}_1 a_1$ of $\vec{a}_1$. Through its point draw the line in the direction of $\vec{a}_2$, and draw through its other end the line in the direction of $\vec{a}_3$. As shown in the right hand figure, you can now draw arrows representing multiples $\vec{a}_2 a_2$ and $\vec{a}_3 a_3$ of $\vec{a}_2$ and $\vec{a}_3$ such that $\vec{a}_1 a_1 + \vec{a}_2 a_2 + \vec{a}_3 a_3 = \vec{0}$. Since, by choice, $a_1 \neq 0$, the sequence $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is linearly dependent. [In the figure, $a_1 > 0$, $a_2 < 0$, and $a_3 < 0$.]

If you vary the construction described above by choosing $\vec{a}_1$ and $\vec{a}_2$ to have the same direction, it will turn out that $a_3 = 0$. In this case $(\vec{a}_1, \vec{a}_2)$ — as well as $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ — is linearly dependent. At this point students should see that if $(\vec{a}_1, \vec{a}_2)$ is linearly dependent then so is any "longer" sequence $(\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots)$. For, one may choose $a_3, \ldots$ all to be 0. This foreshadows Theorem 6-3 on page 222 and Theorems 6-4 and 6-5 on pages 223-224.

Two nonparallel pencils illustrate a sequence $(\vec{a}_1, \vec{a}_2)$ which is not linearly dependent. Three pencils which are not all parallel to any plane illustrate a sequence $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ which is not linearly dependent. A question as to whether one can illustrate with pencils a 4-termed sequence which is not linearly dependent may provoke a discussion of 4-dimensional space.

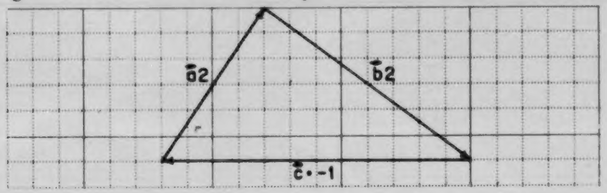
Answers for Part A

1. (a) Since $(\vec{b}6)0 + (\vec{b}0)1 + (\vec{b}4)0 = \vec{0}$, $(\vec{b}6, \vec{b}0, \vec{b}4)$ is linearly dependent. [Similarly, any sequence one of whose terms if $\vec{0}$ is linearly dependent.]
   (b) Since $(\vec{a}2)1 + \vec{a} \cdot -2 = \vec{0}$, $(\vec{a}2, \vec{a})$ is linearly dependent. [It is not difficult to see, then, that any sequence one of whose terms is a multiple of another is linearly dependent.]
   (c) Since $(\vec{a}2)0 + \vec{a}1 + -\vec{a} \cdot 1 + \vec{b}0 = \vec{0}$, $(\vec{a}2, \vec{a}, -\vec{a}, \vec{b})$ is linearly dependent. [This conclusion also follows from the result noted in (b).]
   (d) Linearly dependent. [See part (a).]
   (e) Since $\vec{a} \cdot -1 + \vec{b}3 + (\vec{a} - \vec{b}3)1 = \vec{0}$, $(\vec{a}, \vec{b}, \vec{a} - \vec{b}3)$ is linear dependent.
   (f) Not linearly dependent. Suppose that $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$. Evidently $\vec{a}a + \vec{b}b$ is either a proper translation in some horizontal direction or is $\vec{0}$. Since $\vec{c}c$ is in the vertical direction or is $\vec{0}$, and $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$, $\vec{a}a + \vec{b}b$ cannot be a proper translation in a horizontal direction. So, $\vec{a}a + \vec{b}b = \vec{0}$ and, also $\vec{c}c = \vec{0}$. Since $\vec{a}a + \vec{b}b = \vec{0}$, a similar argument shows that $\vec{a}a = \vec{0}$ and, so, that $\vec{b}b = \vec{0}$. Since $\vec{a}$, $\vec{b}$, and $\vec{c}$ are described as proper translations, a, b, and c must all be 0. Hence, it is not the case that there are numbers a, b, and c, not all 0, such that $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$.
   (g) Not linearly dependent. If $(\vec{a}, \vec{c})$ were linearly dependent then so would be $(\vec{a}, \vec{b}, \vec{c})$.

TC 220 (3)

Answers for Part A [cont.]

2. (a) $a = 2$, $b = 2$, $c = -1$ [Note that $\vec{a} + \vec{b}$ and $\vec{c}$ have the same direction. So, any solution must be such that $a = b$. Counting shows that $\vec{c} = (\vec{a} + \vec{b})2$.]



   (b) [For any nonzero number k, 2k, 2k, and −k are appropriate values for 'a', 'b', and 'c'; there are no others.] Any triangle whose three sides are parallel to the corresponding three sides of the triangle drawn for (a) can be used to illustrate other values for 'a', 'b', and 'c'. Reversing the arrowheads in the figure drawn for (a) illustrates that $\vec{a} \cdot -2 + \vec{b} \cdot -2 + \vec{c}1 = \vec{0}$..

3. (a) [Pencil should be held in any position not parallel to the paper.]

   Since neither $\vec{a}$ nor $\vec{b}$ is a multiple of the other, $(\vec{a}, \vec{b})$ is not linearly dependent; and $\vec{d}$ has been chosen so that it is not a linear combination of $\vec{a}$ and $\vec{b}$. Suppose, now, that $\vec{a}a + \vec{b}b + \vec{d}d = \vec{0}$. Since $\vec{d}$ is not a linear combination of $\vec{a}$ and $\vec{b}$, $d = 0$. So, $\vec{a}a + \vec{b}b = \vec{0}$ and, since $(\vec{a}, \vec{b})$ is not linearly dependent, $a = 0$ and $b = 0$. Hence, by definition, $(\vec{a}, \vec{b}, \vec{d})$ is not linearly dependent.

   (b) $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly dependent however $\vec{d}$ is chosen. For $\vec{a}2 + \vec{b}2 + \vec{c} \cdot -1 + \vec{d}0 = \vec{0}$.

TC 221 (1)

Answers for Part B

1. Fill-ins: 1, 0, 1, 1, the sequence is linearly dependent.
2. This is a theorem. Suppose that $(\vec{a}_1, \ldots, \vec{a}_n)$ is a sequence such that $\vec{a}_i = \vec{a}_j$ where $1 \leq i < j \leq n$. Choose $a_i = 1$, $a_j = -1$, and $a_k = 0$ for $k \neq i, j$. Then, $\vec{a}_1 a_1 + \ldots + \vec{a}_n a_n = \vec{a}_i 1 + \vec{a}_j \cdot -1 = \vec{0}$. Since $1 \neq 0$, not all the numbers $a_1, \ldots, a_n$ are 0. So, by definition, the sequence is linearly dependent.
3. False. For $\vec{a} \neq \vec{0}$, $\vec{a}a = \vec{0}$ only if $a = 0$. So, a 1-termed sequence $(\vec{a})$ is not linearly dependent if $\vec{a} \neq \vec{0}$.
4. Theorem. As in the answer to Exercise 3, if $\vec{a} \neq \vec{0}$ then $(\vec{a})$ is not linearly dependent. So, if $(\vec{a})$ is linearly dependent then $\vec{a} = \vec{0}$. On the other hand, since $\vec{0}1 = \vec{0}$ and $1 \neq 0$, the sequence $(\vec{0})$ is linearly dependent. In other words, if $\vec{a} = \vec{0}$ then $(\vec{a})$ is linearly dependent. Hence, $(\vec{a})$ is linearly dependent if and only if $\vec{a} = \vec{0}$.
5. False. For a counter example, see Exercise 6.

1. If $\vec{0}$ is a term of a sequence of vectors then the sequence is linearly dependent. [*Hint*: This sentence is a theorem, and one proof goes as follows:

   Suppose that $(\vec{a}_1, \ldots, \vec{a}_n)$ is a sequence such that $\vec{a}_i = \vec{0}$, where i is an integer between 1 and n, inclusive. Choose $a_i =$ ____ and, for $j \neq i$, choose $a_j =$ ____. Then $\vec{a}_1a_1 + \ldots + \vec{a}_na_n = \vec{a}_i$ ____ $= \vec{0}$ and, since ____ $\neq 0$, not all the numbers $a_1, \ldots, a_n$ are zero. So, by definition, ...]
2. If two terms of a sequence of vectors are equal then the sequence is linearly dependent.
3. Any 1-termed sequence is linearly dependent.
4. $(\vec{a})$ is linearly dependent if and only if $\vec{a} = \vec{0}$.
5. No 3-termed sequence is linearly dependent.
6. $(B - A, C - A, C - B)$ is linearly dependent.
7. (a) $(\vec{a}, \vec{ac})$ is linearly dependent
   (b) $(\vec{ab}, \vec{ac})$ is linearly dependent
8. (a) If $(\vec{b}, \vec{c})$ is linearly dependent then so is $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$.
   (b) If $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly dependent then so is $(\vec{b}, \vec{c})$.
9. (a) If $(\vec{a}, \vec{b})$ is linearly dependent then so is $(\vec{b}, \vec{a})$.
   (b) If $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly dependent then so is $(\vec{c}, \vec{d}, \vec{b}, \vec{a})$.
10. (a) For $n \geq 2$, if one term [at least] of an n-termed sequence is a linear combination of the others then the sequence is linearly dependent.
    (b) For $n \geq 2$, if an n-termed sequence is linearly dependent then one of its terms [at least] is a linear combination of the others.

*

Some of the theorems in Part B are worth assigning numbers to:

Theorem 6-1
$(\vec{a})$ is linearly dependent $\Longleftrightarrow \vec{a} = \vec{0}$.

Theorem 6-2 For $n \geq 2$,
$(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)$ is linearly dependent

$\Longleftrightarrow$

one of the vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is a linear combination of the others.

Theorems 6-1 and 6-2 can be used to advantage in showing that a given sequence is linearly dependent. [For a 1-termed sequence, use Theorem 6-1; for a "longer" sequence use Theorem 6-2.]

6. By Postulate 3, $(B - A) + (C - B) = C - A$. So, $(B - A)1 + (C - A) \cdot -1 + (C - B)1 = \vec{0}$. Since $1 \neq 0$ it follows, by definition, that $(B - A, C - A, C - B)$ is linearly dependent.
7. Theorems. (b) $(\vec{a}b) \cdot -c + (\vec{a}c)b = \vec{a}(b \cdot -c + c \cdot b) = \vec{a}0 = \vec{0}$. So, if not both $-c$ and b are zero then $(\vec{a}b, \vec{a}c)$ is linearly dependent. Of course, if either $b = 0$ or $c = 0$, the same result follows by Theorem 5-1(a) and Exercise 1. Of course, (a) follows from (b) and Postulate 4_5.
8. (a) Theorem. Suppose that $(\vec{b}, \vec{c})$ is linearly dependent. It follows that there are numbers — say, b and c — not both 0, such that $\vec{b}b + \vec{c}c = \vec{0}$. It follows from this that $\vec{a}0 + \vec{b}b + \vec{c}c + \vec{d}0 = \vec{0}$ and that 0, b, c, and 0 are not all 0. Hence, $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly dependent. Consequently, if $(\vec{b}, \vec{c})$ is linearly dependent then so is $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$.
   (b) False. $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly dependent, for any choice of $\vec{b}$, $\vec{c}$, and $\vec{d}$. And it is possible to choose $\vec{b}$ and $\vec{c}$ so that $(\vec{b}, \vec{c})$ is not linearly dependent. [Actually, the last claim holds up only in vector spaces of dimension at least 2. For 1-dimensional vector spaces — such as $\mathcal{R}$ — (b) is a theorem.]
9. (a) Theorem. $\vec{a}a + \vec{b}b = \vec{b}b + \vec{a}a$ and a and b are not both 0 if and only if b and a are not both 0. So, by definition, $(\vec{a}, \vec{b})$ is linearly dependent if and only if $(\vec{b}, \vec{a})$ is linearly dependent.
   (b) Theorem. [Proof similar to that for (a).]
10. (a) Theorem. Suppose that $(\vec{a}_1, \ldots, \vec{a}_n)$ is a sequence one of whose terms — say $\vec{a}_i$ — is a linear combination of the others. [This is possible only if $n \geq 2$] It follows by Definition 6-1 that there are numbers — say a_j, $j \neq i$ — such that $\vec{a}_i$ is the sum of the vectors $\vec{a}_ja_j$, $j \neq i$. It follows that if $a_i = -1$ then $\vec{a}_1a_1 + \ldots + \vec{a}_na_n = \vec{0}$ and [since $-1 \neq 0$] that not all $a_1, \ldots, a_n$ are 0. So, by Definition 6-2, $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly dependent.
    (b) Theorem. Suppose that $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly dependent. It follows by Definition 6-2 that there are numbers — say $a_1, \ldots, a_n$ — not all 0, such that $\vec{a}_1a_1 + \ldots + \vec{a}_na_n = \vec{0}$. Since $a_1, \ldots, a_n$ are not all 0, at least one of them — say, a_i — is not 0. It follows that $a_i \cdot / a_i = 1$ and that $\vec{a}_1(a_1 \cdot / a_i) + \ldots + \vec{a}_n(a_n \cdot / a_i) = \vec{0} \cdot / a_i = \vec{0}$. Hence, $\vec{a}_i = \vec{a}_i(a_i \cdot / a_i)$ is [in case $n \geq 2$] the sum of the multiples $\vec{a}_j \cdot -(a_j \cdot / a_i)$ for $j \neq i$. So, by Definition 6-1, $\vec{a}_i$ is a linear combination of the vectors $\vec{a}_1, \ldots, \vec{a}_n$ other than $\vec{a}_i$. Hence, for $n \geq 2$, if $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly dependent then one of its terms is a linear combination of the others.

Theorem 6-3

(a) A sequence one of whose terms is $\vec{0}$ is linearly dependent.

(b) A sequence two of whose terms are equal is linearly dependent.

Exercises

Part A

1. Use Theorem 6-2 to prove Theorem 6-3(b). [*Hint*: Is $\vec{a}$ a linear combination of $\vec{a}$?]
2. Use Theorems 6-1 and 6-2 to prove Theorem 6-3(a). [*Hint*: Is $\vec{0}$ a linear combination of $\vec{a}$?]

Part B

1. Suppose that $\vec{a}2 + \vec{b} \cdot -3 + \vec{c}\frac{3}{2} = \vec{0}$.
   (a) Is $(\vec{a}, \vec{b}, \vec{c})$ linearly dependent or not? Explain your answer.
   (b) Complete each of the following (if possible):
      (i) $\vec{a} + \vec{b}$ ____ $+ \vec{c}$ ____ $= \vec{0}$ (ii) $\vec{a} = \vec{b}$ ____ $+ \vec{c}$ ____
      (iii) $\vec{b} = \vec{a}$ ____ $+ \vec{c}$ ____ (iv) $\vec{c} = \vec{a}$ ____ $+ \vec{b}$ ____
2. Suppose that $\vec{a}4 + \vec{b}\frac{2}{7} + \vec{c}0 + \vec{d}\frac{5}{2} = \vec{0}$.
   (a) Is $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ linearly dependent or not? Explain your answer.
   (b) Complete each of the following (if possible):
      (i) $\vec{a} = \vec{b}$ ____ $+ \vec{c}$ ____ $+ \vec{d}$ ____ (ii) $\vec{b} =$ ________
      (iii) $\vec{c} =$ ________ (iv) $\vec{d} =$ ________
   (c) Is $(\vec{a}, \vec{b}, \vec{d})$ linearly dependent?
   (d) Let $\vec{e}$ be any other vector. Is $(\vec{a}, \vec{b}, \vec{d}, \vec{e})$ linearly dependent? Is $(\vec{a}, \vec{e}, \vec{b}, \vec{d})$ linearly dependent? Explain your answers.
3. Suppose that A, B, and C are three points of $\mathscr{E}$ such that $A + \vec{a} = B$ and $A + \vec{a}3 = C$ for some translation $\vec{a}$.
   (a) Draw a picture which illustrates the conditions of this problem.
   (b) Determine whether or not $(C - A, B - A)$ is linearly dependent.
   (c) In your picture for (a), draw a graph of a point D where $D = A + \vec{a} \cdot -2$.
   (d) Determine whether or not $(D - A, C - A)$ is linearly dependent.
4. In addition to the points and the translation $\vec{a}$ of Exercise 3, consider a translation $\vec{b}$ which is not a multiple of $\vec{a}$.
   (a) In your picture for Exercise 3(a), draw the graph of a point E, where $E = A + \vec{b}$.
   (b) Do you think that $(E - A, B - A)$ is linearly dependent? Why?
   (c) Do you think that $(D - A, E - A)$ is linearly dependent? Why?
   (d) In view of your answers in parts (b) and (c), in how many ways can you complete the following sentences to make true statements?

Answers for Part A

1. Suppose that two terms of a sequence are equal. It follows that [either] one of these terms is a linear combination of the other and, so, is a linear combination of all the other terms of the sequence. Hence, by Theorem 6-2, the sequence is linearly dependent. Hence, any sequence two of whose terms are equal is linearly dependent.

2. By Theorem 6-1, a 1-termed sequence whose single term is $\vec{0}$ is linearly dependent. Suppose, now, that a sequence of two or more terms has $\vec{0}$ as one of its terms. Since $\vec{0}$ is a linear combination of any vectors it follows that this term of the sequence is a linear combination of the other terms. So, by Theorem 6-2, the sequence is linearly dependent. Since any sequence is either a 1-termed sequence or has two or more terms it follows that any sequence which has $\vec{0}$ as one of its terms is linearly dependent.

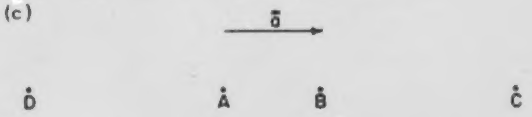
[As given above, neither of these proofs is likely to seem simpler than the proofs of the same theorems given in answer to Exercises 2 and 1, respectively, of Part B on page 221. The principle excuse for asking for them is that two proofs of a given theorem relate this theorem in different ways to other theorems and, so, give a better understanding of the theory under development. Also, the more proofs one has studied, the better one's chances of finding proofs for future theorems.]

Answers for Part B

1. (a) From the given assumption and the fact that 2, −3, and 3/2 are not all 0 it follows, by definition, that $(\vec{a}, \vec{b}, \vec{c})$ is linearly dependent.
   (b) (i) $\cdot -\frac{3}{2}$; $\frac{3}{4}$ (ii) $\frac{3}{2}$; $\cdot -\frac{3}{4}$ (iii) $\frac{2}{3}$; $\frac{1}{2}$ (iv) $\cdot -\frac{4}{3}$; 2

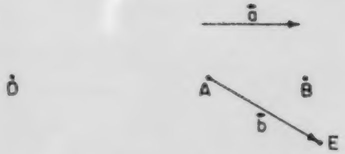
2. (a) Yes. The numbers 4, 2/7, 0, and 5/2 are not all 0.
   (b) (i) $\cdot -\frac{1}{14}$; 0; $\cdot -\frac{5}{8}$ (ii) $\vec{a} \cdot -14 + \vec{c}0 + \vec{d} \cdot -\frac{35}{4}$
   (iii) [impossible] (iv) $\vec{a} \cdot -\frac{8}{5} + \vec{b} \cdot -\frac{4}{35} + \vec{c}0$
   [Other correct answers for (ii) and (iv) may be obtained by permuting terms or omitting '$\vec{c}0$'.]
   (c) Yes.
   (d) Yes [to both questions]. $\vec{a}4 + \vec{b}\frac{2}{7} + \vec{d}\frac{5}{2} + \vec{e}0 = \vec{0}$ and not all of the numbers 4, $\frac{2}{7}$, $\frac{5}{2}$, 0 are 0; $\vec{a}4 + \vec{e}0 + \vec{b}\frac{2}{7} + \vec{d}\frac{5}{2} = \vec{0}$ and not all of the numbers 4, 0, $\frac{2}{7}$, $\frac{5}{2}$ are 0.

3. (a), (c)



   (b) $(C - A, B - A) = (\vec{a}3, \vec{a})$ and is linearly dependent since $(\vec{a}3)1 + \vec{a} \cdot -3 = \vec{0}$ and $1 \neq 0$.
   (d) $(D - A, C - A) = (\vec{a} \cdot -2, \vec{a}3)$ and is linearly dependent since $(\vec{a} \cdot -2)3 + (\vec{a}3)2 = \vec{0}$ and $3 \neq 0$.

4. (a)



(b) No. $(E - A, B - A) = (\vec{a}, \vec{b})$, and $(\vec{a}, \vec{b})$ is not linearly dependent. For, suppose that $\vec{a}a + \vec{b}b = \vec{0}$. Since $\vec{b}$ is not a multiple of $\vec{a}$ it follows that $b = 0$ and, so, that $\vec{a}a = \vec{0}$. Since, in Exercise 3, A, B, and C are three points, $A \neq B$ and so, $\vec{a} \neq \vec{0}$. So, $a = 0$.

(c) No. $(D - A, E - A) = (\vec{a} \cdot -2, \vec{b})$. Suppose that $(\vec{a} \cdot -2)a + \vec{b}b = \vec{0}$. It follows that $\vec{a} \cdot -2a + \vec{b}b = \vec{0}$ and so, by part (b), that $-2a = 0$ and $b = 0$ — that is, that $a = 0$ and $b = 0$. So, $(D - A, E - A)$ is not linearly dependent.

(d) Each can be completed in [essentially] only one way. ['essentially' because, although each blank must be filled by a numeral for 0, 0 has many names.]

[In answering (b) and (c), students may give less compelling reasons — such as '$\vec{a}$ and $\vec{b}$ have different directions' in support of their answers. This is all right, but it should be pointed out to them that the arithmetic justifications given above are not beyond their powers.]

Answers for Part B [cont.]

5. 0; 0

If $\vec{a} = \vec{0}$, the different subsequences of $(\vec{a}, \vec{0}, \vec{0})$ are $(\vec{0})$, $(\vec{0}, \vec{0})$ and $(\vec{0}, \vec{0}, \vec{0})$. There are five types of 3-termed sequences. These are exemplified by $(\vec{a}, \vec{a}, \vec{a})$, $(\vec{a}, \vec{a}, \vec{b})$, $(\vec{a}, \vec{b}, \vec{a})$, $(\vec{b}, \vec{a}, \vec{a})$, and $(\vec{a}, \vec{b}, \vec{c})$, where $\vec{a}$, $\vec{b}$, and $\vec{c}$ are different vectors. A sequence of one of these types is easily seen to have 3, 5, 6, 5, or 7 subsequences, respectively. [This answers the bracketed questions. They are of no importance beyond being thought-provoking and generative of a little combinatorial activity.]

The explanation asked for in connection with Theorem 6-4 is that $\vec{b}b + \vec{d}d = \vec{a}0 + \vec{b}b + \vec{c}0 + \vec{d}d$, and 0, b, 0, and d are not all zero if and only if b and d are not both 0.

The suggested proof of Theorem 6-5 goes as follows: A sequence one of whose terms is a multiple of another has, for some $\vec{a}$ and some c, a subsequence $(\vec{a}, \vec{a}c)$ or a subsequence $(\vec{a}c, \vec{a})$. By Exercises 7 and 9(a) on page 221, any such subsequence is linearly dependent. Hence, by Theorem 6-4, the given sequence is linearly dependent.

* * *

A rigorous proof of Theorem 6-4 [and one of Theorem 6-6] requires the use of mathematical induction on the number of terms in the given sequence. For completeness, we shall outline proofs of these theorems in some detail. As a basis, we need definitions of 'subsequence' and 'permutation':

$$(\vec{b}_1, \ldots, \vec{b}_m) \text{ is a subsequence of } (\vec{a}_1, \ldots, \vec{a}_n)$$
$$\Longleftrightarrow$$
$$\text{there is an order-preserving mapping } f \text{ of } \{1, \ldots, m\} \text{ into } \{1, \ldots, n\} \text{ such that, for } 1 \leq i \leq m,\ \vec{b}_i = \vec{a}_{f(i)}$$

To define 'permutation', merely replace, in the preceding definition, 'subsequence' by 'permutation', replace 'an order-preserving' by 'a one-to-one', and replace 'into' by 'onto'. [In the case of the former definition one can prove that $m \leq n$; in the case of the latter, that $m = n$.] Recall that an order-preserving mapping is one-to-one and that a one-to-one mapping has an inverse. Now, given a one-to-one mapping f of $\{1, \ldots, m\}$ into $\{1, \ldots, n\}$, and given numbers $b_1, \ldots, b_m$, define numbers $a_1, \ldots, a_n$ by:

$$a_j = b_{f^{-1}(j)} \text{ if } j \in \mathfrak{R}f \text{ and } a_j = 0 \text{ if } j \notin \mathfrak{R}f.$$

It follows that, for $1 \leq i \leq m$, $a_{f(i)} = b_i$ and, so, that

$$\vec{a}_{f(1)}a_{f(1)} + \ldots + \vec{a}_{f(m)}a_{f(m)} = \vec{b}_1 b_1 + \ldots + \vec{b}_m b_m.$$

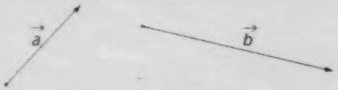
Hence, to prove Theorem 6-4 or Theorem 6-6 amounts to proving that

$$(\star) \quad \vec{a}_1 a_1 + \ldots + \vec{a}_n a_n = \vec{a}_{f(1)}a_{f(1)} + \ldots + \vec{a}_{f(m)}a_{f(m)}$$

under the appropriate assumptions on f — that, besides being one-to-

(i) $(E - A)$ ____ $+ (B - A)$ ____ $= \vec{0}$
(ii) $(D - A)$ ____ $+ (E - A)$ ____ $= \vec{0}$

5. Here is a figure illustrating two vectors $\vec{a}$ and $\vec{b}$ such that $(\vec{a}, \vec{b})$ is *not* linearly dependent.



Given that a and b are real numbers such that $\vec{a}a + \vec{b}b = \vec{0}$, it follows that $a =$ ____ and $b =$ ____.

6.04 Subsequences and Permutations of Sequences

Given a sequence $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$, the sequences

$(\vec{a}, \vec{c})$, $(\vec{b}, \vec{c}, \vec{d})$, and $(\vec{c})$ [as well as some others]

are subsequences of the given sequence. More explicitly,

> a *subsequence* of a given sequence is a sequence whose terms are some [or all] of the terms of the given sequence, in the same order which they have in the given sequence.

For example, the subsequences of $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ are

$$(\vec{a}_1),\ (\vec{a}_2),\ (\vec{a}_3),\ (\vec{a}_1, \vec{a}_2),\ (\vec{a}_1, \vec{a}_3),\ (\vec{a}_2, \vec{a}_3) \text{ and } (\vec{a}_1, \vec{a}_2, \vec{a}_3).$$

If $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is not a sequence of distinct terms then the subsequences listed above may not all be different. For example, the only subsequences of $(\vec{a}, \vec{0}, \vec{0})$ are $(\vec{a})$, $(\vec{0})$, $(\vec{a}, \vec{0})$, $(\vec{0}, \vec{0})$, and $(\vec{a}, \vec{0}, \vec{0})$. These are different from one another if and only if $\vec{a} \neq \vec{0}$. Note that, in this case, neither $(\vec{a}, \vec{a})$ nor $(\vec{0}, \vec{a})$ is a subsequence of the given sequence. [Can you describe a 3-termed sequence which has exactly six subsequences? One which has exactly four subsequences?]

It is almost obvious that if $(\vec{b}, \vec{d})$, say, is linearly dependent then so is $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$. [Explain.] Arguing in the same way we could prove:

> **Theorem 6-4**
> If any subsequence of a given sequence is linearly dependent then the given sequence is linearly dependent.

Using this theorem and two of the theorems you proved in the preceding exercises, it is easy to prove:

---

one, f is order-preserving or onto, respectively. Under the former assumption, (☆) follows from the fact that, for any vector $\vec{a}$, $\vec{a}0 = \vec{0}$ and $\vec{0} + \vec{a} = \vec{a} = \vec{a} + \vec{0}$. Under the latter assumption, (☆) follows from the commutativity and associativity of addition of vectors.

The proof of (☆) under the former assumption is not difficult for one who is familiar with mathematical induction. The initial step consists in noting that in case $n = 1$ it follows that $m = 1$ and $f(1) = 1$. So, in this case (☆) reduces to the valid sentence '$\vec{a}_1a_1 = \vec{a}_1a_1$'. For the inductive step one assumes that $n > 1$ and takes as inductive hypothesis the sentence obtained from (☆) by replacing 'n' by 'n - 1' and — for clarity — 'f' by 'g' and 'm' by 'p' where g is any order-preserving mapping of $\{1, \ldots, p\}$ into $\{1, \ldots, n - 1\}$. There are two cases to consider in deriving (☆) — that in which $n \notin \mathcal{R}f$ and that in which $n \in \mathcal{R}f$. In the first case f is an order-preserving mapping of $\{1, \ldots, m\}$ into $\{1, \ldots, n - 1\}$ and we can replace 'g' in the inductive hypothesis by 'f' and 'p' by 'm'. The proof of (☆) is completed in this case by noting that, since $n \notin \mathcal{R}f$, a_n is, by definition 0 and, so, $\vec{a}_na_n = \vec{0}$ and

$$\vec{a}_1a_1 + \ldots + \vec{a}_na_n = \vec{a}_1a_1 + \ldots + \vec{a}_{n-1}a_{n-1}.$$

In case $n \in \mathcal{R}f$ then, since f is order-preserving, $n = f(m)$. It follows that $\vec{a}_na_n = \vec{a}_{f(m)}a_{f(m)}$ and that the restriction of f to $\{1, \ldots, m - 1\}$ is an order-preserving mapping of the latter into $\{1, \ldots, n - 1\}$. So, replacing 'p' in the inductive hypothesis by 'm - 1', we may take the restriction of f to $\{1, \ldots, m - 1\}$ for g. (☆) now follows from this instance of the inductive assumption and an instance of the valid sentence '$\vec{a} = \vec{b} \Longrightarrow \vec{a} + \vec{c} = \vec{b} + \vec{c}$'. [In case $m = 1$, 'm - 1' may not be substituted for 'p' in the inductive hypothesis. But, in this case, (☆) reduces to a sentence of the form '$\vec{c}_1 + \ldots + \vec{c}_n = \vec{c}_n$' where, for $1 \leq i \leq n - 1$, $\vec{c}_i = \vec{0}$. This can be proved by induction, using the fact that, for any $\vec{a}$, $\vec{0} + \vec{a} = \vec{a}$. (As a consequence of the obvious inductive hypothesis, for $n > 1$, $\vec{c}_1 + \ldots + \vec{c}_{n-1} = \vec{0}$ if $\vec{c}_{n-1}$, as well as $\vec{c}_i$, $1 \leq i \leq n - 2$, is $\vec{0}$. So, if $\vec{c}_i = \vec{0}$ for $1 \leq i \leq n - 1$ then $\vec{c}_1 + \ldots + \vec{c}_n = \vec{0} + \vec{c}_n = \vec{c}_n$.)]

The proof of Theorem 6-6 proceeds, for a time, much like the preceding proof of Theorem 6-4. The initial step and the first case of the inductive step are exactly as above, with 'order-preserving' replaced by 'one-to-one'. The second case of the inductive step is more complicated. Since f is not assumed to be order-preserving we may not conclude in case $n \in \mathcal{R}f$ that $n = f(m)$. All we can be sure of is that, for some k, $n = f(k)$. In case $i = m$ we may proceed as above. If not, we must use commutativity and associativity to show that kth term on the right side of (☆) "may" be shifted to the end of this indicated sum. [This can be done most easily with $(m - k)$ applications of the "switch principle" '$(\vec{a} + \vec{b}) + \vec{c} = (\vec{a} + \vec{c}) + \vec{b}$'. For $k = 1$, first commute then apply the switch principle $m - 2$ times.] We then proceed as before, but instead of taking g to be the restriction of f to $\{1, \ldots, m - 1\}$, we define 'g' so that $g(i) = f(i)$ for $1 \leq i \leq k$ and $g(i) = f(i + 1)$ for $k \leq i \leq m - 1$.

The argument sketched in the preceding paragraph establishes the only if-part of Theorem 6-6. The if-part follows at once since if a given sequence is a permutation of another then this other sequence is a permutation of the given one.

Theorem 6-5
If any term of a given sequence is a multiple of another term then the given sequence is linearly dependent.

Prove Theorem 6-5 now.

*

Given a sequence $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$, such sequences as

$$(\vec{c}, \vec{d}, \vec{b}, \vec{a}),\ (\vec{a}, \vec{c}, \vec{d}, \vec{b}),\ (\vec{d}, \vec{b}, \vec{a}, \vec{c}),\ (\vec{a}, \vec{b}, \vec{c}, \vec{d}), \text{ etc.}$$

are permutations of the given sequence. More explicitly,

> a *permutation* of a given sequence is a sequence whose terms are those of the given sequence, but not necessarily in the same order.

For example, the permutations of $(\vec{a}, \vec{b})$ are $(\vec{a}, \vec{b})$ and $(\vec{b}, \vec{a})$; those of $(\vec{a}, \vec{b}, \vec{c})$ are

$$(\vec{a}, \vec{b}, \vec{c}),\ (\vec{b}, \vec{a}, \vec{c}),\ (\vec{a}, \vec{c}, \vec{b}),\ (\vec{b}, \vec{c}, \vec{a}),\ (\vec{c}, \vec{a}, \vec{b}), \text{ and } (\vec{c}, \vec{b}, \vec{a}).$$

For sequences of distinct terms, a 2-termed sequence has two permutations and a 3-termed sequence has six permutations. If you compare the listing we have given for the permutations of $(\vec{a}, \vec{b}, \vec{c})$ with that for the permutations of $(\vec{a}, \vec{b})$ it should be easy to compute the number of permutations of a 4-termed sequence $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ of distinct terms.

As you saw by two examples in Exercise 9 of Part B on page 221, whether or not a sequence $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly dependent—that is, whether or not there are numbers $x_1, \ldots, x_n$, not all zero, such that

$$\vec{a}_1 x_1 + \ldots + \vec{a}_n x_n = \vec{0}$$

—does not depend on the order of the terms of the sequence. This is so because of two properties of addition of vectors. What two properties? For example, for any vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ and any real numbers a, b, and c,

$$\vec{b}b + \vec{c}c + \vec{a}a = \vec{a}a + \vec{b}b + \vec{c}c. \text{ [Why?]}$$

So,

$$\vec{b}b + \vec{c}c + \vec{a}a = \vec{0} \Longleftrightarrow \vec{a}a + \vec{b}b + \vec{c}c = \vec{0}.$$

Since, obviously,

$$b, c, \text{ and } a \text{ are not all zero} \Longleftrightarrow a, b, \text{ and } c \text{ are not all zero}$$

The permutations of the sequence $(\vec{a}, \vec{b}, \vec{c})$ were listed by modifying the two permutations of $(\vec{a}, \vec{b})$ in three, successive, ways — first by inserting '$\vec{c}$' at the end, then in the middle, and finally at the beginning. The permutations of $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ can be obtained in the same way by modifying those of $(\vec{a}, \vec{b}, \vec{c})$. Since there are 6 of the latter and, in each, 4 places to insert '$\vec{d}$' there are $6 \cdot 4$ permutations of a 4-termed sequence. [As is the case with counting subsequences, counting permutations plays no role in this course.]

It should be intuitively clear that Theorem 6-6 is a consequence of the commutative and associative principles for addition of vectors. Since, as indicated in the immediately preceding commentary, a proof of Theorem 6-6 is quite involved, it would not be sensible to attempt to give one in the text. You might, in class, point out that proving the theorem in the case of 3-termed sequences amounts to proving that, for any vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$,

$$\vec{a} + \vec{b} + \vec{c} = \vec{b} + \vec{a} + \vec{c},\ \vec{a} + \vec{b} + \vec{c} = \vec{a} + \vec{c} + \vec{b},\ \vec{a} + \vec{b} + \vec{c} = \vec{b} + \vec{c} + \vec{a},$$
$$\vec{a} + \vec{b} + \vec{c} = \vec{c} + \vec{a} + \vec{b}, \text{ and } \vec{a} + \vec{b} + \vec{c} = \vec{c} + \vec{b} + \vec{a}.$$

Proving the theorem in the case of 4-termed sequences would amount to proving 23 similarly trivial theorems.

Students are not likely to question Theorem 6-4; and they will have some opportunity to use it, particularly in the guise of its contrapositive:

> If a sequence is not linearly dependent
> then none of its subsequences is linearly dependent.

On the other hand, Theorem 6-6 may raise some questions. Theorem 6-6 is perhaps even more obvious than is Theorem 6-4, but why go to the trouble of introducing sequences — in which order is important — when you can prove a theorem to the effect that order doesn't matter? A partial answer is given on TC 212(1). If one did define linear dependence for sets rather than sequences, a similar theorem would be required, anyway, to justify the definition. A somewhat better answer is that the notion of a sequence of vectors is needed later in contexts in which order is relevant, and the question of linear dependence of such sequences will arise. Such a question cannot be answered by considering the range of the sequence since the sequence may be linearly dependent because it has repeated terms and this cannot be determined by looking at its range. So, if we began by defining the notion of linear dependence of sets of vectors, we should have to start afresh when it became necessary to deal with linear dependence of sequences. Finally, although linear dependence of sequences cannot conveniently be defined in terms of linear dependence of sets, linear dependence of sets can be defined, both conveniently and naturally, in terms of linear dependence of sequences [see Part E on page 230]. [Linear dependence of sequences can be defined in terms of linear dependence of sets:

> A sequence is linearly dependent if and only if either
> it has repeated terms or its range is linearly dependent.

But, "disjunctive" definitions such as this are difficult to use since one is forced, each time, to consider two cases. Also, the suggested definition is somewhat "unnatural" without rather careful motivation.]

Incidentally, Theorem 6-6 can be used to simplify many proofs. As an example, consider Exercise 10(a) on page 221. [This is a proof of the if-part of the very important Theorem 6-2.] Suppose that $(\vec{a}_1, \ldots, \vec{a}_n)$ is a sequence one of whose terms is a linear combination of the other. Let $(\vec{b}_1, \ldots, \vec{b}_n)$ be a permutation of the given sequence such that $\vec{b}_n$ is a linear combination of $\vec{b}_1, \ldots, \vec{b}_{n-1}$. It follows that there are numbers — say, $b_1, \ldots, b_{n-1}$ — such that $\vec{b}_n = \vec{b}_1 b_1 + \ldots + \vec{b}_{n-1} b_{n-1}$. From this [since $\vec{b}_n + \vec{b}_n \cdot -1 = \vec{0}$] it follows that $\vec{b}_1 b_1 + \ldots + \vec{b}_{n-1} b_{n-1} + \vec{b}_n \cdot -1 = \vec{0}$. Since $-1 \neq 0$, $(\vec{b}_1, \ldots, \vec{b}_n)$ is linearly dependent. [Note that the algebraic difficulties have been relegated to the proof of Theorem 6-6. Hence, even if one takes the trouble to prove this theorem, these difficulties need be surmounted only once.]

Presenting this proof to your class may induce respect for Theorem 6-6. You can then ask them to establish the only-if part of Theorem 6-2 in the same manner. [Suppose that $(\vec{a}_1, \ldots, \vec{a}_n)$ is an at least 2-termed linearly dependent sequence. It follows that there is a permutation $(\vec{b}_1, \ldots, \vec{b}_n)$ of this sequence and numbers — say, $b_1, \ldots, b_n$ — such that $b_n \neq 0$ and $\vec{b}_1 b_1 + \ldots + \vec{b}_{n-1} b_{n-1} + \vec{b}_n b_n = \vec{0}$. Etc.]

it follows that

$(\vec{b}, \vec{c}, \vec{a})$ is linearly dependent $\longleftrightarrow$ $(\vec{a}, \vec{b}, \vec{c})$ is linearly dependent.

> **Theorem 6-6**
> A permutation of a given sequence is linearly dependent
>
> if and only if
>
> the given sequence is linearly dependent.

Exercises

Part A

1. Use Theorem 6-4 to prove:
   $(\vec{a}, \vec{b}, \vec{c})$ is not linearly dependent $\longrightarrow$ $(\vec{a}, \vec{c})$ is not linearly dependent
2. Use Theorem 6-6 to prove:
   $(\vec{a}, \vec{b}, \vec{c})$ is not linearly dependent $\longrightarrow$ $(\vec{a}, \vec{c}, \vec{b})$ is not linearly dependent
3. Prove:

   $(\vec{a}, \vec{c}, \vec{b})$ is not linearly dependent $\longleftrightarrow$ $(\vec{a}, \vec{b}, \vec{c})$ is not linearly dependent

4. Show that any inference of either of the forms:

   $$\frac{p \longleftrightarrow q}{\text{not } p \longleftrightarrow \text{not } q} \qquad \frac{\text{not } p \longleftrightarrow \text{not } q}{p \longleftrightarrow q}$$

   is valid.

Part B

As you learned in Chapter 4, the sentence:

(*) $(a \neq 0 \text{ or } b \neq 0) \longleftrightarrow \text{not } (a = 0 \text{ and } b = 0)$

is valid – in particular, the sentences to the left and the right of the '$\longleftrightarrow$' are different ways of saying the same thing. In English we might say, instead of (*):

One [at least] of a and b is not 0 if and only if
not both of a and b are 0.

Since both sides of (*) say the same thing, so do their denials. Hence, using the rules of double denial, the sentence:

(**) $\text{not } (a \neq 0 \text{ or } b \neq 0) \longleftrightarrow (a = 0 \text{ and } b = 0)$

is valid.

Answers for Part A

1. Since $(\vec{a}, \vec{c})$ is a subsequence of $(\vec{a}, \vec{b}, \vec{c})$ it follows from Theorem 6-4 that if $(\vec{a}, \vec{c})$ is linearly dependent then so is $(\vec{a}, \vec{b}, \vec{c})$. Hence [by contraposition] if $(\vec{a}, \vec{b}, \vec{c})$ is not linearly dependent then $(\vec{a}, \vec{c})$ is not linearly dependent.

2. Since $(\vec{a}, \vec{c}, \vec{b})$ is a permutation of $(\vec{a}, \vec{b}, \vec{c})$ it follows by Theorem 6-6 that $(\vec{a}, \vec{c}, \vec{b})$ is linearly dependent if and only if $(\vec{a}, \vec{b}, \vec{c})$ is linearly dependent. Since [trivially] if $(\vec{a}, \vec{b}, \vec{c})$ is not linearly dependent then $(\vec{a}, \vec{b}, \vec{c})$ is not linearly depedent it follows [by biconditional replacement] that if $(\vec{a}, \vec{b}, \vec{c})$ is not linearly dependent then $(\vec{a}, \vec{c}, \vec{b})$ is not linearly dependent. [A proof like that in Exercise 1 can be given, using the only if-part of Theorem 6-6. But, it is wise to get the habit of using biconditional replacement when it is available. Using it simplifies many proofs and, without such a habit, one may forget this.]

3. Since $(\vec{a}, \vec{c}, \vec{b})$ is a permutation of $(\vec{a}, \vec{b}, \vec{c})$ it follows by Theorem 6-6 that $(\vec{a}, \vec{c}, \vec{b})$ is linearly dependent if and only if $(\vec{a}, \vec{b}, \vec{c})$ is linearly dependent. Since [trivially] $(\vec{a}, \vec{b}, \vec{c})$ is not linearly dependent if and only if $(\vec{a}, \vec{b}, \vec{c})$ is not linearly dependent it follows [by biconditional replacement] that $(\vec{a}, \vec{c}, \vec{b})$ is not linearly dependent if and only if $(\vec{a}, \vec{b}, \vec{c})$ is not linearly dependent. [Some students may establish the if-part of this theorem in Exercise 2 by an argument like that given in Exercise 1, then establish the only if-part by a similar argument, and finally combine the two. In outline, their procedure is indicated at the left:

   $$\dfrac{\dfrac{\dfrac{p \Longleftrightarrow q}{q \Longrightarrow p}}{\sim p \Longrightarrow \sim q} \qquad \dfrac{\dfrac{p \Longleftrightarrow q}{p \Longrightarrow q}}{\sim q \Longrightarrow \sim p}}{\sim q \Longleftrightarrow \sim p} \qquad\qquad \dfrac{p \Longleftrightarrow q \qquad \sim p \overset{*}{\Longleftrightarrow} \sim p}{\sim q \Longleftrightarrow \sim p}$$

   That indicated on the right — the '*' indicates that sentences of the indicated form are valid — is very obviously simpler.]

4. For inferences of the first form see the right hand figure, above. Some discussion of the "rule for ignoring valid premisses" may be in order. This rule is discussed in the text on pages 81 and 82 and, more formally, in the commentary for page 75. For inferences of the second form:

   $$\dfrac{\sim\sim q \overset{*}{\Longleftrightarrow} q \qquad \dfrac{\sim\sim p \overset{*}{\Longleftrightarrow} p \qquad \dfrac{\sim p \Longleftrightarrow \sim q}{\sim\sim p \Longleftrightarrow \sim\sim q}}{p \Longleftrightarrow \sim\sim q}}{p \Longleftrightarrow q}$$

1. Give an "English translation" of (**).
2. The sentence '$a_1, \ldots, a_n$ are not all zero' can be restated [in English] in two ways. Complete:
   (a) not all of __________ **(b)** one [at least] of __________
3. The sentence '$a_1, \ldots, a_n$ are all zero' can be restated in two ways. Complete:
   (a) all of __________ **(b)** none of __________

Part C

By definition, to say that a sequence $(\vec{a}, \vec{b}, \vec{c})$ *is* linearly dependent amounts to saying that there are numbers x, y, and z which are not all 0 and which are such that $\vec{a}x + \vec{b}y + \vec{c}z = \vec{0}$.

So, to say that a sequence $(\vec{a}, \vec{b}, \vec{c})$ is *not* linearly dependent amounts to saying that there are no such numbers – that is, that there are no numbers x, y, and z such that x, y, and z are not all 0 and $\vec{a}x + \vec{b}y + \vec{c}z = \vec{0}$. Another way of saying this is to say that, for all numbers x, y, and z, $\vec{a}x + \vec{b}y + \vec{c}z \neq \vec{0}$ unless $(x = 0$ and $y = 0$ and $z = 0)$. So, it turns out that

$$(\vec{a}, \vec{b}, \vec{c}) \text{ is not linearly dependent}$$

$$\Longleftrightarrow$$

$$\text{for all } x, y, \text{ and } z, \text{ if } \vec{a}x + \vec{b}y + \vec{c}z = \vec{0} \\ \text{then } (x = 0 \text{ and } y = 0 \text{ and } z = 0).$$

1. Suppose that $(\vec{a}, \vec{b})$ is linearly dependent – to make it simple, suppose that $\vec{a}3 + \vec{b}2 = \vec{0}$. Let $\vec{c} = \vec{a} - \vec{b}$.
   (a) Show that $\vec{c} = \vec{a}4 + \vec{b}$.
   (b) Find four pairs (a, b) of real numbers [besides $(1, -1)$ and $(4, 1)$] such that $\vec{c} = \vec{a}a + \vec{b}b$.
   (c) If someone told you that he knew a pair (a, b) such that $\vec{c} = \vec{a}a + \vec{b}b$, would you stand much chance of guessing what numbers he had in mind?
2. Suppose that $(\vec{a}, \vec{b})$ is *not* linearly dependent and that $\vec{c} = \vec{a} + \vec{b}2$. If a friend tells you that he knows a pair (a, b) such that $\vec{c} = \vec{a}a + \vec{b}b$, how much will you be willing to bet that the second of his numbers is 2? [*Hint*: Your friend's numbers must, of course, satisfy the equation '$\vec{a}a + \vec{b}b = \vec{a} + \vec{b}2$'. Find an equivalent equation whose right side is '$\vec{0}$', and remember that $(\vec{a}, \vec{b})$ is not linearly dependent.]

6.05 Linearly Independent Sequences

In Part B of the preceding exercises you discovered an important property of sequences of vectors which are not linearly dependent.

Answers for Part B

1. Neither a nor b is not 0 if and only if both a and b are 0.
2. (a) not all of $a_1, \ldots, a_n$ are 0
   (b) one of $a_1, \ldots, a_n$ is not 0
3. (a) all of $a_1, \ldots, a_n$ are 0
   (b) none of $a_1, \ldots, a_n$ is not 0

Answers for Part C

[The discussion is a somewhat informal proof of the case n = 3 of Theorem 6-7 on page 227. A different proof of this very basic theorem is given on pages 227 and 228. The present proof is formalized on pages 267 - 269. It may help you to see formalizations of the sentences in the second paragraph:

$(\vec{a}, \vec{b}, \vec{c})$ is not linearly dependent

$$\Longleftrightarrow \text{not } \exists_x \exists_y \exists_z \text{ (not } (x = 0 \text{ and } y = 0 \text{ and } z = 0) \text{ and } \vec{a}x + \vec{b}y + \vec{c}z = \vec{0})$$

$$\Longleftrightarrow \forall_x \forall_y \forall_z\ (\vec{a}x + \vec{b}y + \vec{c}z \neq \vec{0} \text{ or } (x = 0 \text{ and } y = 0 \text{ and } z = 0))$$

$$\Longleftrightarrow \forall_x \forall_y \forall_z\ [\vec{a}x + \vec{b}y + \vec{c}z = \vec{0} \Longrightarrow (x = 0 \text{ and } y = 0 \text{ and } z = 0)]$$

An alternative intermediate step is:

$$\forall_x \forall_y \forall_z \text{ not } (\vec{a}x + \vec{b}y + \vec{c}z = \vec{0} \text{ and not } (x = 0 \text{ and } y = 0 \text{ and } z = 0))$$

— in English:

> For all x, y, and z, never $\vec{a}x + \vec{b}y + \vec{c}z = \vec{0}$
> without (x = 0 and y = 0 and z = 0).

Understanding the argument depends on realizing that 'not for some [any] x [so-and-so]' amounts to the same thing as 'for each x not [so-and-so]' and that 'not (not p and q)' amounts to the same thing as 'if q then p'. The last can be mediated by 'not q or p' [or, as in the text, 'not q unless p']. As in the case of proofs earlier in this chapter, time spent in clarifying these points is likely to result in saving time later.]

1. (a) $\vec{c} = \vec{a} - \vec{b} = (\vec{a} - \vec{b}) + \vec{0} = (\vec{a} - \vec{b}) + (\vec{a}3 + \vec{b}2) = \vec{a}4 + \vec{b}$
   (b) [For any k, (3k + 1, 2k - 1) is such a pair; there are no others.]
   (c) No.
2. Bet any amount you can get covered; it's a sure thing. [Since $\vec{a}a + \vec{b}b = \vec{a} + \vec{b}2$, $\vec{a}(a - 1) + \vec{b}(b - 2) = \vec{0}$. Since $(\vec{a}, \vec{b})$ is not linearly dependent it follows that a - 1 = 0 and b - 2 = 0.]

[And, in earlier exercises you discovered that there are such sequences.] In the remainder of the course we shall have a great deal to do with sequences which are not linearly dependent. So, it will be convenient to have a name for such sequences.

> Definition 6-3
>
> A sequence is *linearly independent*
>
> if and only if
>
> it is not linearly dependent.

As you saw in Part B [in the case of a 3-termed sequence] it follows from Definitions 6-2 and 6-3 that

> Theorem 6-7
>
> $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly independent
>
> $\Longleftrightarrow$
>
> for all real numbers $x_1, \ldots, x_n$,
>
> $\vec{a}_1x_1 + \ldots + \vec{a}_nx_n = \vec{0} \Longrightarrow (x_1 = 0, \ldots, \text{and } x_n = 0)$.

A formal proof of Theorem 6-7 [either as a tree or as a column] requires knowledge of some rules of logic which we shall discuss only later in this chapter. Nevertheless, your intuitive understanding of the phrases 'there are' and 'for all' will be sufficient to make convincing the paragraph proofs we shall give for the two parts ["if" and "only if"] of this theorem. It will be easy later, if we wish, to put these proofs in column-form, using the rules we shall adopt for 'there are' and 'for all'.

We first prove the if-part of Theorem 6-7:

Suppose that, for all real numbers $x_1, \ldots, x_n$,

$$\text{if } \vec{a}_1x_1 + \ldots + \vec{a}_nx_n = \vec{0} \text{ then } (x_1 = 0, \ldots, \text{and } x_n = 0).$$

Now, if $(\vec{a}_1, \ldots, \vec{a}_n)$ were linearly dependent, it would follow [*by* Definition 6-2] that there are numbers—say, $a_1, \ldots, a_n$—*which are not all zero* and are such that $\vec{a}_1a_1 + \ldots + \vec{a}_na_n = \vec{0}$. This is not the case because, by assumption, if $\vec{a}_1a_1 + \ldots + \vec{a}_na_n = \vec{0}$ then $(a_1 = 0, \ldots, \text{and } a_n = 0)$. So, $(\vec{a}_1, \ldots, \vec{a}_n)$ is not linearly dependent.

Consequently, if, for all $x_1, \ldots, x_n$,

$$\vec{a}_1x_1 + \ldots + \vec{a}_nx_n = \vec{0} \Longrightarrow (x_1 = 0, \ldots, \text{and } x_n = 0)$$

then $(\vec{a}_1, \ldots, \vec{a}_n)$ is not linearly dependent.

From the form of Theorem 6-7 it is obvious that it might be taken as a definition of the phrase 'linearly independent'. Then, 'linearly dependent' could be defined as an abbreviation for 'not linearly independent'. The motivation for the procedure adopted in this text is, of course, that the concept of linear dependence is intuitively the simpler of the two, as well as the more easily illustrated. Nevertheless, the characterization of linear independence which is furnished by Theorem 6-7 is by far the most important one.

Of the two parts of Theorem 6-7, the only if-part is the more frequently used. A brief explanation of a common sort of use is in order here. As we shall see, it is frequently possible to reduce the solution of a geometric problem — or the proof of a geometric theorem — to that of finding all real number solutions of a vector equation of the form:

$$(\star) \qquad \vec{a}_1x_1 + \ldots + \vec{a}_nx_n = \vec{0}$$

where it is known that $(\vec{a}_1, \ldots, \vec{a}_n)$ is a linearly independent sequence of translations. [In studying 3-dimensional geometry, n will necessarily be at most 3.] In a given problem, the 'x_1', ..., 'x_n' will be replaced by given real number terms. [For examples of such equations, see Exercise 1 on page 228.] The only if-part of Theorem 6-7 tells us that the only solutions of ($\star$) are the solutions of the corresponding system of real number equations:

$$(\star\star) \qquad x_1 = 0, \ldots, x_n = 0$$

Since, trivially, each solution of this system is a solution of ($\star$), by finding all solutions of ($\star\star$) we find all solutions of ($\star$), thereby solving the geometric problem [or proving the theorem]. More specifically, a problem concerning a triangle can often be solved by using the fact that — by a later definition — A, B, and C are vertices of a triangle if and only if (B - A, C - A) is linearly independent, and by translating the problem into that of solving an equation of the form:

$$(B - A)x + (C - A)y = \vec{0}$$

[The real "problem" is, of course, to find this equation. Theorem 6-7 then tells us how to solve it.]

As remarked during the preceding discussion, it is a trivial matter to justify replacing the '$\Longrightarrow$' in Theorem 6-7 by a second '$\Longleftrightarrow$'. Since this is trivial, and since stating the theorem in this stronger form would complicate the ensuing discussion, we have not made the replacement in the text. That it might be done should be brought out in class discussion.

We shall have less use for the if-part of Theorem 6-7.

If your students have attained a fairly good understanding of the proof of Theorem 6-7 in the case n = 3 which is given on page 226 it would be possible to review it by using the same argument to prove the theorem itself. In that case the proof given on pages 227 and 228 might be skipped. [As previously remarked, however, two proofs are always better than one.]

In giving the proofs, it has seemed best to use 'not linearly dependent' rather than 'linearly independent'. The switch from one to the other by way of Definition 6-3 and biconditional replacement is a trivial matter.

The proof given for the if-part of Theorem 6-7 is, we hope, intuitively appealing. It — in a slightly streamlined form — is analyzed on pages 255-256, where, also, the rules of logic implicit in it are justified. In column-form it looks like this [merely to save space we take $n = 1$]:

(1) $\forall_x\ [\vec{a}x = \vec{0} \Longrightarrow x = 0]$ [assumption]*
(2) $\vec{a}a = \vec{0} \Longrightarrow a = 0$ [(1)]
(3) not $(\vec{a}a = \vec{0}$ and $a \neq 0)$ [(2)]
(4) not $\exists_x\ (\vec{a}x = \vec{0}$ and $x \neq 0)$ [(3)]
(5) $(\vec{a})$ is linearly dependent $\Longleftrightarrow$
$\exists_x\ (\vec{a}x = \vec{0}$ and $x \neq 0)$ [Definition 6-2]
(6) $(\vec{a})$ is not linearly dependent [(5), (4)]
(7) (1) $\Longrightarrow$ (6) [(6), *(1)]

From (1) to (2) is by the elimination rule for '$\forall$'; (2) to (3) is by an inference from a sentence of the form '$p \Longrightarrow q$' to one of the form 'not (p and not q)' — justification of such inferences requires something equivalent to proof by contradiction; (3) to (4) is by a rule more or less equivalent to the elimination rule for '$\exists$'. It is because of a restriction on the use of this latter rule that we must start with (1) rather than (2) as an assumption. Were it not for this restriction we could "prove":

$$[\vec{a}a = \vec{0} \Longrightarrow a = 0] \Longrightarrow (\vec{a}) \text{ is not linearly dependent}$$

from this deduce its instance:

$$[\vec{a}0 = \vec{0} \Longrightarrow 0 = 0] \Longrightarrow (\vec{a}) \text{ is not linearly dependent}$$

and, deducing the antecedent from the valid sentence '$0 = 0$', end up with a "proof" that any 1-termed sequence is not linearly dependent. [The argument would generalize to show that no sequence — of whatever length — is linearly dependent. This illustrates the difference between sentences like (1) $\Longrightarrow$ (6) and (2) $\Longrightarrow$ (6) and points up the need for quantifiers. [On this point, see the exercises on page 234.]

The proof given for the only if-part of Theorem 6-7 is analyzed on pages 254 and 255. The only unfamiliar rules are the introduction rules for '$\exists$' and '$\forall$'.

(1) $(\vec{a})$ is not linearly dependent [assumption]***
(2) $\vec{a}a = \vec{0}$ [assumption]**
(3) $a \neq 0$ [assumption]*
(4) $\exists_x\ (\vec{a}x = \vec{0}$ and $x \neq 0)$ [(2), (3)]
(5) $(\vec{a})$ is linearly dependent $\Longleftrightarrow$ (4) [Definition 6-2]
(6) $(\vec{a})$ is linearly dependent [(5), (4)]
(7) $a \neq 0 \Longrightarrow (\vec{a})$ is linearly dependent [(6), *(3)]
(8) $a = 0$ [(7), (1)]
(9) $\vec{a}a = \vec{0} \Longrightarrow a = 0$ [(8), **(2)]
(10) $\forall_x\ [\vec{a}x = \vec{0} \Longrightarrow x = 0]$ [(9)]
(11) (1) $\Longrightarrow$ (10) [(10), ***(1)]

In this case we could as well have proved (1) $\Longrightarrow$ (9); but (1) $\Longrightarrow$ (10) is what is needed if we are to combine our conclusion with that of the preceding proof to obtain a biconditional sentence.

Here, now, is a proof of the only if-part of Theorem 6-7:

Suppose that $(\vec{a}_1, \ldots, \vec{a}_n)$ is not linearly dependent and that $a_1, \ldots, a_n$ are numbers such that $\vec{a}_1a_1 + \ldots + \vec{a}_na_n = \vec{0}$. If it were the case that even one of the numbers $a_1, \ldots, a_n$ were different from zero then it would follow [by Definition 6-2] that $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly dependent. Since, by assumption, this is not the case it follows that each of the numbers $a_1, \ldots, a_n$ is 0. Hence, under our assumption that $(\vec{a}_1, \ldots, \vec{a}_n)$ is not linearly dependent, it follows [for any numbers $a_1, \ldots, a_n$] that

$$\text{if } \vec{a}_1a_1 + \ldots + \vec{a}_na_n = \vec{0} \text{ then } (a_1 = 0, \ldots, \text{and } a_n = 0).$$

Consequently, if $(\vec{a}_1, \ldots, \vec{a}_n)$ is not linearly dependent then, for all $x_1, \ldots, x_n$,

$$\vec{a}_1x_1 + \ldots + \vec{a}_nx_n = \vec{0} \Longrightarrow (x_1 = 0, \ldots, \text{and } x_n = 0).$$

Exercises

Part A

1. Given that $(\vec{a}, \vec{b})$ is linearly independent, determine all pairs (a, b) which satisfy:
   - **(a)** $\vec{a}(5a + 2) + \vec{b}(7b - 10) = \vec{0}$
   - **(b)** $\vec{a}(2a + 5b) + \vec{b}(4a - b) = \vec{0}$
   - **(c)** $\vec{a}(a - b) + \vec{b}(a + b) = \vec{0}$
   - **(d)** $\vec{a}(6 - 12a) - \vec{b}(3b) = \vec{b}(a - b)$
   - **(e)** $\vec{a}\left(a - \frac{15}{a + 2}\right) + \vec{b}(4b - 3) = \vec{0}$
   - **(f)** $\vec{a}(a^2 - 9) + \vec{b}(a^2 + 5a + 6) = \vec{0}$
2. Suppose that $(B - A)(a + 5) + (C - B)(b - 2) + (A - C)(c - 1) = \vec{0}$. Find three ordered triples (a, b, c) which satisfy the given sentence. Draw figures to show that the values you selected "work" when A, B, and C are collinear—that is, on a line—as well as when A, B, and C are noncollinear.
3. Suppose that $(\vec{a}, \vec{b})$ is linearly dependent. Draw arrows to represent $\vec{a}$ and $\vec{b}$ and mark a point O. Draw a picture, and give a description in words, of the set of all points X such that

   $$X = O + \vec{y}$$

   for some linear combination $\vec{y}$ of $\vec{a}$ and $\vec{b}$.
4. In Exercise 4, replace the word 'dependent' by 'independent' and repeat the exercise.

• 5. Draw arrows to represent translations $\vec{a}$ and $\vec{b}$ such that $(\vec{a}, \vec{b})$ is

Parts A - H require more than one class-homework assignment to complete. Because of the importance of linear independence throughout the remainder of the course, these exercises deserve due consideration by all students. Part A could be used as in-class practice exercises to illustrate linear independence. Parts B and C make a reasonable homework assignment. The theorems of Part D can constitute a second class activity and Parts F and G a second homework assignment. The third class period should include a discussion of Parts F and G, including as many alternate solutions for each exercise as possible. Part H represents a third homework assignment. We believe that as the course continues you will find this time well spent.

Answers for Part A

1. (a) $(-2/5, 10/7)$ (b) $(0, 0)$ (c) $(0, 0)$

   (d) $(1/2, -1/4)$ (e) $(-5, 3/4)$, $(3, 3/4)$

   (f) $(-3, b)$ is a solution for any b. [Note that changing '+ 6' to '- 6' in (f) would yield an equation with no solutions.]

   [Be sure students realize that Theorem 6-7 tells them that there are no solutions other than these listed. That these are, in fact, solutions follows from the fact that $\vec{a}0 + \vec{b}0 = \vec{0}$.]

2. [Obviously, one solution is $(-5, 2, 1)$. Since, however [mostly by Postulate 3], $(B - A)k + (C - B)k + (A - C)k = \vec{0}$, for any number k, it follows that $(-5 + k, 2 + k, 1 + k)$ is a solution. If A, B, and C are noncollinear, these are the only solutions. (See Theorem 6-12 on page 234.) In this case the figure for any such solution will be a triangle similar to $\triangle ABC$ with ratio of similitude $|k|$. For A, B, C collinear, there will be additional solutions. What these are will depend on how the points are chosen.]

3. [If neither $\vec{a}$ nor $\vec{b}$ is $\vec{0}$ then both will have the same direction and the picture should represent the line in this direction through O. If one of $\vec{a}$ and $\vec{b}$ is $\vec{0}$ and the other not, the picture should represent the line through O in the direction of the non-$\vec{0}$ vector. If $\vec{a} = \vec{0} = \vec{b}$ then the set in question is the singleton $\{O\}$.]

4. the plane containing the lines through O in the directions of $\vec{a}$ and $\vec{b}$

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5. [This is a re-wording of Exercise 4 and has the same answer.]

6. the set of all points of $\mathcal{E}$

[The answers for exercises like Exercises 3 - 5 will serve as motivation for later definitions of 'line' and 'plane'. That for Exercise 6 will suggest the adoption of new parts fo Postulate 4 which will specify the dimension of $\mathcal{T}$.]

Answers for Part B

[Theorem 6-8 is useful as a shortcut to geometric results which could otherwise be obtained by applying the only if-part of Theorem 6-7. Since this is the reason for stating it we have refrained from replacing 'If ... then' by '... if and only if'. You should, however, point out that, as in the case of Theorem 6-7, the '$\Longrightarrow$' might easily be replaced by '$\Longleftrightarrow$'. Proof of the "group theorem" is discussed below.]

linearly independent, and mark a point O. Draw a picture, and give a description in words, of

$$\{X: \exists_y (X = O + \vec{y} \text{ and } (\vec{a}, \vec{b}, \vec{y}) \text{ is linearly dependent})\}.$$

⋆ **6.** Suppose that O is a point and $(\vec{a}, \vec{b}, \vec{c})$ is a linearly independent sequence of translations. Describe, in words,

$$\{X: \exists_y (X = O + \vec{y} \text{ and } (\vec{a}, \vec{b}, \vec{c}, \vec{y}) \text{ is linearly dependent})\}.$$

Part B

1. From the fact that translations form a group with respect to addition it follows that

$$\vec{a}_1 a_1 + \ldots + \vec{a}_n a_n = \vec{a}_1 b_1 + \ldots + \vec{a}_n b_n \Longrightarrow (\vec{a}_1 a_1 - \vec{a}_1 b_1) + \ldots + (\vec{a}_n a_n - \vec{a}_n b_n) = \vec{0}.$$

Use this result in proving:

Theorem 6–8 If $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly independent then $[\vec{a}_1 a_1 + \ldots + \vec{a}_n a_n = \vec{a}_1 b_1 + \ldots + \vec{a}_n b_n \Longrightarrow (a_1 = b_1, \ldots, \text{and } a_n = b_n)]$

2. Suppose that $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent. Find all triples (a, b, c) which satisfy these equations.

(a) $\vec{a}a + \vec{b}b + \vec{c}c = \vec{a} + \vec{b}2 - \vec{c}$
(b) $\vec{a}a + \vec{b}b + \vec{c}c = \vec{a}b + \vec{b}c + \vec{c}(a + b)$
(c) $\vec{a}a + \vec{b}b + \vec{c}c = -\vec{a}7 + \vec{b} \cdot (5 + c)$
(d) $\vec{a}a + \vec{b}b + \vec{c}c = \vec{a}(c - b) + \vec{b}(b + a)$

Part C

By Definition 6–3 [and the replacement rule for biconditional sentences] the phrase 'is not linearly dependent' may always be replaced by 'is linearly independent', and *vice versa*. Similarly, the phrase 'is not linearly independent' is interchangeable with 'is linearly dependent'. [Explain. (Is Part A on page 225 relevant?)] Each of the following exercises suggests an inference, but one premiss or the conclusion is missing. Find the missing sentence.

1. Premiss: If two terms of $(\vec{a}_1, \ldots, \vec{a}_n)$ are equal then $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly dependent.
   Premiss: $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly independent.
2. Conclusion: If a sequence is linearly independent then no term is a multiple of another.
3. Premiss: If $(\vec{a})$ is linearly dependent then $\vec{a} = \vec{0}$.

---

1. Since $\vec{a}_i a_i - \vec{a}_i b_i = \vec{a}_i(a_i - b_i)$ it follows from the given theorem that

$$\vec{a}_1 a_1 + \ldots + \vec{a}_n a_n = \vec{a}_1 b_1 + \ldots + \vec{a}_n b_n \Longrightarrow \vec{a}_1(a_1 - b_1) + \ldots + \vec{a}_n(a_n - b_n) = \vec{0}.$$

Assuming that $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly independent and that $\vec{a}_1 a_1 + \ldots + \vec{a}_n a_n = \vec{a}_1 b_1 + \ldots + \vec{a}_n b_n$ it follows that $\vec{a}_1(a_1 - b_1) + \ldots + \vec{a}_n(a_n - b_n) = \vec{0}$ and, by Theorem 6-7, that $a_1 - b_1 = 0$, ..., and $a_n - b_n = 0$. So, under the same assumptions, $a_1 = b_1$, ..., and $a_n = b_n$. Hence, the theorem.

2. (a) (1, 2, −1) (b) (0, 0, 0) (c) (−7, 5, 0) (d) (0, 0, 0)

* * *

The "group theorem" stated in Exercise 1 is an instance of:

$$(☆) \quad \vec{a}_1 + \ldots + \vec{a}_n = \vec{b}_1 + \ldots + \vec{b}_n \Longrightarrow (\vec{a}_1 - \vec{b}_1) + \ldots + (\vec{a}_n - \vec{b}_n) = \vec{0}$$

The initial step of an inductive proof of (☆) — the proof of '$\vec{a} = \vec{b} \Longrightarrow \vec{a} - \vec{b} = \vec{0}$' — is easy. Before proceeding to the inductive step it is convenient to prove:

$$(1) \quad (\vec{a} + \vec{b}) - (\vec{c} + \vec{d}) = (\vec{a} - \vec{c}) + (\vec{b} - \vec{d})$$

This will be used in the inductive step. Also, instances of (1) and of the theorem '$\vec{a} - \vec{b} = \vec{0} \Longleftrightarrow \vec{a} = \vec{b}$' take care of the somewhat special case $n = 2$ of (☆).

For the inductive step the inductive hypothesis is (☆) with '$n - 1$' substituted for 'n'. [The hypothesis is that this holds for any choice of '$(\vec{a}_1, \ldots, \vec{a}_{n-1})$ and $(\vec{b}_1, \ldots, \vec{b}_{n-1})$, and this can best be brought out by using other letters — say 'c', and 'd' — rather than 'a and 'b'.] To derive (☆) one begins by assuming that $\vec{a}_1 + \ldots + \vec{a}_n = \vec{b}_1 + \ldots + \vec{b}_n$. We are interested only in the case $n \geq 3$ and, for this, '$\vec{a}_1 + \ldots + \vec{a}_n$' is an abbreviation for '$(\vec{a}_1 + \ldots + \vec{a}_{n-1}) + \vec{a}_n$'. So, by the associative principle, $\vec{a}_1 + \ldots + \vec{a}_n = \vec{a}_1 + \ldots + (\vec{a}_{n-1} + \vec{a}_n)$. It follows, then, from the assumption just made [and the associative principle] that

$$\vec{a}_1 + \ldots + (\vec{a}_{n-1} + \vec{a}_n) = \vec{b}_1 + \ldots + (\vec{b}_{n-1} + \vec{b}_n).$$

From this and an instance of the inductive hypothesis it follows that

$$(\vec{a}_1 - \vec{b}_1) + \ldots + ((\vec{a}_{n-1} + \vec{a}_n) - (\vec{b}_{n-1} + \vec{b}_n)) = \vec{0}.$$

From this and an instance of (1) it follows that

$$(\vec{a}_1 - \vec{b}_1) + \ldots + ((\vec{a}_{n-1} - \vec{b}_{n-1}) + (\vec{a}_n - \vec{b}_n)) = \vec{0}.$$

From this and an instance of the associative principle it follows that

$$(\vec{a}_1 - \vec{b}_1) + \ldots + (\vec{a}_{n-1} - \vec{b}_{n-1}) + (\vec{a}_n - \vec{b}_n) = \vec{0}.$$

Hence, (☆).

Answers for Part

1. Conclusion: No two terms of $(\vec{a}_1, \ldots, \vec{a}_n)$ are equal. [Note that 'Two terms of $(\vec{a}_1, \ldots, \vec{a}_n)$ are not equal.' is, at best ambiguous, at worst incorrect. An alternative correct answer is '$(\vec{a}_1, \ldots, \vec{a}_n)$ is a sequence of distinct terms.']
2. Premiss: If one term of a sequence is a multiple of another then the sequence is linearly dependent.
3. Conclusion: If $\vec{a} \neq \vec{0}$ then $(\vec{a})$ is linearly independent.

4. Premise: If $(\vec{a}, \vec{b})$ is linearly independent and $\vec{a}a + \vec{b}b = \vec{0}$ then $(a = 0$ and $b = 0)$.
   Premise: $\vec{a}3 = \vec{b}5$
5. Premise: If $(\vec{a}, \vec{b})$ is linearly dependent and $\vec{a} \neq \vec{0}$ then $\vec{b}$ is a multiple of $\vec{a}$.
   Conclusion: $\vec{a} \neq \vec{0} \Longrightarrow (\vec{a}, \vec{b})$ is linearly independent.
6. Conclusion: If $(\vec{a}, \vec{b})$ is linearly independent then $(\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0})$.

Part D

Prove each of the following theorems.

1. *Theorem 6–9* If a sequence is linearly independent then any of its subsequences is linearly independent. [*Hint*: See Part A on page 225.]
2. *Theorem 6–10* Any permutation of a linearly independent sequence is linearly independent. [*Hint*: This is a short way of saying that if a given sequence is linearly independent then any permutation of the given sequence is linearly independent.]
3. *Theorem 6–11* Any linearly independent sequence is a sequence of distinct, non-$\vec{0}$, terms. [*Hint*: This is a short way of saying that if a sequence is linearly independent then no two of its terms are equal and none of its terms is $\vec{0}$. Recall Theorem 6–3 and note that any inference of the form:

$$\frac{p \Longrightarrow q \qquad p \Longrightarrow r}{p \Longrightarrow (q \text{ and } r)}$$

is valid.]
4. Show that inferences of the form referred to in the preceding hint are valid.

***Part E**

A *set* S of vectors is said to be linearly independent if and only if every [finite] sequence of distinct terms from S is linearly independent. Prove:

Any subset of a linearly independent set is linearly independent.

Part F

Consider the following sentence:

(S) If there are three linearly independent vectors then there are two linearly independent vectors.

and its contrapositive:

(C) If there are not two linearly independent vectors then there are not three linearly independent vectors.

In each of the following exercises, you are given two premisses. You are to state what conclusion [if any] follows from these premisses.

1. **(a)** **(i)** S
   (ii) There are three linearly independent vectors.
   (b) **(i)** C
   (ii) There are three linearly independent vectors
2. **(a)** **(i)** S
   (ii) There are not two linearly independent vectors.
   (b) **(i)** C
   (ii) There are not two linearly independent vectors
3. **(a)** **(i)** S
   (ii) There are two linearly independent vectors.
   (b) **(i)** C
   (ii) There are two linearly independent vectors.
4. **(a)** **(i)** S
   (ii) There are not three linearly independent vectors.
   (b) **(i)** C
   (ii) There are not three linearly independent vectors.

Part G

Consider the sentence:

(S) If there are two linearly independent vectors then there is a vector $\vec{x}$ such that $\vec{x} \neq \vec{0}$.

and its converse:

(V) If there is a vector $\vec{x}$ such that $\vec{x} \neq \vec{0}$ then there are two linearly independent vectors.

You are to state a conclusion [if any] which follows from the two given premisses.

1. **(a)** **(i)** S
   (ii) There are two linearly independent vectors.
   (b) **(i)** V
   (ii) There are two linearly independent vectors.
2. **(a)** **(i)** S
   (ii) There is no vector $\vec{x}$ such that $\vec{x} \neq \vec{0}$.
   (b) **(i)** V
   (ii) There is no vector $\vec{x}$ such that $\vec{x} \neq \vec{0}$.
3. **(a)** **(i)** S
   (ii) There is a vector $\vec{x}$ such that $\vec{x} \neq \vec{0}$.
   (b) **(i)** V
   (ii) There is a vector $\vec{x}$ such that $\vec{x} \neq \vec{0}$.
4. **(a)** **(i)** S
   (ii) There are not two linearly independent vectors.
   (b) **(i)** V
   (ii) There are not two linearly independent vectors

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4. Conclusion: $(\vec{a}, \vec{b})$ is linearly dependent.

5. Premiss: $\vec{b}$ is not a multiple of $\vec{a}$.

6. Premiss: If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ then $(\vec{a}, \vec{b})$ is linearly dependent.

Answers for Part D

1. It follows [by contraposition] from Theorem 6-4 that if a sequence is not linearly dependent then any of its subsequences is not linearly dependent. So, by Definition 6-3, if a sequence is linearly independent then any of its subsequences is linearly independent.

2. It follows [by contraposition] from the only if-part of Theorem 6-6 that if a sequence is not linearly dependent then any permutation of it is not linearly dependent. So, by Definition 6-3, ...

3. By Theorem 6-3(b), a sequence which is not linearly dependent does not have two equal terms; by Theorem 6-3(a), such a sequence does not have a term which is $\vec{0}$. Hence, a linearly independent sequence is a sequence of distinct, non-$\vec{0}$, terms.

4. $$\cfrac{\cfrac{\cfrac{\overset{*}{p} \qquad p \Longrightarrow q}{q} \qquad \cfrac{\overset{*}{p} \qquad p \Longrightarrow r}{r}}{q \text{ and } r}}{p \Longrightarrow (q \text{ and } r)}\,*$$

Answers for Part E

Each sequence whose terms belong to a given subset of S is a sequence whose terms belongs to S. So, if each sequence of the latter kind is linearly independent then so is each sequence of the former kind. Hence, if S is linearly independent then so is any subset of S.

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Answers for Part F

1. (a) There are two linearly independent vectors.
   (b) There are two linearly independent vectors.
2. (a) There are not three linearly independent vectors.
   (b) There are not three linearly independent vectors.
3. (a) [no conclusion]
   (b) [no conclusion]
4. (a) [no conclusion]
   (b) [no conclusion]

Answers for Part G

1. (a) There is a vector $\vec{x}$ such that $\vec{x} \neq \vec{0}$.
   (b) [no conclusion]
2. (a) There are not two linearly independent vectors.
   (b) [no conclusion]
3. (a) [no conclusion]
   (b) There are two linearly independent vectors.
4. (a) [no conclusion]
   (b) There is no vector $\vec{x}$ such that $\vec{x} \neq \vec{0}$.

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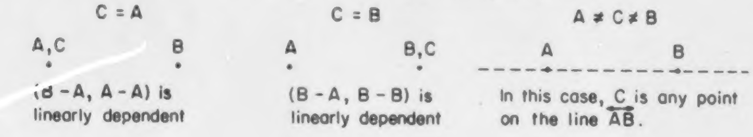
Answers for Part H

1. No. Since A and B are two points, B ≠ A; so, by Theorem 3-2(b), $B - A \neq \vec{0}$.

2. Your students should have pictures something like this:



(B - A, A - A) is linearly dependent

(B - A, B - B) is linearly dependent

In this case, C is any point on the line $\overleftrightarrow{AB}$.

3. Suppose that $B - A \neq \vec{0}$ and $(B - A)a + (C - A)b = \vec{0}$. It follows that if $b = 0$ then $(B - A)a = \vec{0}$ and, so, $a = 0$. Hence, if a and b are not both 0 then $b \neq 0$.

4. By Exercise 3, $b \neq 0$ and, so, $C - A = (B - A) \cdot -a/b$.

5. (a) One.; One.
   (b) Infinitely many.; All of them. That is, each plane which contains A and B also contains C.

6. (a) That $p = 0$ and $r = 0$. Whether or not $S - Q$ is a linear combination of $P - Q$ and $R - Q$ is not determined by the data.
   (b) Yes. $S - Q = (P - Q) \cdot -p/s + (R - Q) \cdot -r/s$.

7. (a) If the points P, Q, and R are not distinct, then $(P - Q, R - Q)$ is linearly dependent. Since, by hypothesis, $(P - Q, R - Q)$ is linearly independent, it follows that P, Q, and R are distinct. No line contains P, Q, and R, for intuitively, if there is a line containing P, Q, and R, then one of $P - Q$ and $R - Q$ is a multiple of the other. But in that event, $(P - Q, R - Q)$ is linearly dependent.
   (b) Exactly one.; None, for in the event that $s = 0$, the equation '$(P - Q)p + (R - Q)r + (S - Q)s = \vec{0}$' is satisfied by any point S; One, for if $s \neq 0$ then $S - Q$ is a linear combination of $P - Q$ and $R - Q$. Intuitively this means that S can be "reached" from Q by applying to Q that linear combination of $P - Q$ and $R - Q$, and since any such point arrived at from Q is in the plane of P, Q, and R, S is in this plane.

Part H

Consider *two* points A and B and let C be a third point such that $(B - A, C - A)$ is linearly dependent.

1. Can $B - A$ be $\vec{0}$? [Explain.]
2. Draw figures to illustrate the three cases in which $C = A$, in which $C = B$, and in which $A \neq C \neq B$.
3. Show that if [as above] $B - A \neq \vec{0}$ and $(B - A)a + (C - A)b = \vec{0}$, where not both a and b are 0, then $b \neq 0$. [*Hint*: Assuming that $(B - A)a + (C - A)b = \vec{0}$, suppose that $b = 0$. What may you conclude about $(B - A)a$? [Why?] Assuming that $B - A \neq \vec{0}$, what may you conclude about a? [Why?] From your results so far it follows, under the assumptions you have made, that if $b = 0$ then (________ and ________). Assuming that not both a and b are 0, what follows about b?]
4. Show that if $B - A \neq \vec{0}$ and $(B - A)a + (C - A)b = \vec{0}$, where not both a and b are 0, then $C - A$ is a multiple of $B - A$. [*Hint*: Use what you proved in Exercise 3.]
5. **(a)** How many lines contain both A and B? How many of these lines also contain C?
   (b) How many planes contain both A and B? How many of these planes also contain C?
6. Suppose that $(P - Q, R - Q)$ is linearly independent and that p, r, and s are numbers such that $(P - Q)p + (R - Q)r + (S - Q)s = \vec{0}$, for some p, r, and s.
   (a) Let $s = 0$. What can you say about p and r? Is $S - Q$ a linear combination of $P - Q$ and $R - Q$? Explain.
   (b) Suppose that $s \neq 0$. Is $S - Q$ a linear combination of $P - Q$ and $R - Q$? Explain.
7. **(a)** Explain why the points P, Q, and R described in Exercise 6 must be distinct. How many lines contain all three of the points P, Q, and R?
   (b) How many planes contain all three of the points P, Q, and R? For $s = 0$, how many of these planes must contain S? For $s \neq 0$, how many of these planes must contain S?

6.06 A Useful Theorem about Linearly Independent Vectors

Using the definitions of linearly dependent and independent sequences together with the fact that $\mathcal{T}$ is a vector space, we have obtained quite a few special properties of sequences of vectors. Another such property, and one which will be quite useful, is:

> Suppose that $(\vec{a}, \vec{b})$ is linearly independent and that $\vec{a}$, $\vec{b}$,
> (*) and $\vec{c}$ are translations such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Then,
> $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$ if and only if $a = b = c$.

Before we try to prove (*), it is probably a good idea to see just what it is that (*) is saying.

As might be guessed, (☆) is another variant of the only if-part of the case $n = 2$ of Theorem 6-7. It will turn out to be quite a useful one. Students should be challenged to establish (☆) before studying the proof sketched on page 233. That the only if-part of the conclusion follows from the given assumptions is almost immediate. What remains is to derive '$a = b = c$' from:

$$(\vec{a}, \vec{b}) \text{ is linearly independent,}$$
$$\vec{a} + \vec{b} + \vec{c} = \vec{0}, \text{ and}$$
$$\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}.$$

[Analogy with solving systems of equations might suggest eliminating '$\vec{c}$' from the two equations and comparing the result with the first of the three assumptions.]

It turns out to be relatively easy to derive the biconditional conclusion of (☆) — "in one piece" — from the assumptions in (☆). This is what is done on pages 233-234.

Sample Quiz

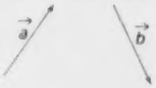
[Notice that the solutions of items such as the following not only require an understanding of current concepts but in addition require a good deal of skill in handling notions from earlier work. Similar items can be constructed to provide needed drill and review.]

1. Given that $(\vec{a}, \vec{b})$ is linearly independent and that $\vec{a}(2p - 3q + 5) = \vec{b}(5 - 3p - 2q)$ determine all of the values of 'p' and 'q'.
2. Given that $(\vec{a}, \vec{b})$ is linearly independent and that $\vec{a}(p + q - 3) + \vec{b}(p^2 - 3q + 9) = \vec{0}$ determine all of the values of 'p' and 'q'.
3. $[\vec{a}]$ is the set of all linear combinations of $\vec{a}$. Show that if $(\vec{b} \in [\vec{a}]$ and $\vec{c} \in [\vec{a}])$ then $(\vec{b}, \vec{c})$ is linearly dependent.

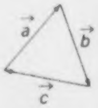
Key to Sample Quiz

1. Since $(\vec{a}, \vec{b})$ is linearly independent, it follows that $2p - 3q + 5 = 0$ and $3p + 2q - 5 = 0$. Solving this system, we find that $p = \frac{5}{13}$ and $q = \frac{25}{13}$.
2. From the hypothesis, it follows that $p + q - 3 = 0$ and $p^2 - 3q + 9 = 0$. Solving this system, we find that $(p = 0$ and $q = 3)$ or $(p = -3$ and $q = 6)$.
3. Suppose that $\vec{b} \in [\vec{a}]$ and $\vec{c} \in [\vec{a}]$. Then, $\vec{b} = \vec{a}b$ and $\vec{c} = \vec{a}c$ for some b and c. For $\vec{a} = \vec{0}$, it follows that $\vec{b}$ and $\vec{c}$ are both $\vec{0}$ so that $(\vec{b}, \vec{c})$ is linearly dependent. For $\vec{a} \neq \vec{0}$, it is clear that for either $\vec{b} = \vec{0}$ or $\vec{c} = \vec{0}$ the sequence $(\vec{b}, \vec{c})$ is linearly dependent. For $\vec{b} \neq \vec{0}$, it follows that $b \neq 0$ so that $\vec{a} = \vec{b} \cdot /b$. Then, $\vec{c} = (\vec{b} \cdot /b) \cdot c = \vec{b}(c/b)$ so that $\vec{b}(c/b) + \vec{c} \cdot -1 = \vec{0}$. Since $-1 \neq 0$, $(\vec{b}, \vec{c})$ is linearly dependent. A similar argument may be given for $\vec{c} \neq \vec{0}$. So, $(\vec{b}, \vec{c})$ is linearly dependent in any case.

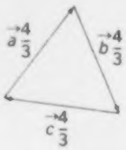
First, to say that $(\vec{a}, \vec{b})$ is linearly independent is to say that $\vec{a}$ and $\vec{b}$ have different directions. [If they had the same direction, then one of them, say $\vec{a}$, would be a multiple of the other, $\vec{b}$, and so $\vec{a}$ would be dependent on $\vec{b}$.] Here is a diagram of two such vectors:

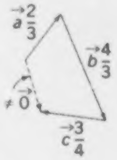


Next, we are given that $\vec{a}$, $\vec{b}$, and $\vec{c}$ are translations such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Here is a diagram appropriate for this together with the first condition:



The theorem tells us that if we wish to consider linear combinations of $\vec{a}$, $\vec{b}$, and $\vec{c}$ which map each point on itself $[\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}]$ then just those linear combinations for which each of the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is multiplied by the same real number will do the job.





To prove (*) let's suppose that

$$(\vec{a}, \vec{b}) \text{ is linearly independent and } \vec{a} + \vec{b} + \vec{c} = \vec{0}$$

and try to show that it follows that

$$\vec{a}a + \vec{b}b + \vec{c}c = \vec{0} \Longleftrightarrow a = b = c.$$

[This turns out to be pretty simple. The key consists in noting that, since $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ it follows that $\vec{c} = -(\vec{a} + \vec{b})$ and, so, that $\vec{c}c = \vec{a} \cdot -c + \vec{b} \cdot -c$. From this it follows that

$$\vec{a}a + \vec{b}b + \vec{c}c = \vec{a}(a - c) + \vec{b}(b - c).$$

You can fill in the details now, or do it in Exercise 1, below. Assuming that this has been done, we will go on with the proof of (*).]

Since $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ it follows that $\vec{a}a + \vec{b}b + \vec{c}c = \vec{a}(a - c) + \vec{b}(b - c)$. So,

$$\vec{a}a + \vec{b}b + \vec{c}c = \vec{0} \Longleftrightarrow \vec{a}(a - c) + \vec{b}(b - c) = \vec{0}$$

Since $(\vec{a}, \vec{b})$ is linearly independent it follows that

$$\vec{a}(a - c) + \vec{b}(b - c) = \vec{0} \Longleftrightarrow (a - c = 0 \text{ and } b - c = 0)$$

Finally,

$$(a - c = 0 \text{ and } b - c = 0) \Longleftrightarrow (a = c \text{ and } b = c)$$

[that is, $\Longleftrightarrow (a = b = c)$]. So,

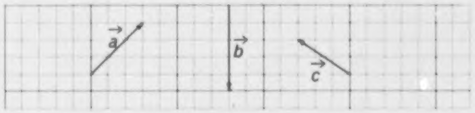
$$\vec{a}a + \vec{b}b + \vec{c}c = \vec{0} \Longleftrightarrow (a = b = c).$$

Hence, we have proved:

> **Theorem 6-12**
> $(\vec{a}, \vec{b})$ is linearly independent and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$
>
> $$\Longrightarrow$$
>
> $$[\vec{a}a + \vec{b}b + \vec{c}c = \vec{0} \Longleftrightarrow a = b = c]$$

Exercises

1. Complete the proof of Theorem 6-12.
2. Suppose that $\vec{a}$, $\vec{b}$, and $\vec{c}$ are as pictured below:



   (a) Check that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.
   (b) Draw arrows representing the translations $\vec{a}3$, $\vec{b}3$, $\vec{c}3$, and check that $\vec{a}3 + \vec{b}3 + \vec{c}3 = \vec{0}$.
   (c) Choose any number $r \neq 3$. Draw arrows representing $\vec{a}3$, $\vec{b}3$, and $\vec{c}r$. Check that $\vec{a}3 + \vec{b}3 + \vec{c}r \neq \vec{0}$.

3. Suppose that we change Theorem 6-12 "slightly" by eliminating the hypothesis that $(\vec{a}, \vec{b})$ is linearly independent. Would we still have a theorem? In other words, we are asking whether or not the following sentence is a theorem:

   If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$,
   then $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$ if and only if $a = b = c$.

   Actually we are asking whether the following two sentences are theorems:
   (a) If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$ then $a = b = c$.
   (b) If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $a = b = c$ then $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$.
   Prove, or give a counter-example, for (a) and for (b).

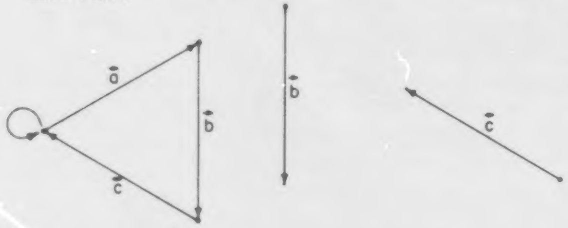
Answers for Exercises

1. "Completing the proof" amounts merely to filling in some algebraic details. Since $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ it follows that $\vec{c} = -(\vec{a} + \vec{b})$ and, so, that

$$\begin{aligned} \vec{c}c &= -(\vec{a} + \vec{b})c \\ &= (\vec{a} + \vec{b}) \cdot -c \\ &= \vec{a} \cdot -c + \vec{b} \cdot -c. \end{aligned}$$

Hence,
$$\begin{aligned} \vec{a}a + \vec{b}b + \vec{c}c &= (\vec{a}a + \vec{b}b) + (\vec{a} \cdot -c + \vec{b} \cdot -c) \\ &= (\vec{a}a + \vec{a} \cdot -c) + (\vec{b}b + \vec{b} \cdot -c) \\ &= \vec{a}(a + -c) + \vec{b}(b + -c) \\ &= \vec{a}(a - c) + \vec{b}(b - c) \end{aligned}$$

2. (a) To make this check, the students might trace the arrows for $\vec{a}$, $\vec{b}$, and $\vec{c}$ and make use of parallel rulers to draw the resultant translation $\vec{a} + \vec{b} + \vec{c}$. [Hopefully, the drawing errors will not be too great.] Doing so they will have pictures something like this:



(b) [The picture for this part should be of a triangle whose sides are parallel to and three times as long as the corresponding sides of the triangle pictured in part (a).]

(c) Notice that anything besides $\vec{c}3$ will not allow the figure in (b) to "close".

3. (a) is not a theorem. There are several easily given counterexamples.

(i) $\vec{a} = \vec{b} = \vec{c} = \vec{0}$, and any choice of a, b, and c such that not (a = b = c)

(ii) $\vec{b} = -\vec{a}$, $\vec{c} = \vec{0}$, and any choice of a, b, and c such that $a = b \neq c$.

(iii) Any choice of $\vec{a}$, $\vec{b}$, a and b such that a and b are not both zero and $\vec{a}a + \vec{b}b = \vec{0}$, any choice of $\vec{c}$, and $c = 0$.

(b) is a theorem. Suppose that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $a = b = c$. Then,

$$\vec{a}a + \vec{b}b + \vec{c}c = \vec{a}a + \vec{b}a + \vec{c}a = (\vec{a} + \vec{b} + \vec{c})a = \vec{0}a = \vec{0}.$$

The purpose of the exploration exercises is to point out that, while our convention for using "open sentences" to express universal generalities is sufficient to allow for expressing many generalities, something more is needed. This need — and how to meet it — has already been discussed in the commentary on logic at the end of Chapter 1.

Parts A and B of the exercises are, roughly, parallel to one another. The discoveries students should make while doing them are discussed on pages 236 - 239 of the text. You should use these pages, yourself, as basic commentary on the exercises.

It would seem best to do Parts A and B in class discussion rather than to assign them as homework.

At the beginning of Part A, the emphasis should be on what Theorem 2-2 says about an arbitrary point A, for a given translation $\vec{a}$ and a given translation $\vec{b}$. [In stating Theorem 2-2 here we use ordinary functional notation to facilitate comparing it and its two parts ["if" and "only if"] with sentence (a) of Exercise 3 and sentences (a) and (b) of Exercise 2.] Preliminary discussion should bring out the point that the if-part of Theorem 2-2 tells us that in case $\vec{a} = \vec{b}$ then, no matter what point A we may choose, we can be sure that this point has the same image under $\vec{a}$ as it does under $\vec{b}$. Obviously this is a pretty trivial theorem — in fact, it is a valid sentence:

$$\dfrac{\dfrac{\vec{a} \overset{*}{=} \vec{b} \qquad \vec{a}(A) = \vec{a}(A)}{\vec{a}(A) = \vec{b}(A)}}{\vec{a} = \vec{b} \Longrightarrow \vec{a}(A) = \vec{b}(A)}$$

and, since 'A' refers only to arguments of the mappings to which '$\vec{a}$' refers, '$\vec{a}(A) = \vec{a}(A)$ is a valid sentence.

On the other hand, the only if-part of Theorem 2-2 is very far from trivial. It tells us that, no matter what point A we may choose, if this point has the same image under both $\vec{a}$ and $\vec{b}$ then $\vec{a} = \vec{b}$ — and, so, each point is bound to have the same image under $\vec{a}$ as it does under $\vec{b}$. This is a very special property of the set of translations and its proof [for which, see TC 102(2), answer for Exercise 2 on page 102] makes use of Postulate 2(b).

In ordinary speech one might state the if-part more explicitly by saying:

> If $\vec{a}$ is $\vec{b}$ then each point has the
> same image under $\vec{a}$ as under $\vec{b}$.

A similar statement of the only if-part is:

> If some point has the same image
> under $\vec{a}$ as under $\vec{b}$ then $\vec{a}$ is $\vec{b}$.

If you have already accustomed your students to the use of quantifiers, you can point out that the if-part says just what is said by:

(1) $$\vec{a} = \vec{b} \Longrightarrow \forall_X\ \vec{a}(X) = \vec{b}(X)$$

while the only if-part has the same content as:

(2) $$\exists_X\ \vec{a}(X) = \vec{b}(X) \Longrightarrow \vec{a} = \vec{b}$$

[Note that, while each of the sentences just displayed has the same content ["says the same thing"] as the corresponding part of

Exploration Exercises

Part A

If we use ordinary function notation to refer to the image of a point under a translation, we can rewrite Theorem 2–2 as:

$$\vec{a}(A) = \vec{b}(A) \Longleftrightarrow \vec{a} = \vec{b}$$

The if-part of this biconditional sentence is:

$$\vec{a} = \vec{b} \Longrightarrow \vec{a}(A) = \vec{b}(A)$$

and its only if-part is:

$$\vec{a}(A) = \vec{b}(A) \Longrightarrow \vec{a} = \vec{b}$$

The if-part is rather trivial; the only if-part says something about translations which is not true of all mappings of $\mathscr{E}$ onto itself.

1. Explain the remarks in the preceding sentence.
2. Suppose that 'f' and 'g' are variables whose values are functions with $\mathscr{R}$ as domain. Here are two sentences about such functions:
   (a) $f = g \Longrightarrow f(a) = g(a)$ **(b)** $f(a) = g(a) \Longrightarrow f = g$
   Is (a) true? How about (b)?
3. Here are two biconditional sentences about functions of the kind referred to in Exercise 2:
   (a) $f = g \Longleftrightarrow f(a) = g(a)$
   (b) $f = g \Longleftrightarrow$ for each real number x, $f(x) = g(x)$
   Is (a) true? How about (b)?
4. Write out the if-part of sentence (b) of Exercise 3 and compare what it means to you with the meaning of sentence (b) of Exercise 2. [*Suggestion*: The sentence you wrote is a conditional sentence, so, in reading it aloud, the first word you say is 'if'. Underline the antecedent of this conditional sentence.]
5. Compare the meaning of sentence (b) of Exercise 2 with that of the false sentence:
   If there is a real number x such that $f(x) = g(x)$ then $f = g$.

Part B

1. Here are two sentences about linearly dependent sequences of vectors:

(a)
$$\left|\begin{array}{c} (\vec{a}a + \vec{b}b = \vec{0} \text{ and } (a \neq 0 \text{ or } b \neq 0)) \\ \Longrightarrow \\ (\vec{a}, \vec{b}) \text{ is linearly dependent} \end{array}\right.$$

(b)
$$\left|\begin{array}{c} (\vec{a}, \vec{b}) \text{ is linearly dependent} \\ \Longrightarrow \\ (\vec{a}a + \vec{b}b = \vec{0} \text{ and } (a \neq 0 \text{ or } b \neq 0)) \end{array}\right.$$

Theorem 2-2 it is not equivalent [in the sense of being mutually replaceable with] to the latter. This is because the corresponding parts of Theorem 2-2 have the same content as the displayed sentences only if these parts are understood as assertions about all values of 'A'.]

Note that, although the if and only if-parts of Theorem 2-2 are consequences of one another, this is not the case with (1) and (2). One of the great advantages of using variables to express universal generalities is that it allows us to say what is said by two sentences like (1) and (2) by asserting a biconditional sentence like Theorem 2-2. [This paves the way for applications of the replacement rule for biconditional sentences.] This advantage is gained at the cost of exposing oneself to the danger of making the elementary error of believing that Theorem 2-2 has the same content as '$\forall_X\ \vec{a}(X) = \vec{b}(X) \Longleftrightarrow \vec{a} = \vec{b}$'. What it does have the same content as is:

$$\forall_X\ [\vec{a}(X) = \vec{b}(X) \Longleftrightarrow \vec{a} = \vec{b}]$$

— a much stronger result.

Answers for Part A

1. [See the preceding commentary.]
2. (a) is true; (b) is false. [Not only the truth but the validity of (a) is established just as is that of the if-part of Theorem 2-2. Counter-examples of (b) are rife. For example, choose for f and g any two linear functions with different slopes, and for a the real number at which these functions have the same value. Or, choose the squaring and cubing functions. These have the same value at 0 and, also, at 1, but are very different functions. It may be well to be very explicit at this point as to the effect of such a counterexample. Letting 'sq' and 'cu' be names for the squaring and cubing functions, one instance of sentence (b) is:

   $$\mathrm{sq}(1) = \mathrm{cu}(1) \Longrightarrow \mathrm{sq} = \mathrm{cu}$$

   Since its antecedent is true [$1^2 = 1^3$] and its consequence is false, this instance of (b) is false. Hence, (b) is false.]
3. (a) is false; it has as one of its consequences the false sentence (b) of Exercise 2.

   (b) is true; it says precisely what we mean by saying that f is the same function as g.
4. The if-part of sentence (b) of Exercise 3 is:

   for each real number x, $f(x) = g(x) \Longrightarrow f = g$

   This means that I can show that f is g by showing that, at each real number, f has the same value as g does. Sentence (b) of Exercise 2 means that I can show that f is g by finding some real number at which f and g have the same value.

[In discussing the if-part of sentence (b) of Exercise 3, distinguish between this conditional sentence and the "universal generalization sentence:

$$\text{for each real number } x\ [f(x) = g(x) \Longrightarrow f = g]$$

This latter has the same content as sentence (b) of Exercise 2. The brackets in the latter sentence are an important clue. They tell us that the quantifying phrase 'for each real number x' "belongs to" the entire sentence. Since this point frequently confuses students we recommend another example at this point. Have students translate into words sentences like:

$$(1)\ \forall_X\ \vec{a}(X) = \vec{b}(X) \Longrightarrow \vec{a} = \vec{b}$$
$$(2)\ \forall_X\ [\vec{a}(X) = \vec{b}(X) \Longrightarrow \vec{a} = \vec{b}]$$
$$(3)\ \exists_X\ \vec{a}(X) = \vec{b}(X) \Longrightarrow \vec{a} = \vec{b}$$
$$(4)\ \exists_X\ [\vec{a}(X) = \vec{b}(X) \Longrightarrow \vec{a} = \vec{b}]$$

The word translations for (1) and (3) start with the word 'if'. This is followed by a quantifying phrase which "belongs to" the antecedent. The word translations for (2) and (4) both begin with quantifying phrases, followed by 'if'. Again notice that brackets were an important signal. This point is brought out again in section 6.07 when we illustrate the way we read sentence (b′).]

5. These two sentences do mean the same thing.

Sample Quiz

1. Consider the sentence:

   $(\vec{m}, \vec{m}x, \vec{a})$ is linearly dependent
   for each real number x.

   If you think that this sentence is a theorem, write 'Yes.' and prove it. If you think that it is not a theorem, write 'No.' and give a counterexample.

2. Assume that $(\vec{a}, \vec{b})$ is linearly independent and that

   $$\vec{a}(x + y + 1) + \vec{b}(2x - y + 3) = \vec{0}.$$

   If it is possible to determine values for 'x' and 'y', do so. If not, explain.

Key to Sample Quiz

1. Yes. Either $x = 0$ or $x \neq 0$. For $x = 0$, the given sequence has $\vec{0}$ as its second term and, so, is linearly dependent. If $x \neq 0$, then $\vec{m}x + (\vec{m}x)\cdot -1 + \vec{a}0 = \vec{0}$ and, since $-1 \neq 0$, the given sequence is linearly dependent. Hence, the given sequence is linearly dependent for each x.

2. By definition, the given equation holds if and only if $x + y + 1 = 0$ and $2x - y + 3 = 0$, for some x and y. And, the latter is the case if and only if $x = -4/3$ and $y = 1/3$.

The preceding discussion should bring out the need for quantifying phrases [in English] and prepare the way for the discussion of quantifiers in section 6.07. [Part B will, then, serve as a check, and to point out the relevance of the discussion to the present chapter.] If, however, you feel in need of another example, you might consider the analogue of the only if-part of Theorem 2-2 for linear functions:

If f and g are linear functions, $a \neq b$, and
$f(a) = g(a)$ and $f(b) = g(b)$ then $f = g$.

Compare this with:

If f and g are linear functions with
slope 1 and $f(a) = g(a)$ then $f = g$.

and with:

If f and g are functions with domain $\mathcal{R}$
and, for each x, $f(x) = g(x)$ then $f = g$.

It is possible that students [or you] may wonder how to prove:

$$(\star)\qquad \forall_x\ f(x) = g(x) \Longrightarrow f = g$$

and where a similar attempt to prove (b) of Exercise 2 breaks down. To prove ($\star$), we derive its consequent from its antecedent and, since functions are sets of ordered pairs, we can infer '$f = g$' if we are able to assert:

$$(a, b) \in f \Longleftrightarrow (a, b) \in g$$

We need to know that, by the definition of function notation,

$$b = g(a) \Longleftrightarrow (a, b) \in g \quad \text{and} \quad b = f(a) \Longleftrightarrow (a, b) \in f$$

To save space, we shall not repeat these below. Instead, we use inferences whose validity follows from the validity of these biconditionals and the replacement rule.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall_x\ f(x)^{\dagger} = g(x)}{f(a) = g(a)} \qquad \dfrac{(a, b^{*}) \in g}{b = g(a)}}{b = f(a)}}{(a, b) \in f}}{(a, b) \in g \Longrightarrow (a, b) \in f}\,{*} \qquad \begin{array}{c} \boxed{\text{By a similar argument,}} \\ \downarrow \\ (a, b) \in f \Longrightarrow (a, b) \in g \end{array}}{(a, b) \in f \Longleftrightarrow (a, b) \in g}}{f = g}\,{\dagger}$$
$$\forall_x\ f(x) = g(x) \Longrightarrow f = g$$

Note that the validity of the next-to-last inference depends on treating its premiss as an assertion about all values of 'a' and all values of 'b'. So, after accepting this inference, we are forbidden by the deduction rule, to discharge an assumption in which either of these variables occurs. This is no actual restriction here since the only such premisses, '$(a, b) \in g$' and '$(a, b) \in f$' [in the unwritten part of the proof] have already been discharged. It explains, however, why we needed '$\forall_x\ f(x) = g(x)$' as an assumption rather than '$f(a) = g(a)$'. The former can be discharged at the end of the proof, since it contains neither 'a' nor 'b'. The latter could not have been.

*

Is (a) true? How about (b)? [*Hint*: If (b) is true then so is each of its instances. Consider the instance for which $\vec{b} = \vec{a}$ and $a = b = 0$. Is the antecedent of this instance true? Is its consequent true?]

2. Write new sentences (a′) and (b′) by replacing the antecedent of (a) and the consequent of (b) by:

   there are real numbers x and y such that
   $(\vec{a}x + \vec{b}y = \vec{0}$ and $(x \neq 0$ or $y \neq 0))$

   Compare the meanings of (a) and (a′). Of (b) and (b′).

3. Write new sentences (a″) and (b″) by replacing the antecedent of (a) and the consequent of (b) by:

   for all real numbers x and y, $(\vec{a}x + \vec{b}y = \vec{0}$ and $(x \neq 0$ or $y \neq 0))$

   Compare the meanings of (a) and (a″). Of (b) and (b″).

4. Complete:
   Of the two sentences (a′) and (a″), only ____ says precisely what (a) says.
   Of the two sentences (b′) and (b″), only ____ says precisely what (b) says.

6.07 Quantifiers

In Chapter 2 you learned that, because we adopted the substitution rule, we can make an assertion about all points [or all translations, or all real numbers] by asserting an appropriate sentence which contains a variable which has all points for its values [or all translations, or all real numbers]. Since then, and up to the beginning of this chapter, we have not needed any other way of making "general assertions". The preceding exploration exercises may have shown you, first, that there is now a need for another way of asserting generalities and, second, how this need is taken care of in English grammar. To be quite clear as to what is involved let's look again at the sentences:

$$\text{(a) } f = g \Longrightarrow f(a) = g(a) \quad \text{and:} \quad \text{(b) } f(a) = g(a) \Longrightarrow f = g$$

in Exercise 2 of Part A.

To simplify matters, let's suppose that each of the letters 'f' and 'g' in (a) and (b) is a name of some particular function whose domain is $\mathscr{R}$. By the substitution rule, each of (a) and (b) implies each of its instances. For example,

(a) implies:	and (b) implies:
$f = g \Longrightarrow f(0) = g(0)$	$f(0) = g(0) \Longrightarrow f = g$
$f = g \Longrightarrow f(3.5) = g(3.5)$	$f(3.5) = g(3.5) \Longrightarrow f = g$
$f = g \Longrightarrow f(\pi) = g(\pi)$	$f(\pi) = g(\pi) \Longrightarrow f = g$
etc.	etc.

Answers for Part B

1. (a) is true. This sentence is comparable with the only if-part of Theorem 2-2. It asserts that $(\vec{a}, \vec{b})$ is linearly dependent if we can find numbers a and b, not both zero such that $\vec{a}a + \vec{b}b = \vec{0}$.

   (b) is false. This sentence is comparable with the if-part of Theorem 2-2. It asserts that if $(\vec{a}, \vec{b})$ is linearly dependent then, whatever numbers a and b we choose, it will turn out that $\vec{a}a + \vec{b}b = \vec{0}$ and that these numbers are not both zero! Any choice of a linearly dependent sequence $(\vec{a}, \vec{b})$ gives a counter-example if we also choose a = 0 and b = 0.

2. Sentences (a) and (a′) have the same meaning; (b) and (b′) do not. In fact, (b) is false and (b′) is true-by-definition.

3. Sentences (a) and (a″) do not have the same meaning; but (b) and (b″) do [and are both false]. [(a″) is true — you cannot find a counter-example — but this is because the antecedent has no true instances.]

4. (a′); (b″)

*

The next two sections, 6.07 and 6.08, contain important discussions concerning the roles of universally and existentially quantified sentences in our formal system. These sections are the last which are concerned with matters of logic. Up to now in the course, it was important that the students understood how to use open sentences to make assertions about all values of the variables in those sentences. It is equally important that they learn how to make use of universally and existentially quantified sentences.

A superficial glance at the material in these sections might give one the impression that it is much too formal for this level. However, by reading the text in class, discussing the ideas presented, and answering questions as they arise, you should find that this material is well within the grasp of the students.

As was suggested in the commentary for page 235, the first pages of section 6.07 serve as commentary for the exercises on pages 235 and 236. Used as such, these pages can be covered in short order, giving you, and the class, a "flying start" into section 6.07.

What (a) says can be put into English by saying:

For any real number a, if f is g then $f(a)$ is $g(a)$.

What (b) says can be put into English by saying:

For any real number a, if $f(a)$ is $g(a)$ then f is g.

It should be obvious from this that (a) is true and that (b) is false. [As to the latter, the identity mapping $i_{\mathcal{R}}$ and the squaring mapping $\{(x, y): y = x^2\}$ have the same value at 0, but are quite different functions.]

There is another way of putting into English what sentence (a) says:

(1) If f is g then f and g have the same value at *each* real number.

Also, there is another way of saying what (b) says:

(2) If f and g have the same value at *some* real number then f is g.

Of course, as before, the first of these sentences is true and the second is false. If we wish to have a sentence which, like (b), ends with 'then f is g' but which, unlike (b), is true then we must say something like:

(3) If f and g have the same value at *each* real number then f is g.

This sentence is the converse of (1) and, combining the two into a biconditional sentence we have:

f is g if and only if f and g have the same value at each real number.

Note that although (a) and (1) convey the same information, their converses (b) and (3) do not. By the use of the word 'each', English manages to express ideas which we cannot express with our variables and substitution rule. Since we need to be able to express these ideas we shall introduce two symbols '∀' and '∃' into our language. You have met these informally before now. They are called 'quantifiers'. The first—*the universal quantifier*—is read as 'for each', and the second—*the existential quantifier*—is read as 'there exists a'. Using these we can write sentences like (1) and (2):

$$\text{(a}'\text{)}\ f = g \Longrightarrow \forall_x f(x) = g(x) \qquad \text{(b}'\text{)}\ \exists_x f(x) = g(x) \Longrightarrow f = g$$

[Read (a′) as 'if $f = g$ then, for each real number x, $f(x) = g(x)$'; read (b′) as 'if there exists a real number x such that $f(x) = g(x)$ then $f = g$'.] As we have seen, (a) and (a′) convey the same information [and are both true]; (b) and (b′) convey the same information [and are both false]. Although (b′) is false, the converse of (a′):

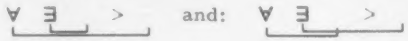
$$\forall_x f(x) = g(x) \Longrightarrow f = g$$

is true. [What is the first word in a word translation of this sentence?] Since (a′) is true it follows that the biconditional sentence:

$$f = g \Longleftrightarrow \forall_x f(x) = g(x)$$

is also true. [See Exercise 3 of Part A.]

Note that indices, unlike variables, are not subject to the substitution rule. Their "proper" role is that of linking the quantifiers '∀' and '∃' with argument places. In this role they could be replaced by arbitrarily chosen meaningless marks or by horizontal brackets, as in:



The other role assigned them in the text — that of cueing one as to how to read a quantifier itself — might be better served by a separate device. More explicitly, the 'x' in a '$\forall_x$' tells one that, in the context in question, '∀' is to be read as 'for each real number', and the $\vec{x}$ in a '$\forall_{\vec{x}}$' tells one that, in that context, '∀' is to be read as 'for each translation'. One might, with advantage, release indices from this duty by using '$\forall\mathcal{R}$' and '$\forall\mathcal{T}$' as different universal quantifiers and make no distinction as to the indices to be used with them.

* * *

The preceding brings out the fact that reading '$\forall_x$' as 'for each real number x', and interpreting the latter in a manner analogous to one's way of interpreting 'for the real number z', does some violence to the proper meaning of '∀'. It is well to remember that our formal language is a written one which is intended to be read rather than spoken. What sounds one makes — or imagines himself to make — when reading such a language to himself is a matter of convention. The meaning of the sentences themselves is, properly, garnered from the rules of logic which apply to them. Of course, such a puristic attitude is not justified in the present circumstances. Here, reading a formal sentence aloud amounts to translating it into the most nearly equivalent English sentence. But, since the translation is seldom quite equivalent to the original, it is important to realize that the rules of logic adopted for the formal language are the final authority in deciding to what extent customary interpretations of the English translations of formal sentences are appropriate.

*

There is sometimes a question concerning the use of the word 'generalization'. To some, 'universal generalization' is a reasonable phrase, but 'existential generalization' is not. Point out, if necessary, that a generalization sentence says "something general" about the members of a set. A universal generalization may be understood as imputing some property to each member of the set. The corresponding existential generalization may be understood as imputing the same property to some member of the set. Both are equally "general" statements. The meaning of 'general' here is the opposite of that of 'particular'. '$\forall_x\ x + 0 = x$' and '$\exists_x\ x + 0 = x$' are general statements; '$2 + 0 = 2$' is a particular statement.

A somewhat similar objection is sometimes made to the use of 'instance' with reference to existential generalizations. The two objections may be related, and eliminating the first may get rid of the second. Another approach is to say that 'instance' is a technical word and, in consequence, it means [in this context at least] what the authors say it means! [But, you might add that it is very generally used in this sense — this is not merely another quirk of the authors.]

The letter 'x' which is attached to the quantifier in the preceding sentence is called *an index*. Just as we have agreed to use letters 'a', 'b', 'c', etc. from the beginning of the alphabet as real number variables, we shall use letters 'z', 'y', 'x', etc. from the end of the alphabet as real number indices. So, one purpose of the letter 'x' in (a′) and (b′) is to tell us that the quantifiers refer to real numbers. [In contrast, '$\forall_{\vec{x}}$' should be read 'for each translation $\vec{x}$', and '$\exists_X$' should be read as 'there exists a point X such that'.] The only other purpose of indices is to *link* each quantifier with the proper places in the expression which follows it. To see the need for such "linking", compare the two sentences:

$$\forall_x \exists_y\, y > x \quad \text{and:} \quad \forall_y \exists_x\, y > x$$

These differ only as to which quantifier is linked to which "side" of '$>$'. The first sentence tells us that, no matter what real number we choose, there is a greater real number – that is, it tells us that there is no greatest real number. On the other hand, the second sentence tells us that there is no *least* real number.

Clearly, it makes no difference what index we choose to use with a given quantifier; the sentences '$\forall_y\, y + 0 = y$' and '$\forall_x\, x + 0 = x$' say the same thing [and each says just what we mean to say when we *assert* either '$a + 0 = a$' or '$b + 0 = b$'].

As another example of the use of quantifiers, consider the sentences in Exercise 1 of Part B on page 235. The first of these:

$$\text{(a)}\quad \left|\begin{array}{c} (\vec{a}a + \vec{b}b = \vec{0} \text{ and } (a \neq 0 \text{ or } b \neq 0)) \\ \Longrightarrow \\ (\vec{a}, \vec{b}) \text{ is linearly dependent} \end{array}\right.$$

is similar to the sentence (b) of Part A. It tells us that $(\vec{a}, \vec{b})$ is linearly dependent if we can find even one pair (a, b) of numbers which are not both 0 and are such that $\vec{a}a + \vec{b}b = \vec{0}$. We can say the same thing by using existential quantifiers:

(a′) $\exists_x \exists_y (\vec{a}x + \vec{b}y = \vec{0}$ and $(x \neq 0$ or $y \neq 0)) \Longrightarrow (\vec{a}, \vec{b})$ is linearly dependent

The converse of (a′) is:

(b′) $(\vec{a}, \vec{b})$ is linearly dependent $\Longrightarrow \exists_x \exists_y (\vec{a}x + \vec{b}y = \vec{0}$ and $(x \neq 0$ or $y \neq 0))$

Both (a′) and (b′) are true. In fact, they are the if-part and only if-part, respectively, of the definition of linear dependence for 2-termed sequences. On the other hand, the converse of (a):

$$\text{(b)}\quad \left|\begin{array}{c} (\vec{a}, \vec{b}) \text{ is linearly dependent} \\ \Longrightarrow \\ (\vec{a}a + \vec{b}b = \vec{0} \text{ and } (a \neq 0 \text{ or } b \neq 0)) \end{array}\right.$$

is, as you have seen in Part B, false. What it asserts can be said by using quantifiers:

(b″) $(\vec{a}, \vec{b})$ is linearly dependent $\Longrightarrow \forall_x \forall_y (\vec{a}x + \vec{b}y = \vec{0}$ and $(x \neq 0$ or $y \neq 0))$ Comparison of (b′) with (b″) should make quite clear the difference in meaning between the converse (b′) of (a′) and the converse (b) of (a).

As in the case of our other logical symbols ['$\Longrightarrow$', 'not', etc.] we need rules of logic for dealing with sentences which contain '$\forall$' and '$\exists$'. To discover these rules, let's begin by considering the sentences:

$$a + 0 = a \quad \text{and:} \quad \forall_x\, x + 0 = x$$

We shall call the second of these *a universal generalization sentence* – specifically, it is the universal generalization of the first sentence with respect to 'a', using 'x' as index. We shall call the first of the two sentences *a general instance* of the second. Finally, sentences which, like '$2 + 0 = 2$' and '$(b + 1) + 0 = b + 1$', are substitution-instances of the first sentence are called *instances* of the second.

We shall use a similar terminology in discussing '$\exists$'. Of the two sentences:

$$2 + a = 2 \quad \text{and:} \quad \exists_x\, 2 + x = 2$$

the second is *the existential generalization* of the first with respect to 'a', using 'x' as index, and the first is *a general instance* of the second. Finally, the substitution-instances of the first sentence are called *instances* of the second.

Exercises

1. Consider the universal generalization sentence:

$$(*) \qquad \forall_x\, (x - 5)^2 + 10x = x^2 + 25$$

(a) Is (∗) true?
(b) Give a general instance of (∗). [Use variable 'a'.]
(c) Give two instances of (∗).

2. Consider the sentence:

$$(**) \qquad (a + 5)^2 - 10(5 - a) = a^2 - 25$$

(a) Give a universal generalization of (∗∗) with respect to 'a'.
(b) When viewed as an assertion is (∗∗) true? Is the universal generalization you wrote for (a) true?
(c) Give two instances of the universal generalization you wrote in (a).

3. **(a)** Give an existential generalization of (∗∗) with respect to 'a'.

(b) Is the existential generalization written in (a) true? Explain your answer.

(c) In Exercise 1(b) you wrote a general instance of (*). Give an existential generalization [of this general instance] with respect to 'a'. Is it true?

*

It is now easy to give two of the rules we need – one for '∀' and one for '∃'. To see what they should be, consider the two inferences:

$$\frac{\forall_x\ x + 0 = x}{2 + 0 = 2} \quad \text{and:} \quad \frac{2 + 0 = 2}{\exists_x\ 2 + x = 2}$$

[Note that the conclusion of the first inference is an instance of its *premiss*, and that the premiss of the second inference is an instance of its *conclusion*.]

Since the premiss of the first inference says that each number satisfies the equation '$a + 0 = a$' and the conclusion says that 2 satisfies this equation, the premiss of this inference implies its conclusion. In other words, the first inference is valid.

Since the conclusion of the second inference says that there exists a number which satisfies the equation '$2 + a = 2$' and the premiss says that 0 satisfies this equation, the premiss of this inference implies its conclusion. In other words, the second inference is valid.

These two examples suggest the rules:

(1) A universal generalization implies any of its instances.

(2) An existential generalization is implied by any of its instances.

These are two of the four rules we shall adopt for dealing with generalization sentences. But, before adopting them, it will be well to be quite sure that what they say is intuitively acceptable.

For this purpose – and to simplify the statement of these and other rules – we need a simpler way of talking "in general terms" about generalization sentences and their instances. We already have such a way of talking about conditional sentences, denial sentences, etc. For example, here is a not very simple way of stating one of our rules:

> Any conditional sentence, together with the denial of its consequent, implies the denial of its antecedent.

Here is another way of stating the same rule:

Any inference of the form:

$$\frac{p \longrightarrow q \qquad \text{not } q}{\text{not } p}$$

is valid.

TC 239

Answers for Exercises

1. (a) Yes. (b) $(a - 5)^2 + 10a = a^2 + 25$

(c) $(b - 5)^2 + 10b = b^2 + 25$, $(7 - 5)^2 + 10\cdot 7 = 7^2 + 25$, etc.

[Note that '$2^2 + 10\cdot 7 = 7^2 + 25$', for example, is not an instance of (☆). It is a consequence of the second of the instances given in answer to (c) and the sentence '$7 - 5 = 2$'. Anyone who knows what an instance is, and knows that '7' is a name for a real number, can give the answer to (c); and he should accept this instance if he accepts (☆). Such a person may, conceivably, not know that $7 - 5 = 2$ and, if so, he will not see any connection between '$(7 - 5)^2 + \ldots$' and '$2^2 + \ldots$'.]

2. (a) Using [for variety] 'y' as index:

$$\forall_y\ (y + 5)^2 - 10(5 - y) = y^2 - 25$$

[Any other real number index would do as well.]

(b) No.; No.

(c) $(0 + 5)^2 - 10(5 - 0) = 0^2 - 25$, $((a + 2) + 5)^2 - 10(5 - (a + 2)) = (a + 2)^2 - 25$, etc. [Call attention to the fact that, even though (☆☆) — as an assertion — is false, (☆☆) does have some true instances.]

3. (a) $\exists_x\ (x + 5)^2 - 10(5 - x) = x^2 - 25$

TC 240 (1)

(b) Yes. 0 is "such a number" — that is, the instance '$(0 + 5)^2 - 10(5 - 0) = 0^2 - 25$' is true.

(c) $\exists_x\ (x - 5)^2 + 10x = x^2 + 25$. This sentence is true. As a matter of fact, its truth follows from that of (☆) — if each number has a given property then [since there are numbers] some number has that property.

As students discuss this text material, we suggest constructing charts similar to the following on the chalkboard.

Universal Generalization	$\forall_x$ Fx	$\forall_x$ xb = c	$\forall_x$ [x + x = x $\Longrightarrow$ x = 0]
General Instance	Fa	ab = c	a + a = a $\Longrightarrow$ a = 0
Instance	F2 F5	2b = c 5b = c	2 + 2 = 2 $\Longrightarrow$ 2 = 0 5 + 5 = 5 $\Longrightarrow$ 5 = 0

Existential Generalization	$\exists_x$ Fx	$\exists_x$ 2 + x = 2	$\exists_x$ [x + x = x $\Longrightarrow$ x = 0]
General Instance	Fa	2 + a = 2	a + a = a $\Longrightarrow$ a = 0
Instance	F0 F7	2 + 0 = 2 2 + 7 = 2	0 + 0 = 0 $\Longrightarrow$ 0 = 0 7 + 7 = 7 $\Longrightarrow$ 7 = 0

In order to deal equally simply with rules about generalization sentences we need some notation, analogous to the letters 'p' and 'q', which will enable us to indicate any generalization sentence, any *general* instance of the same sentence, and any instance of it whatever. What we shall do is use expressions like 'Fa', 'Gc', etc. as place-holders for sentences which contain variables which we wish to refer to, and '$\forall_x Fx$', '$\exists_y Gy$', etc. as place-holders for corresponding generalizations with respect to these variables. For example, if '$\forall_x Fx$' occurs in the statement of some rule, and we wish to obtain an example of the rule by replacing 'Fa' by '$ab = c$', then we should replace '$\forall_x Fx$' by '$\forall_x xb = c$'. If '$F2$' appears in the same rule then, in obtaining the same example, we should replace it by '$2b = c$'.

As another example, let's consider the pattern sentence:

$$F0 \Longrightarrow \exists_x Fx$$

To obtain an example of a sentence of this form, we need to choose what to take for 'Fa'. If we choose, say, '$2 + a = 2$' then we should replace '$F0$' by '$2 + 0 = 2$' and replace '$\exists_x Fx$' by '$\exists_x 2 + x = 2$'. Doing so we obtain the sentence:

$$2 + 0 = 2 \Longrightarrow \exists_x 2 + x = 2$$

Exercises

1. We have agreed that 'Fa' is a place-holder for sentences which contain the variable 'a'. Choosing '$2a = 2$' for 'Fa', obtain the corresponding sentence of the form '$F1 \Longrightarrow \exists_x Fx$'. [*Hint*: $2 \cdot 1 = 2 \Longrightarrow \exists_x \ldots$.]
2. Repeat Exercise 1, choosing '$|1 - 3a| = 2$' for 'Fa'.
3. In each of the following, you are given a choice for 'Fa' and a form for sentences. Write the sentence of the given form which corresponds with the given choice for 'Fa'.
   (a) $2a = 10$; $F5$ (b) $2a = 10$; $\forall_x Fx$
   (c) $10/a = 2$; $F5 \Longrightarrow \exists_x Fx$ (d) $-a^2 = a \cdot a$; $\exists_x Fx$
   (e) $-a^2 = a \cdot a$; $\exists_x Fx \Longrightarrow F2$ (f) $-a^2 = a \cdot a$; $F0 \Longrightarrow \exists_x Fx$
4. Directions: Replace 'Fa' by '$G\vec{a}$' in Exercise 3 and do these problems.
   (a) $B - A = \vec{a}$; $G\vec{g}$ (b) $B - A = \vec{a}$; $G\vec{g} \Longrightarrow \exists_{\vec{x}} G\vec{x}$
   (c) $B = A + \vec{a}$; $G\vec{0} \Longrightarrow \exists_{\vec{x}} G\vec{x}$ (d) $B = A + \vec{a}$; $\exists_{\vec{x}} G\vec{x} \Longrightarrow G\vec{0}$

*

Now, let's consider sentences of the form:

$$\forall_x Fx \Longrightarrow Fa$$

TC 240 (2)

The example is chosen to illustrate a point connected with the new notation. In order to construct a sentence of the given form one must begin by choosing a sentence of the form 'Fa' [or 'Fb', etc.] from which to obtain a replacement for '$F0$' by substitution and a replacement for '$\exists_x Fx$' by existential generalization. In making this initial choice, it is well to use a variable which does not occur in the given "pattern sentence". For example, the sentence:

$$a + 0 = a \Longrightarrow \exists_x x + 0 = a$$

is of the form:

$$Fa \Longrightarrow \exists_x Fx$$

To show this, choose '$b + 0 = a$' as a replacement for 'Fb'. Then 'Fa' is to be replaced by '$a + 0 = a$' and '$\exists_x Fx$' by '$\exists_x x + 0 = a$'. Uses of the new symbolism like that just illustrated may be difficult at first for students. It is probably best to disregard the problem until it arises naturally — say, through a student's saying "I don't see why they think that this is of that form."

TC 241

Answers for Exercises

1. $2 \cdot 1 = 2 \Longrightarrow \exists_x 2x = 2$
2. $|1 - 3 \cdot 1| = 2 \Longrightarrow \exists_x |1 - 3x| = 2$
3. (a) $2 \cdot 5 = 10$ (b) $\forall_x 2x = 10$
   (c) $10/5 = 2 \Longrightarrow \exists_x 10/x = 2$ (d) $\exists_x -x^2 = x \cdot x$
   (e) $\exists_x -x^2 = x \cdot x \Longrightarrow -2^2 = 2 \cdot 2$ (f) $-0^2 = 0 \cdot 0 \Longrightarrow \exists_x -x^2 = x \cdot x$
4. (a) $B - A = \vec{g}$ (b) $B - A = \vec{g} \Longrightarrow \exists_{\vec{x}} B - A = \vec{x}$
   (c) $B = A + \vec{0} \Longrightarrow \exists_{\vec{x}} B = A + \vec{x}$ (d) $\exists_{\vec{x}} B = A + \vec{x} \Longrightarrow B = A + \vec{0}$
   [Note that, taken as assertions, (c) is true and (d) is false.]

For any sentence of this form there is a corresponding English sentence of the form:

(*) For any *a*; if, for each *x*, *Fx* then *Fa*.

Certainly, no matter what sentence you chose to replace '*Fa*' by, you would be willing to accept the resulting sentence of the form (*). Your willingness to do so would have nothing to do with what the sentence you chose for '*Fa*' might say. You would accept the resulting sentence just because it had the form (*). In short, any sentence of the form (*) is not only true but *valid*. Its truth follows from its form, independently of what sentences replacing '*Fa*' may say.

Precisely the same remarks may be made about any sentence of the form:

(**) For any *a*; if *Fa* then, for *some x*, *Fx*.

Any sentence of the form (**) is valid, no matter what sentence replaces '*Fa*'.

Translating (*) and (**) into our notation we obtain the rule:

Any sentence of either of the forms:

$$\forall_x Fx \Longrightarrow Fa \qquad Fa \Longrightarrow \exists_x Fx$$

is valid.

Obviously, in this and similar rules we could just as well write '*c*' instead of '*a*', or '*y*' instead of '*x*'; and we could write '$\vec{a}$' instead of '*a*' if we also wrote '$\vec{x}$' or '$\vec{y}$', say, instead of '*x*'. In short '*a*' can be replaced by any variable and '*x*' by any index which refers to the domain of that variable.

We are now almost ready to reformulate the two rules on page 240 about generalization sentences. The first of these is:

(1) A universal generalization implies any of its instances.

We have seen that any sentence of the form '$\forall_x Fx \Longrightarrow Fa$' is valid and [using this and modus ponens], it is easy to conclude that any inference of the form:

$$\frac{\forall_x Fx}{Fa}$$

is valid. This means, precisely, that a universal generalization implies any of its *general* instances. To complete our task we must get rid of the restrictive word 'general', and state the resulting rule in a simple way. To do so, let's agree that when we are given expressions '*Fa*' and '*Ft*' we are to choose, as before, a sentence as a replacement for '*Fa*', to choose a *term* as a replacement for '*t*', and are to replace '*Ft*' by the sentence which results from the chosen one when the chosen term is substituted for '*a*'. What this amounts to is that, whatever sentence replaces '$\forall_x Fx$', we can arrange things so as to replace '*Ft*' by any instance of this generalization we wish merely by making the appropriate choice for '*t*'. In these terms, the rule (1) may be stated:

Any inference of the form:

$$\frac{\forall_x Fx}{Ft}$$

is valid.

The rule can be justified by the following scheme:

$$\dfrac{\forall_x Fx \quad \dfrac{\forall_x Fx \Longrightarrow Fa}{\forall_x Fx \Longrightarrow Ft}\,\text{(Subst)}}{Ft}\,\text{(Modus ponens)}$$

Since substitution-inferences and modus ponens inferences are valid it follows that any inference of the form:

$$\frac{\forall_x Fx \quad \forall_x Fx \Longrightarrow Fa}{Ft}$$

is valid. So, since any sentence of the form '$\forall_x Fx \Longrightarrow Fa$' is valid it follows that any inference of the form:

$$\frac{\forall_x Fx}{Ft}$$

is valid—that is, a universal generalization implies any of its instances.

Exercises

1. In the following, you are given sentences of the form '$\forall_x Fx$'. In each case, write the instance *F*1.
   (a) $\forall_x\, x^2 - 1 = (x + 1)(x - 1)$ (b) $\forall_x\, x^2 - x = 0$
   (c) $\forall_x\, (2x - 1)^2 = 2x^2 - 1$ (d) $\forall_x\, \sqrt{(-x)^2} = |x|$
2. In the following, you are given sentences of the form '$\forall_{\vec{x}}\, G\vec{x}$'. In each case, write the instance $G(B - A)$.
   (a) $\forall_{\vec{x}}\, [\vec{x} = B - A \Longrightarrow A + \vec{x} = B]$ (b) $\forall_{\vec{x}}\, [A + \vec{x} = A \Longrightarrow \vec{x} = \vec{0}]$
   (c) $\forall_{\vec{x}}\, [C - \vec{x} = B - \vec{x} \Longrightarrow C = B]$ (d) $\forall_{\vec{x}}\, \vec{x} - \vec{x} = \vec{0}$

3. We have restated rule (1) on page 240 as an inference rule and, in the paragraph just preceding these exercises, we have justified it by using part of the rule on page 242 and other rules of logic. Do the same for rule (2) on page 240.
4. The if-part and the only if-part of Theorem 2-2 are:

   (a) $\vec{a} = \vec{b} \longrightarrow A + \vec{a} = A + \vec{b}$ **(b)** $A + \vec{a} = A + \vec{b} \longrightarrow \vec{a} = \vec{b}$

   Here are two closely related sentences:
   (a′) $\vec{a} = \vec{b} \longrightarrow \forall_X X + \vec{a} = X + \vec{b}$
   (b′) $\exists_X X + \vec{a} = X + \vec{b} \longrightarrow \vec{a} = \vec{b}$
   Using our new rules for generalization sentences [and our rules for conditional sentences] it is easy to show that sentence (a′) implies sentence (a) and that sentence (b′) implies sentence (b). Do so. [*Hint*: (a) and (b) are conditional sentences; and one way to derive a conditional sentence from given premisses is to use its antecedent as an extra premiss and try to derive its consequent.]
5. Use the same procedure as in Exercise 4 to show that any inference of either of the forms:

$$\frac{p \longrightarrow \forall_x Fx}{p \longrightarrow Fa} \qquad \frac{\exists_x Fx \longrightarrow q}{Fa \longrightarrow q}$$

   is valid.

*

The two rules for generalization sentences which we have adopted may be stated together as:

> Any inference of either of the forms:
>
> $$\frac{\forall_x Fx}{Ft} \qquad \frac{Ft}{\exists_x Fx}$$
>
> is valid.

To understand the remaining two rules we need to recall the distinction between using a sentence as an assertion and using it as an assumption. [This distinction was brought out in Part B on pages 86 and 87 and, as was pointed out on pages 87 and 88, is essential for our statement of the deduction rule.] Recognizing this distinction led us to insert the underlined words in the statement of the substitution rule:

> Any sentence which is used to make an assertion about all values of some variable implies each of its substitution-instances with respect to this variable.

Answers for Exercises

1. (a) $1^2 - 1 = (1 + 1)(1 - 1)$ (b) $1^2 - 1 = 0$
   (c) $(2 \cdot 1 - 1)^2 = 2 \cdot 1^2 - 1$ (d) $\sqrt{(-1)^2} = |-1|$
   [In connection with (d), lead students to recall that '$\forall_x \sqrt{x^2} = x$' is false.]

2. (a) $B - A = B - A \Longrightarrow A + (B - A) = B$
   (b) $A + (B - A) = A \Longrightarrow B - A = \vec{0}$
   (c) $C - (B - A) = B - (B - A) \Longrightarrow C = B$
   (d) $(B - A) - (B - A) = \vec{0}$

TC 244

3.

$$\cfrac{Ft \qquad \cfrac{Fa \Longrightarrow \exists_x Fx \leftarrow \text{valid}}{Ft \Longrightarrow \exists_x Fx}}{\exists_x Fx}$$

This justifies the rule:

Any inference of the form:

$$\frac{Ft}{\exists_x Fx}$$

is valid.

4. To show that sentence (a′) implies sentence (a):

$$\cfrac{\cfrac{\cfrac{\vec{a} \overset{*}{=} \vec{b} \qquad \vec{a} = \vec{b} \Longrightarrow \forall_X X + \vec{a} = X + \vec{b}}{\forall_X X + \vec{a} = X + \vec{b}}}{A + \vec{a} = A + \vec{b}}}{\vec{a} = \vec{b} \Longrightarrow A + \vec{a} = A + \vec{b}}\,*$$

To show that sentence (b′) implies sentence (b):

$$\cfrac{\cfrac{\cfrac{A + \vec{a} \overset{*}{=} A + \vec{b}}{\exists_X X + \vec{a} = X + \vec{b}} \qquad \exists_X X + \vec{a} = X + \vec{b} \Longrightarrow \vec{a} = \vec{b}}{\vec{a} = \vec{b}}}{A + \vec{a} = A + \vec{b} \Longrightarrow \vec{a} = \vec{b}}\,*$$

5.

$$\cfrac{\cfrac{\cfrac{\overset{*}{p} \qquad p \Longrightarrow \forall_x Fx}{\forall_x Fx}}{Fa}}{p \Longrightarrow Fa}\,* \qquad \cfrac{\cfrac{\cfrac{\overset{*}{Fa}}{\exists_x Fx} \qquad \exists_x Fx \Longrightarrow q}{q}}{Fa \Longrightarrow q}\,*$$

Incidentally, we can restate this rule to advantage in terms of our new notation:

> Any inference of the form:
> $$\frac{Fa}{Ft}$$
> is valid, provided that its premiss is used as an assertion about all values of the indicated variable.

The need to distinguish between assertions and assumptions arose in applying the deduction rule. Roughly, what we learned was that inferences [such as substitution] whose validity depends on their premisses being understood as assertions about all values of some variable should not be used in deriving consequences of *assumptions* in which this variable occurs. [For a more precise statement, see (*) on page 87.] The two kinds of inference which remain to be considered are of this kind.

The first has to do with universal generalizations and we shall again consider the sentences:

$$a + 0 = a \quad \text{and:} \quad \forall_x\, x + 0 = x$$

When we use the first of these as an assertion we mean to say that, no matter what number we choose, the result of adding 0 to it is the number itself. But, this is precisely what the second sentence says. So, *when the first is used as an assertion* it implies the second. [On the other hand, if we started a proof by saying '*Suppose* that $a + 0 = a$.' it would certainly not be reasonable to continue with 'It follows that $\forall_x\, x + 0 = x$.' The use of '$a + 0 = a$' *as an assumption* would merely give notice that we intended to ignore any values of 'a' which do not satisfy this equation.] So, in case '$a + 0 = a$' is used as an *assertion* the inference:

$$\frac{a + 0 = a}{\forall_x\, x + 0 = x}$$

is valid. Similar considerations apply to any general instance of any universal generalization:

> Any sentence which is used to make an assertion about all values of some variable implies any universal generalization of itself with respect to this variable.

More simply put:

> Any inference of the form:
> $$\frac{Fa}{\forall_x\, Fx}$$
> is valid, provided that its premiss is used as an assertion about all values of the indicated variable.

[As remarked on page 242, in applying this rule, 'a' can be replaced by any variable and 'x' by any index which refers to the domain of that variable.]

To illustrate the use of this new rule let's consider again the sentences:

$$\text{(a)}\ \vec{a} = \vec{b} \longrightarrow A + \vec{a} = A + \vec{b} \qquad \text{(a}'\text{)}\ \vec{a} = \vec{b} \longrightarrow \forall_X\, X + \vec{a} = X + \vec{b}$$

of Exercise 4 on page 244. In that exercise you used our first rule for universal generalizations to derive (a) from (a′). Now we can use our new rule to derive (a′) from (a). Here's how:

$$\dfrac{\dfrac{\dfrac{\overset{*}{\vec{a} = \vec{b}} \qquad \vec{a} = \vec{b} \longrightarrow A + \vec{a} = A + \vec{b}}{A + \vec{a} = A + \vec{b}}}{\forall_X\, X + \vec{a} = X + \vec{b}}}{\vec{a} = \vec{b} \longrightarrow \forall_X\, X + \vec{a} = X + \vec{b}}\,^{*}$$

We need the new rule to tell us that the middle one of the three inferences is valid. According to the rule, this inference is valid *provided that its premiss* '$A + \vec{a} = A + \vec{b}$' is used as an assertion about all values of 'A'. To see whether this is the case we must look at the sentences from which we derived '$A + \vec{a} = A + \vec{b}$'. These sentences are the two premisses of the derivation and, of these, only the second one contains 'A'. Since we have not used *this* premiss as an assumption, we are using it to make an assertion about all values of 'A' and, so, the same is true of '$A + \vec{a} = A + \vec{b}$'. Put in another way, since the first premiss of the derivation '$\vec{a} = \vec{b}$' does not contain the variable 'A' which has been "generalized on" in the second inference, we are justified in treating this premiss as an assumption and in discharging it when, at the end of the proof, we use the deduction rule. In contrast, since the second premiss '$\vec{a} = \vec{b} \longrightarrow A + \vec{a} = A + \vec{b}$' does contain the variable 'A', we would not be justified to treat it as an assumption, and to discharge it, once the variable 'A' has been generalized on.

For comparison, let's consider the derivation of (a) form (a′):

$$\cfrac{\cfrac{\cfrac{\overset{*}{\vec{a} = \vec{b}} \quad \vec{a} = \vec{b} \longrightarrow \forall_X X + \vec{a} = X + \vec{b}}{\forall_X X + \vec{a} = X + \vec{b}}}{A + \vec{a} = A + \vec{b}}}{\vec{a} = \vec{b} \longrightarrow A + \vec{a} = A + \vec{b}}\,*$$

This is much like the derivation we have just given of (a′) from (a). The difference is that the middle inference is to be justified by our first rule for universal generalizations, rather than by the second. Since this first rule is not restricted as the second one is, the justification of the inference requires us only to note that its conclusion is, indeed, an instance of its premiss.

The point which has been brought out in connection with these two derivations can be seen very simply by considering the following two derivations:

$$\cfrac{\cfrac{\overset{*}{\forall_x Fx}}{Fa}}{\forall_x Fx \longrightarrow Fa}\,* \qquad \cfrac{\cfrac{\overset{*}{Fa}}{\forall_x Fx}}{Fa \longrightarrow \forall_x Fx}\,*$$

Each of these "claims" that its conclusion is valid. In the case of the first derivation the claim is justified; in the case of the second *it is not.* For the first inference in the second derivation to be valid, its premiss 'Fa' must be taken as an assertion about all values of 'a'. So, it cannot be taken as an assumption and discharged, as indicated, on the authority of the deduction rule. In short [in the second derivation], either the first inference is invalid or the deduction rule has been improperly applied. *In either case the derivation is invalid.* [This is fortunate. For, if derivations of this kind were valid, we could prove, for example, '$a = 0 \longrightarrow \forall_x x = 0$'. From this and '0 = 0' we could deduce '$\forall_x x = 0$'!]

If you study the discussion on page 244 of the derivation of (a′) from (a) you will see that what this derivation illustrates is that any inference of the form:

$$\frac{p \longrightarrow Fa}{p \longrightarrow \forall_x Fx}$$

is valid provided that its premiss is taken as an assertion about all values of the indicated variable *and* provided that this variable does not occur in the sentence which is taken for 'p'. [Recall that in the derivation of (a′) from (a) it was important that 'A' did not occur in '$\vec{a} = \vec{b}$'.]

Pages 245 through 247 contain some of our basic rules concerning universally and existentially quantified sentences. Students probably should not be left to their own devices to read this material. Rather, reading and discussion should be done in class.

* * *

An intuitive test for whether a valid inference does or does not block a variable consists in substituting for the variable in question in both the premisses and conclusion of the inference. If, for some substitution, the resulting inference is invalid then the given inference is of the first kind; otherwise it is of the second kind. For example, a typical substitution inference:

$$\frac{a + b = b + a}{1 + b = b + 1}$$

is of the first kind because, for, example, the inference:

$$\frac{2 + b = b + 2}{1 + b = b + 1}$$

obtained by substituting '2' for 'a' is invalid. On the other hand, a modus ponens inference such as:

$$\frac{a \neq 0 \qquad a \neq 0 \Longrightarrow [\vec{a}a = \vec{0} \Longrightarrow \vec{a} = \vec{0}]}{\vec{a}a = \vec{0} \Longrightarrow \vec{a} = \vec{0}}$$

is of the second kind since the result of any substitution for 'a' or for '$\vec{a}$' is again a modus ponens inference and, so, is valid. [This test is, of course, purely intuitive. Its only use is in motivating our choice as to which rules must be encumbered by provisos like that in the boxed statement on page 245 of the substitution rule.]

As noted in the text, the distinction discussed in the preceding paragraph is vital for proper use of the deduction rule.

Up to now, the only basic inferences of the first kind are substitution inferences. These together with the two new kinds of inference about to be introduced are the only inferences of the first kind which will be taken as basic in our development of logic. There will, however, be other inferences of this kind among those whose justifying schemes contain basic inferences of one or more of the three types just mentioned. [For examples, see Exercises 1 and 2 of Part C on page 251.]

The most immediate effect of provisos such as that included in the substitution rule is illustrated by the fact that although, for example:

$$\frac{a + b = b + a}{1 + b = b + 1}$$

is a valid inference, it does not [fortunately] follow that:

$$a + b = b + a \Longrightarrow 1 + b = b + 1$$

is a valid sentence. [''Fortunately'' because if the sentence in question were valid then so would be '$b + b = b + b \Longrightarrow 1 + b = b + 1$' and, with it, '1 + b = b + 1'. Although true, the last is certainly not valid. The validity of the inference shows that the truth of '1 + b = b + 1' follows from that of the commutative principle for addition; but the truth of '1 + b = b + 1' is evidently not merely a consequence of logical principles.] A more striking example is given on page 247.

Exercises

1. In which of the following arguments is the conclusion a valid one. Why?
   (i) Suppose that for each translation $\vec{x}$, $A + \vec{x} = A$. It follows, in particular, that $A + \vec{a} = A$. Hence if, for each $\vec{x}$, $A + \vec{x} = A$ then $A + \vec{a} = A$.
   (ii) Suppose that $A + \vec{a} = A$. It follows that, for each $\vec{x}$, $A + \vec{x} = A$. Hence, if $A + \vec{a} = A$ then, for each $\vec{x}$, $A + \vec{x} = A$.
2. In each part, you are given an inference of the following form:

$$\frac{p \longrightarrow Fa}{p \longrightarrow \forall_x Fx}$$

You are to decide, in each case, whether or not the inference is valid, and to give a reason for your decision.

(a) $$\frac{A + \vec{a} = A + \vec{b} \longrightarrow \vec{a} = \vec{b}}{A + \vec{a} = A + \vec{b} \longrightarrow \forall_{\vec{x}}\, \vec{x} = \vec{b}}$$

(b) $$\frac{\vec{a} = \vec{b} \longrightarrow A + \vec{a} = A + \vec{b}}{\vec{a} = \vec{b} \longrightarrow \forall_X\, X + \vec{a} = X + \vec{b}}$$

(c) $$\frac{A = B \longrightarrow A + \vec{b} = B + \vec{b}}{A = B \longrightarrow \forall_{\vec{x}}\, A + \vec{x} = B + \vec{x}}$$

(d) $$\frac{A + \vec{b} = B + \vec{b} \longrightarrow A = B}{A + \vec{b} = B + \vec{b} \longrightarrow \forall_X\, X = B}$$

*

To arrive at our final rule, let's consider as an example the two sentences:

$$A + \vec{a} = A + \vec{b} \longrightarrow \vec{a} = \vec{b} \quad \text{and:} \quad \exists_X\, X + \vec{a} = X + \vec{b} \longrightarrow \vec{a} = \vec{b}$$

The first of these is the only if-part of Theorem 2-2 and [when used as an assertion] is true. It tells us that

> for any point A, if A has the same
> image under both $\vec{a}$ and $\vec{b}$ then $\vec{a}$ is $\vec{b}$.

In other words,

> if there is any point which has the same
> image under both $\vec{a}$ and $\vec{b}$ then $\vec{a}$ is $\vec{b}$.

This last is precisely what the second of the two sentences says. This being so, the inference:

$$\frac{A + \vec{a} = A + \vec{b} \longrightarrow \vec{a} = \vec{b}}{\exists_X\, X + \vec{a} = X + \vec{b} \longrightarrow \vec{a} = \vec{b}}$$

TC 245,247 (2)

Note that corresponding sentences of the form 'Fa' and '$\forall_x$ Fx' furnish an example of sentences which are not equivalent but — when the former is asserted — have the same content. [This distinction is discussed on TC 155(2).]

Note also that the substitution rule, as stated on page 245, can be derived from the two rules for universal generalizations which are given on pages 243 and 244. So, the substitution rule need be taken as a primitive rule only in languages which, like that used in Chapters 1 - 5, contain variables but do not make use of quantifiers.

Inferences of the form displayed near the foot of the page can be justified by the scheme:

$$\dfrac{\dfrac{\dfrac{\overset{*}{p} \qquad p \Longrightarrow Fa}{Fa}}{\forall_x\ Fx}}{p \Longrightarrow \forall_x\ Fx}\,*$$

The validity of the second inference requires that its premiss be taken as an assertion about all values of 'a'. For this to be the case, the premiss '$p \Longrightarrow Fa$' must be taken as such an assertion. Also, 'p' must be taken as such an assertion if 'a' occurs in 'p'. In this case, however, 'p' could not be discharged as indicated. Hence, for the validity of the total scheme we must require that 'a' not occur in its first premiss [p] and that its second premiss be taken as an assertion about all values of 'a'.

*

TC 248

Answers for Exercises

1. Argument (i) is valid, and its conclusion is a valid sentence. [Of course, neither argument (i) nor its conclusion is interesting, since the assumption '$\forall_{\vec{x}}\ A + \vec{x} = A$' is not satisfied by any point.]

   Argument (ii) is invalid since the first inference precludes the use made of the deduction rule. The conclusion of the argument is false since it, together with the theorem '$A + \vec{0} = A$' implies the false statement '$\forall_{\vec{x}}\ A + \vec{x} = A$'.

2. (a) invalid (b) valid (c) valid (d) invalid
   Each of the four inferences is of the required form, but the proviso that "the variable which is generalized in the consequent must not occur in the antecedent" is satisfied only in the case of (b) and (c).

should certainly be reckoned as valid—at least when its premiss is used as an assertion. Similar considerations apply to similar pairs of sentences:

$$\frac{a + c = b + c \longrightarrow a = b}{\exists_x\, a + x = b + x \longrightarrow a = b}$$

$$\begin{array}{c}(a\vec{a} + b\vec{b} = \vec{0} \text{ and } (a \neq 0 \text{ or } b \neq 0)) \longrightarrow (\vec{a}, \vec{b}) \text{ is linearly dependent}\\ \hline \exists_y\, (a\vec{a} + \vec{b}y = \vec{0} \text{ and } (a \neq 0 \text{ or } y \neq 0)) \longrightarrow (\vec{a}, \vec{b}) \text{ is linearly dependent}\\ \hline \exists_x\exists_y (\vec{a}x + \vec{b}y = \vec{0} \text{ and } (x \neq 0 \text{ or } y \neq 0)) \longrightarrow (\vec{a}, \vec{b}) \text{ is linearly dependent}\end{array}$$

$$\frac{f(a) = g(a) \longrightarrow f = g}{\exists_x\, f(x) = g(x) \longrightarrow f = g}$$

are all valid inferences if their premisses are taken as assertions. [The last inference is valid—if its premiss is asserted—even though its premiss and conclusion are both false. Anyone who does assert the premiss should, then, be willing to accept the conclusion.] In general, any inference of the form:

$$\frac{Fa \longrightarrow q}{\exists_x\, Fx \longrightarrow q}$$

is valid provided that its premiss is taken as an assertion about all values of the indicated variable *and* provided that this variable does not occur in the sentence which is taken for '*q*'. [Compare this with the rule stated in the text on page 247.]

The intuitive justification of this rule may be seen more clearly if we state it in a slightly different [but equivalent] form:

> Any inference of the form:
>
> $$\frac{\exists_x\, Fx \quad Fa \longrightarrow q}{q}$$
>
> is valid provided that its conditional premiss be taken as an assertion about all values of the indicated variable, and provided that this variable does not occur in the sentence which is taken for '*q*'.

[This second rule follows from the one first given and modus ponens. The second rule, together with the deduction rule, justifies the first rule.] This second form of the rule can be stated intuitively as follows:

> In case we know that, whatever *a* may be, if *Fa* then *q* it follows, in case there is an *x* such that *Fx*, that *q*.

The reasonableness of the proviso concerning inferences of the form:

(☆) $$\frac{Fa \Longrightarrow q}{\exists_x\, Fx \Longrightarrow q}$$

— and the validity of such inferences subject to this proviso — may be argued in the following way. Let's choose sentences by which to replace 'Fa' and 'q', and consider the inference:

(☆☆) $$\frac{Fa \qquad Fa \Longrightarrow q}{q}$$

If we substitute for 'a', throughout this inference (☆☆), any numeral we choose, the resulting inference will be valid. Moreover, if 'a' does not occur in the sentence chosen for 'q', the conclusion of the resulting inference will be the same no matter what numeral we choose to substitute for 'a'. Should we be fortunate enough to find a substitution such that both premisses of the resulting inference are acceptable, then we would be justified in accepting the conclusion. In working toward this end, our task will be lightened considerably if our choice of replacements for 'Fa' and 'q' has been such that we can accept the resulting 'Fa ⇒ q' when this is taken as an assertion about all values of 'a'. For, in this case, we need only search for numbers which satisfy the sentence chosen for 'Fa'. If we are successful in finding such a number, we shall be justified in accepting the conclusion of (☆☆). Now, in such circumstances as these, any grounds we may have for believing that there is a number which satisfies the sentence chosen for 'Fa' are equally strong grounds for believing the sentence chosen for 'q'. But, this is sufficient intuitive ground for agreeing that the inference:

$$\frac{\exists_x\, Fx \qquad Fa \Longrightarrow q}{q}$$

is valid — presuming of course that its second premiss is taken as an assertion about all values of 'a', and that 'a' does not occur in its conclusion. Since 'a' certainly does not occur in the first premiss of this inference, we can now apply the deduction rule to show that (☆) is valid — subject of course, to the conditions just stated.

It is not strictly necessary to show that the limitations imposed on the validity of inferences of the form (☆) are unavoidable. Still, doing so has a certain independent interest. If 'a' is allowed to occur in the replacement for 'q' then inferences of the form:

$$\frac{Fa \Longrightarrow Fa}{\exists_x\, Fx \Longrightarrow Fa}$$

come into question. If such inferences were valid then so would be sentences of the form:

(☆☆☆) $$\exists_x\, Fx \Longrightarrow Fa$$

The same result would ensue if it were permissible to treat the premiss of a (☆)-type inference as an assumption:

$$\dfrac{Fa \Longrightarrow Fa \qquad \dfrac{\dfrac{\dfrac{Fa \overset{*}{\Longrightarrow} q}{\exists_x\ Fx \Longrightarrow q}}{[Fa \Longrightarrow q] \Longrightarrow [\exists_x\ Fx \Longrightarrow q]}\,*}{[Fa \Longrightarrow Fa] \Longrightarrow [\exists_x\ Fx \Longrightarrow Fa]}\,\text{(Subst)}}{\exists_x\ Fx \Longrightarrow Fa}$$

If, now, in (☆☆☆), we choose 'a = 0' for 'Fa', it would follow that the sentence:

$$\exists_x\ x = 0 \Longrightarrow a = 0$$

is valid. Since the antecedent of this is a consequence of the valid sentence '0 = 0' it would then follow that 'a = 0' is valid. So, if either part of the proviso concerning (☆) were deleted, we should be able to prove — by "logic" alone — that there are no numbers other than zero! Such a "logic" would, of course, be unacceptable.

The equivalence of the boxed rule with that stated previously is easy to show:

$$\dfrac{\exists_x\ Fx \qquad \dfrac{Fa \Longrightarrow q}{\exists_x\ Fx \Longrightarrow q}}{q} \qquad\qquad \dfrac{\dfrac{\overset{*}{\exists_x\ Fx} \qquad Fa \Longrightarrow q}{q}}{\exists_x\ Fx \Longrightarrow q}\,*$$

The application of the deduction rule in the second scheme is justified by the fact that, although the second premiss of the scheme must be taken as an assertion about all values of 'a', 'a' does not occur in the first premiss — and it is this premiss which is discharged by the application of the deduction rule.

With the four rules for quantifiers, we have completed the list of basic rules of the logic of sentence connectives and quantifiers. If, as is often done, set-theory is taken to be a part of logic then there are other rules of logic. Of these we cite only two. The first serves as a definition of brace notation:

> Any sentence of the form:
>
> $$a \in \{x:\ Fx\} \Longleftrightarrow Fa$$
>
> is valid.

The second is a form of the principle of extensionality:

> Any inference of the form:
>
> $$\dfrac{a \in S \Longleftrightarrow a \in T}{S = T}$$
>
> is valid provided that its premiss is taken as an assertion about all values of 'a'.

Here, 'S' and 'T' are to be understood as variables whose values are subsets of the domain of the variable 'a'. A more usual and clearly equivalent [but less convenient] form of the principle of extensionality is:

> Any sentence of the form:
>
> $$\forall_x\ [x \in S \Longleftrightarrow x \in T] \Longrightarrow S = T$$
>
> is valid.

Although we have, in our discussion of generalization sentences, arrived at a number of different-appearing rules, all follow easily from four of them by using modus ponens or the deduction rule. Here are the four rules which we shall take as basic:

An inference of any of the following forms is valid:

$\dfrac{\forall_x Fx}{Ft}$ [Elimination rule for '∀', (E∀)]

$\dfrac{Fa}{\forall_x Fx}$ [Introduction rule for '∀', (I∀)]

$\dfrac{\exists_x Fx \quad Fa \longrightarrow q}{q}$ [Elimination rule for '∃', (E∃)]

$\dfrac{Ft}{\exists_x Fx}$ [Introduction rule for '∃', (I∃)]

In the (I∀) and (E∃) inferences the premiss in which the indicated variable occurs must be taken as an assertion about all values of this variable. Also, in (E∃) this variable must not occur in the sentence taken for '*q*'.

Exercises

Part A

1. Using the elimination rule for '∀' and the deduction rule it is easy to show that the sentence '$\forall_{\vec{x}} A + \vec{x} = B + \vec{x} \longrightarrow A + \vec{c} = B + \vec{c}$' is valid. Do so.
2. Now, show that any sentence of the form '$\forall_x Fx \longrightarrow Ft$' is valid. [Your derivation in Exercise 1 may suggest a way to proceed.]
3. Using the result of Exercise 2 and modus ponens it is easy to justify the elimination rule for '∀'. Do so.
4. As in Exercises 2 and 3, relate the introduction rule for '∃' to the validity of sentences of the form '$Ft \longrightarrow \exists_x Fx$'.

Part B

1. Making use of modus ponens, the elimination rule for '∀', and the deduction rule, show that the inference:

$$\frac{C + \vec{a} = C + \vec{b} \longrightarrow \forall_X X + \vec{a} = X + \vec{b}}{C + \vec{a} = C + \vec{b} \longrightarrow P + \vec{a} = P + \vec{b}}$$

is valid. [*Hint*: Since the conclusion of the given inference is a conditional sentence, begin by adopting the antecedent of this sentence as an assumption.]

2. Now, show that any inference of the form:

$$\frac{p \longrightarrow \forall_x Fx}{p \longrightarrow Ft}$$

Answers for Part A

1.
$$\cfrac{\cfrac{\forall_{\vec{x}}\, A + \overset{*}{\vec{x}} = B + \vec{x}}{A + \vec{a} = B + \vec{a}}}{\forall_{\vec{x}}\, A + \vec{x} = B + \vec{x} \Longrightarrow A + \vec{a} = B + \vec{a}}\,*$$

2.
$$\cfrac{\cfrac{\overset{*}{\forall_x Fx}}{Ft}\,(E\forall)}{\forall_x Fx \Longrightarrow Ft}\,*$$

3.
$$\frac{\forall_x Fx \qquad \forall_x Fx \overset{*}{\Longrightarrow} Ft}{Ft}$$

['*' indicates that, assuming the result of Exercise 2, the indicated premiss is a valid sentence.]

[Exercises 2 and 3 show that, given the deduction rule and modus ponens, it makes no difference whether we choose to consider (E∀) or the rule justified in Exercise 2 as our basic elimination rule for '∀'.]

4.
$$\cfrac{\cfrac{Ft}{\exists_x Fx}\,(I\exists)}{Ft \Longrightarrow \exists_x Fx}\,* \qquad\qquad \frac{Ft \qquad Ft \overset{*}{\Longrightarrow} \exists_x Fx}{\exists_x Fx}$$

Answers for Part B

1.
$$\cfrac{\cfrac{\cfrac{C + \vec{a} \overset{*}{=} C + \vec{b} \qquad C + \vec{a} = C + \vec{b} \Longrightarrow \forall_X X + \vec{a} = X + \vec{b}}{\forall_X X + \vec{a} = X + \vec{b}}}{P + \vec{a} = P + \vec{b}}\,(E\forall)}{C + \vec{a} = C + \vec{b} \Longrightarrow P + \vec{a} = P + \vec{b}}\,*$$

2.
$$\cfrac{\cfrac{\cfrac{\overset{*}{p} \qquad p \Longrightarrow \forall_x Fx}{\forall_x Fx}}{Ft}\,(E\forall)}{p \Longrightarrow Ft}\,*$$

3.
$$\cfrac{\forall_x Fx \qquad \cfrac{\forall_x Fx \overset{*}{\Longrightarrow} \forall_x Fx}{\forall_x Fx \Longrightarrow Ft}\,[\text{Ex. 2}]}{Ft}$$

4.
$$\cfrac{\cfrac{\cfrac{\overset{*}{Ft}}{\exists_x Fx}\,(I\exists) \qquad \exists_x Fx \Longrightarrow q}{q}}{Ft \Longrightarrow q}\,* \qquad\qquad \cfrac{Ft \qquad \cfrac{\exists_x Fx \overset{*}{\Longrightarrow} \exists_x Fx}{Ft \Longrightarrow \exists_x Fx}\,[\text{by preceding scheme}]}{\exists_x Fx}$$

is valid. [Your derivation in Exercise 1 should suggest a way to proceed.]

3. The elimination rule for '∀' can be justified if one assumes that inferences of the kind treated in Exercise 2 are valid. Show how. [*Hint*: In Exercise 2, replace '*p*' by '$\forall_x Fx$' and recall Exercise 2 of [illegible]]

4. As in Exercises 2 and 3, relate the introduction rule for '∃' with the validity of inferences of the form:

$$\frac{\exists_x Fx \Longrightarrow q}{Ft \Longrightarrow q}$$

Part C

1. As pointed out in the text, the elimination rule for '∃' can be restated by saying that any inference of the form:

$$(*) \qquad \frac{Fa \Longrightarrow q}{\exists_x Fx \Longrightarrow q}$$

is valid provided that its premiss is taken as an assertion about all values of the indicated variable and provided that this variable does not occur in the sentence which is taken for '*q*'. Show that either form of the elimination rule for '∃' can be justified on the basis of the other and either modus ponens or the deduction rule.

2. The introduction rule for '∀' can be used to justify the rule:

Any inference of the form:

$$\frac{p \Longrightarrow Fa}{p \Longrightarrow \forall_x Fx}$$

is valid provided that its premiss is taken as an assertion about all values of the indicated variable and provided that this variable does not occur in the sentence taken for '*p*'.

Do this. [*Hint*: Use three inferences—modus ponens, the introduction rule for '∀' and the deduction rule. Explain why the provisions stated in the rule are needed.]

* 3. The rule you justified in Exercise 2 can, alternatively, be used in justifying the introduction rule for '∀'. Do so. [*Hint*: In preparation for using an inference of the kind described in Exercise 2, begin with a conditionalizing inference:

$$\frac{Fa}{p \Longrightarrow Fa}$$

Supposing that the sentence chosen to replace '*p*' does not contain the indicated variable, you can now add an inference of the kind

[Summary. The elimination rule for '∀' can be stated in any of three equivalent forms depending on whether we take as basic the validity of:

$$\frac{\forall_x Fx}{Ft}, \quad \forall_x Fx \Longrightarrow Ft, \quad \text{or:} \quad \frac{p \Longrightarrow \forall_x Fx}{p \Longrightarrow Ft}$$

Similarly, the introduction rule for '∃' can be stated in terms of the validity of:

$$\frac{Ft}{\exists_x Fx}, \quad Ft \Longrightarrow \exists_x Fx, \quad \text{or:} \quad \frac{\exists_x Fx \Longrightarrow q}{Ft \Longrightarrow q}$$

Note that none of these six rules requires any restriction as to premisses being taken as assertions or to the occurrence or nonoccurence of variables.]

Answers for Part C

1.

$$\frac{\dfrac{\overset{*}{\exists_x Fx} \qquad Fa \Longrightarrow q}{q}\,(E\exists)}{\exists_x Fx \Longrightarrow q}\,* \qquad\qquad \frac{\exists_x Fx \qquad \dfrac{Fa \Longrightarrow q}{\exists_x Fx \Longrightarrow q}\,(*)}{q}$$

2.

$$\frac{\dfrac{\dfrac{\overset{*}{p} \qquad p \Longrightarrow Fa}{Fa}}{\forall_x Fx}\,(I\forall)}{p \Longrightarrow \forall_x Fx}\,*$$

Since, for the second inference to be valid, its premiss must be taken as an assertion about all values of 'a', the second premiss of the derivation must also be taken as such an assertion. So, by the proviso to the deduction rule, 'a' may not occur in the first premiss of the derivation if this is, as indicated, to be treated as an assumption.

3.

$$\frac{p \qquad \dfrac{\dfrac{Fa}{p \Longrightarrow Fa}}{p \Longrightarrow \forall_x Fx}\,(\text{Ex. } 2)}{\forall_x Fx}$$

This scheme justifies inferences of the form:

$$\frac{p \qquad Fa}{\forall_x Fx}$$

supposing that 'a' does not occur in the first premiss and that the second premiss is taken as an assertion about all values of 'a'. Choosing for 'p' the sentence '$\forall_x\ x = x$' — or any other valid sentence in which 'a' does not occur, we obtain a justification of (I∀).

[Summary. The introduction rule for '∀' can be stated in either of two equivalent forms depending on whether we take as basic the validity of:

$$\frac{Fa}{\forall_x Fx} \quad \text{or:} \quad \frac{p \Longrightarrow Fa}{p \Longrightarrow \forall_x Fx}$$

In either case, the premiss must be taken as an assertion about all values of 'a' and, in the second case, 'a' must not occur in the sentence taken for 'p'.]

described in Exercise 2. Supposing that the sentence chosen to replace 'p' is a valid sentence, you can complete the job of showing that '$\forall_x Fx$' is a consequence of 'Fa' – provided, of course, that 'Fa' is taken as an assertion about all values of 'a'.]

Part D

Consider the sentence '$a + b = b + a$'. Since this is one of our postulates, we intend to use it as an assertion about all values of 'a' and 'b'. This being so, we may apply the introduction rule for '$\forall$' in either of two ways, depending on which of the two variables we choose to generalize with respect to. Each of the inferences:

$$\frac{a + b = b + a}{\forall_y\, a + y = y + a} \qquad \frac{a + b = b + a}{\forall_x\, x + b = b + x}$$

is valid. Clearly, we should be prepared to accept the conclusion of either of these inferences as an assertion about all values of the variable –'a' or 'b' – which occurs in it. So, we may apply the introduction rule for '$\forall$' again:

$$\frac{\forall_y\, a + y = y + a}{\forall_x\forall_y\, x + y = y + x} \qquad \frac{\forall_x\, x + b = b + x}{\forall_y\forall_x\, x + y = y + x}$$

As this illustrates, given any theorem we may infer from it a sentence obtained by universally generalizing with respect to each of as many of the variables in it as we wish, doing so in any order. From such a "multiple universal generalization" we may get back to the given sentence by using the elimination rule for '$\forall$' as many times as is needed to "strip off" the universal quantifiers.

To make it easy to discuss situations involving several quantifiers we need to extend our notation by agreeing to use expressions like 'Fab', '$Gabc$', etc. as place-holders for sentences which contain two or more variables which we wish to take note of. For example, we might in applying a rule, replace 'Fab' by '$a + b = b + a$'. If we did then we should be prepared to replace '$\forall_x Fax$' by '$\forall_x\, a + x = x + a$', etc. The inferences displayed above illustrate [when tacked together, two by two] the schemes:

$$\dfrac{\dfrac{Fab}{\forall_y Fay}}{\forall_x\forall_y Fxy} \quad \text{and:} \quad \dfrac{\dfrac{Fab}{\forall_x Fxb}}{\forall_y\forall_x Fxy}$$

The following scheme, which involves both basic kinds of inference for universal generalizations, shows that the order in which '$\forall$'s are "attached" doesn't matter:

Answers for Part D

1. $$\dfrac{\dfrac{\forall_x\forall_y\ Fxy}{\forall_y\ Fsy}}{Fst}$$

2. Fab is taken as an assertion about all values of 'a' and 'b'

3. $$\dfrac{\dfrac{Fst}{\exists_y\ Fsy}}{\exists_x\ \exists_y\ Fxy}$$

4. $$\dfrac{\exists_x\exists_y\ Fxy \qquad \dfrac{\dfrac{Fab \Longrightarrow q}{\exists_y\ Fay \Longrightarrow q}}{\exists_x\exists_y\ Fxy \Longrightarrow q}}{q} \quad \text{or:} \quad \dfrac{\exists_x\exists_y\ Fxy \qquad \dfrac{Fab \Longrightarrow q}{\exists_y\ Fay \Longrightarrow q}}{q}\,(E\exists)$$

*

The "preamble" to the preceding exercises can be extended slightly to show that any sentence of the form:

$$\forall_x\forall_y\ Fxy \Longrightarrow \forall_y\forall_x\ Fxy$$

is valid. Recalling from page 242 that statements about validity like the preceding one are to be interpreted as applying for any choice of indices and variables, it is clear that the converse of a sentence of the form in question is a sentence of the same form. So, the '$\Longrightarrow$' can immediately be replaced by a '$\Longleftrightarrow$'. Similarly, Exercise 4, with '$\exists_y\ \exists_x\ Fxy$' for 'q', can be extended to show that any sentence of the form:

$$\exists_x\ \exists_y\ Fxy \Longrightarrow \exists_y\ \exists_x\ Fxy$$

is valid. And, again, the '$\Longrightarrow$' may, without further argument, be replaced by a '$\Longleftrightarrow$'.

These results on the permutability of "like quantifiers" suggest making comparisons between sentences of the forms:

$$\exists_x\ \forall_y\ Fxy \quad \text{and:} \quad \forall_y\ \exists_x\ Fxy$$

As examples, consider the pair:

(1) $\exists_x\ \forall_y\ y + x = y$ (2) $\forall_y\ \exists_x\ y + x = y$

and the pair:

(3) $\exists_x\ \forall_y\ y + x = 0$ (4) $\forall_y\ \exists_x\ y + x = 0$

Sentence (1) is a consequence of '$a + 0 = a$' [if this sentence is taken as an assertion]:

$$\dfrac{\dfrac{a + 0 = a}{\forall_y\ y + 0 = y}\,(I\forall)}{\exists_x\ \forall_y\ y + x = y}\,(I\exists)$$

$$\left.\begin{array}{c}\forall_x\forall_y Fxy\\ \hline \forall_y Fay\\ \hline Fab\end{array}\right\}\text{Elimination rule for '}\forall\text{'}$$

$$\left.\begin{array}{c}Fab\\ \hline \forall_x Fxb\\ \hline \forall_y\forall_x Fxy\end{array}\right\}\text{Introduction rule for '}\forall\text{'}$$

1. It has essentially been shown in the preceding discussion that, on the basis of the rules collected on page 250, any inference of the form:

$$\frac{\forall_x\forall_y Fxy}{Fst}$$

is valid. To test your understanding, give a scheme for showing that this is the case.

2. One of the schemes given above shows that any inference of the form:

$$\frac{Fab}{\forall_x\forall_y Fxy}$$

is valid *provided that* . . . . [Complete.]

3. Use the introduction rule for '∃' to show that any inference of the form:

$$\frac{Fst}{\exists_x\exists_y Fxy}$$

is valid.

4. Show that any inference of the form:

$$\frac{\exists_x\exists_y Fxy \quad Fab \longrightarrow q}{q}$$

is valid provided that its conditional premiss is taken as an assertion about all values of both the indicated variables and provided that neither of these variables occurs in the sentence which is taken for '*q*'. [*Hint*: Begin by using two inferences of type (*) in Exercise 1 of Part C.]

*

Exercises 1 through 4 may be summarized by saying that blocks of quantifiers *of the same kind* [all '∀'s or all '∃'s] can be handled just like single quantifiers. For, as the exercises show, such blocks satisfy rules which are entirely analogous to the four basic rules for single quantifiers,

Sentence (4) is a consequence of 'a + −a = 0' [if this is taken as an assertion]:

$$\dfrac{\dfrac{a + -a = 0}{\exists_x\ a + x = 0}(I\exists)}{\forall_y\ \exists_x\ y + x = 0}(I\forall)$$

Sentences (1) and (4) are, then, theorems. They say pretty much what is said by the PAO and the IPO, respectively, when the latter are [as postulates are] taken as assertions. The only difference is that (1) does not make available to us the numeral '0' and (4) does not make available the operator '−'. What (1) says is that there is some [unspecified] number which may, "without effect", be added to any number one chooses. What (2) says is much less — given any number, there is some [unspecified] number which may, again without effect, be added to the given number. The difference is that, as far as (2) is concerned, different numbers may "have different zeros". The difference between (3) and (4) is similar. According to (4), each number has an opposite and different numbers may have different opposites — as, indeed, they do. According to (3) there is some number which will do equally well as an opposite of any number. Since (3) is false and (4) is true it follows that (3) is not a consequence of (4). So, generally, a sentence of the form:

$$\exists_x\ \forall_y\ Fxy$$

is not a consequence of the corresponding sentence of the form:

$$\forall_y\ \exists_x\ Fxy$$

More succinctly, not all sentences of the form:

$$\forall_y\ \exists_x\ Fxy \Longrightarrow \exists_x\ \forall_y\ Fxy$$

are valid. [As is usual in similar cases, only "uninteresting" sentences of this form are valid.] On the other hand:

> Any sentence of the form:
>
> $$\exists_x\ \forall_y\ Fxy \Longrightarrow \forall_y\ \exists_x\ Fxy$$
>
> is valid.

This rule is justified by the scheme:

$$\dfrac{\exists_x\ \forall_y\ Fxy \qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall_y\ \overset{*}{F}ay}{Fab}(E\forall)}{\exists_x\ Fxb}(I\exists)}{\forall_y\ \exists_x\ Fxy}(I\forall)}{\forall_y\ Fay \Longrightarrow \forall_y\ \exists_x\ Fxy}\,*}{\forall_y\ \exists_x\ Fxy}(E\exists)$$

6.08 Using Quantifiers in Proofs

You have seen that in order to explain the notions of linear dependence and linear independence it is necessary to use the ideas expressed by the words 'each' and 'there exists'. We have chosen to use the quantifiers '∀' and '∃' to express these ideas. To make use of the notions of linear dependence and independence one needs to know how to deal with universal and existential generalizations, however they may be expressed. The easiest way to learn this is to discover the rules for eliminating and introducing quantifiers and then to see how these rules can be used in proving theorems. As in the case of other techniques of proof, once you have learned how to use the rules you can usually see what to do without bothering to make formal use of them. If, sometimes, you can't then you still have the rules to fall back on. [This is something like learning to find sums of whole numbers. Once, you may have counted on your fingers to find that 3 + 5 is 8. You probably don't do this now but, if you forgot the sum of 3 and 5, you still have your fingers for a last resort.]

As a first illustration, let's consider the proof of the only if-part of Theorem 6-7. To simplify matters we shall consider the case $n = 2$:

$$(\vec{a}, \vec{b}) \text{ is not linearly dependent}$$
$$\Longrightarrow$$
$$\forall_x \forall_y\, [\vec{a}x + \vec{b}y = \vec{0} \longrightarrow (x = 0 \text{ and } y = 0)]$$

In this case the proof given on page 227 goes as follows:

Suppose that $(\vec{a}, \vec{b})$ is not linearly dependent. Suppose, also, that a and b are any numbers such that

(1) $$\vec{a}a + \vec{b}b = \vec{0}.$$

It follows from (1) that

if not $(a = 0$ and $b = 0)$ then $(\vec{a}a + \vec{b}b = \vec{0}$ and not $(a = 0$ and $b = 0))$.

So [by the introduction rule for '∃']

if not $(a = 0$ and $b = 0)$ then $\exists_x \exists_y\, (\vec{a}x + \vec{b}y = \vec{0}$ and not $(x = 0$ and $y = 0))$.

By Definition 6-2 it follows that

if not $(a = 0$ and $b = 0)$ then $(\vec{a}, \vec{b})$ is linearly dependent.

Note that the use of the deduction rule is permissible in spite of the previous use of (I∀) because the variable 'b' does not occur in the discharged premiss. Any attempt to justify inferences of the converse kind will fail. For example, in:

$$\dfrac{\dfrac{\forall_x\, \exists_y\ Fxy}{\exists_y\ Fay}\,(E\forall) \qquad \dfrac{\dfrac{\dfrac{\overset{*}{Fab}}{\forall_y\ Fay}\,(I\forall)}{\exists_x\, \forall_y\ Fxy}\,(I\exists)}{Fab \Longrightarrow \exists_x\, \forall_y\ Fxy}\,*}{\exists_x\, \forall_y\ Fxy}\,(E\exists)$$

is invalid because the premiss 'Fab' must be treated as an assertion about all values of 'b' if the (I∀)-inference is to be valid but must be treated as an assumption about values of 'b' [as well as values of 'a'] if the deduction rule is to be applied as indicated. Such schizophrenic behavior is outlawed by the proviso to the deduction rule.

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The text of section 6.08 consists of an analysis of the roles played by the new rules for quantifiers in the proofs previously given for the two parts of Theorem 6-7 and for Theorem 6-13. Exercises in which students can test their understanding of the use of these rules begin on page 260.

At the beginning of the section we reiterate our point that the reason for one's learning the rules of logic — and how to use them — is to enable him to argue easily and correctly without making conscious use of them. We wish students who complete this course to be able to understand and apply the rules of reasoning in the format of paragraph arguments. We are not necessarily interested in turning out large numbers of students who are expert at tree-form derivations, especially if in achieving this goal the student has lost interest in or has acquired a poor attitude toward mathematics generally.

Our experience has been that detailed studies of the rules of reasoning from the point of view we have taken here are best presented by the teacher. This is especially true when analyzing complicated tree-form derivations. Such discussions as that found on pages 254 - 260 should be understood by the teacher well enough for him to develop the ideas within the framework of a class discussion. A good teaching style can do much to cover the ideas in the text in a nontedious fashion.

The first step in the argument ["It follows from (1) that"] makes use of an inference of the form:

$$\frac{q}{p \Longrightarrow (q \text{ and } p)}$$

whose validity is an immediate consequence of the introduction rule for 'and' and the deduction rule.

An explanation of the next step is given following the proof.

The step indicated by "By Definition 6-2 it follows that" makes use of the replacement rule for biconditional sentences.

The remaining steps illustrate modus tollens, the deduction rule, the introduction rule for '∀' [twice], and the deduction rule.

Since, by assumption, $(\vec{a}, \vec{b})$ is not linearly dependent it follows that $a = 0$ and $b = 0$. Hence [discharging (1)],

$$\text{if } \vec{a}a + \vec{b}b = \vec{0} \text{ then } (a = 0 \text{ and } b = 0).$$

So [by the introduction rule for '∀'],

$$\forall_x \forall_y\, [\vec{a}x + \vec{b}y = \vec{0} \Longrightarrow (x = 0 \text{ and } y = 0)].$$

Hence, the theorem.

Notice that in order to follow the "natural" way of writing, we have used an inference of the form:

$$\frac{p \Longrightarrow Fab}{p \Longrightarrow \exists_x \exists_y Fxy}$$

and cited the introduction rule for '∃'. Such inferences are easily justified by this rule:

$$\dfrac{\dfrac{\dfrac{\overset{*}{p} \quad p \Longrightarrow Fab}{Fab}}{\exists_x \exists_y Fxy} \longleftarrow \text{Exercise 3 of Part D}}{p \Longrightarrow \exists_x \exists_y Fxy}\,{}^{*}$$

As our second illustration we shall take the proof of the if-part of Theorem 6-7—again in the case $n = 2$:

$$\forall_x \forall_y\, [\vec{a}x + \vec{b}y = \vec{0} \Longrightarrow (x = 0 \text{ and } y = 0)]$$
$$\Longrightarrow$$
$$(\vec{a}, \vec{b}) \text{ is not linearly dependent}$$

The proof given on pages 227 and 228 goes as follows:

Suppose that

$$\forall_x \forall_y\, [\vec{a}x + \vec{b}y = \vec{0} \Longrightarrow (x = 0 \text{ and } y = 0)].$$

It follows [by the elimination rule for '∀'] that, for any numbers a and b,

(1) $\vec{a}a + \vec{b}b = \vec{0} \Longrightarrow (a = 0 \text{ and } b = 0)$.

Hence, it is *not* the case—for any a and b—that

(2) $\vec{a}a + \vec{b}b = \vec{0}$ and not $(a = 0 \text{ and } b = 0)$.

It follows that

(3) not $\exists_x \exists_y\, (\vec{a}x + \vec{b}y = \vec{0}$ and not $(x = 0 \text{ and } y = 0))$.

So, by Definition 6-2, $(\vec{a}, \vec{b})$ is not linearly dependent. Hence, the theorem.

The steps from (1) to (2) and from (2) to (3) probably seem reasonable enough but both depend on rules of logic which we have not taken up explicitly. The step from (1) to (2) is an inference of the form:

(*) $$\frac{p \Longrightarrow q}{\text{not } (p \text{ and not } q)}$$

[Assuming that if p then q, you can't very well have p without q.] The step from (2) to (3) is an inference of the form:

$$\frac{\text{not } Fab}{\text{not } \exists_x \exists_y Fxy}$$

[Assuming that, *whatever a and b are*, it is not the case that Fab, it follows that there do not exist numbers x and y such that Fxy.] As in Part D, it is clear that this second kind of inference is just a combination of two inferences of the form:

(**) $$\frac{\text{not } Fa}{\text{not } \exists_x Fx}$$

and so it will be sufficient to discuss inferences of this kind. As we shall see, the validity of such inferences [provided that their premisses are taken as assertions about all values of the indicated variable] is equivalent to the elimination rule for '∃'.

The justification of inferences of the form (*) can be based on the idea that the denial of a sentence can be inferred from the fact that the sentence itself implies a contradiction. More formally, any inference of the form:

$$\frac{p \Longrightarrow (q \text{ and not } q)}{\text{not } p}$$

is valid. This follows easily from modus tollens and the law of noncontradiction [page 164] according to which any sentence of the form 'not (q and not q)' is valid. A more convenient rule—which comes to the same thing—is that any inference of the form:

$$\frac{p \Longrightarrow q \quad p \Longrightarrow \text{not } q}{\text{not } p}$$

is valid. We can now justify inferences of type (*) by showing that if we assume that $p \longrightarrow q$ then the additional assumption that (p and not q) leads to a contradiction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{*}{p \text{ and not } q}}{p} \qquad p \longrightarrow q}{q}}{(p \text{ and not } q) \longrightarrow q}\,* \qquad \dfrac{\dfrac{\overset{*}{p \text{ and not } q}}{\text{not } q}}{(p \text{ and not } q) \longrightarrow \text{not } q}\,*}{\text{not } (p \text{ and not } q)}$$

The justification of inferences of type (**) is like the answer you may have given for Exercise 3 of Part C on page 251. To see how it goes, consider the following scheme:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{not } Fa}{p \Longrightarrow \text{not } Fa}}{Fa \Longrightarrow \text{not } p}}{\exists_x Fx \Longrightarrow \text{not } p} \quad p}{\text{not } \exists_x Fx}$$

If we can find a replacement for 'p' such that each of the four inferences is valid then, with this replacement,

$$\dfrac{\text{not } Fa \quad p}{\text{not } \exists_x Fx}$$

will be valid. If, at the same time, we can choose as a replacement for 'p' a valid sentence then, as a premiss, this sentence can be ignored, and we shall have justified (**). Now, of the four inferences in question, the first, second, and fourth will be valid no matter what choice we make for 'p'. The third will be valid providing that its premiss [$Fa \Longrightarrow$ not p] is taken as an assertion about all values of 'a' and that 'a' does not occur in the chosen replacement for 'p'. It follows that, for 'p', we need a valid sentence in which 'a' does not occur. This is easy. We might choose '$0 = 0$', or we might choose, in each application, our replacement for '$\exists_x Fx \longrightarrow \exists_x Fx$'. Any number of other choices are available. There still remains the question as to when we can be sure that the premiss of the third inference is taken as an assertion. Again the answer is easy. This is sure to be the case if the premiss [not Fa] of the first inference is taken as an assertion. The final result, then, is that any inference of the form:

$$\dfrac{\text{not } Fa}{\text{not } \exists_x Fx}$$

Here are three schemes referred to implicitly on these pages:

$$\text{(A):} \quad \dfrac{p \Longrightarrow (q \text{ and not } q) \qquad \text{not } (q \overset{*}{\text{and}} \text{ not } q)}{\text{not } p}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{*}{p} \quad p \Longrightarrow q}{q} \qquad \dfrac{\overset{*}{p} \quad p \Longrightarrow \text{not } q}{\text{not } q}}{q \text{ and not } q}}{p \Longrightarrow (q \text{ and not } q)}\,*}{\text{not } p}\ \text{[by (A)]}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{*}{p} \quad p \Longrightarrow (q \text{ and not } q)}{q \text{ and not } q}}{q}}{p \Longrightarrow q}\,* \qquad \dfrac{\dfrac{\dfrac{\overset{*}{p} \quad p \Longrightarrow (q \text{ and not } q)}{q \text{ and not } q}}{\text{not } q}}{p \Longrightarrow \text{not } q}\,*}{\text{not } p}$$

The first scheme establishes the validity of the first of two displayed inferences. The other two schemes shows how either of the displayed inferences can be used to justify the other.

Note that by using the abbreviation introduced on page 88 in connection with the deduction rule, the two rules of inference whose equivalence is shown above could be stated:

Any inference of either of the forms:

$$\dfrac{\begin{matrix}[p]\\ q \text{ and not } q\end{matrix}}{\text{not } p} \qquad \dfrac{\begin{matrix}[p]\\ q\end{matrix} \quad \begin{matrix}[p]\\ \text{not } q\end{matrix}}{\text{not } p}$$

is valid.

The advantage to doing so is that, for example, the scheme at the foot of the page could be abbreviated to:

$$\dfrac{\dfrac{\dfrac{\overset{*}{p \text{ and not } q}}{p} \qquad p \Longrightarrow q}{q} \qquad \dfrac{\overset{*}{p \text{ and not } q}}{\text{not } q}}{\text{not } p \text{ and not } q}\,*$$

is valid provided that its premiss is taken as an assertion about all values of the indicated variable.

As our final illustration of the use of the rules for '∃' let's consider the proof of:

Theorem 6–13 $(\vec{a}, \vec{b})$ is linearly dependent and $\vec{a} \neq \vec{0}$ $\Longrightarrow$ $\vec{b} \in [\vec{a}]$

This is a theorem for which we shall have some use later. You have already studied it in Part H on page 232. In fact, by solving Exercise 4 of Part H you have already done most of the work required to prove Theorem 6–13.

To prove this theorem it is natural to begin by assuming that

(1) $(\vec{a}, \vec{b})$ is linearly dependent and $\vec{a} \neq \vec{0}$

and to attempt to show that it follows from this assumption, and various theorems about translations, that

(2) $\vec{b} \in [\vec{a}]$.

If we can do this then we can show that Theorem 6–13 *is* a theorem by using the deduction rule. [Explain.] Our problem, then, is to get from (1) to (2) with the help of whatever theorems prove useful. Among these theorems will certainly be Definition 6–2 [of linear dependence] and Definition 5–1 [of $[\vec{a}]$]. Both of these definitions contain '∃'.

A good way to begin our progress from (1) to (2) is to note that it follows from the first part of (1) and Definition 6–2 that

(*) $\exists_x \exists_y (\vec{a}x + \vec{b}y = \vec{0}$ and not $(x = 0$ and $y = 0))$.

Now, the elimination rule for '∃' tells us that (2) follows from (*) and

(**) $(\vec{a}a + \vec{b}b = \vec{0}$ and not $(a = 0$ and $b = 0)) \Longrightarrow \vec{b} \in [\vec{a}]$

provided that (**) depends on no special assumptions about a and b. [If this is the case then we can take (**) as an assertion about all values of 'a' and 'b'. This assertion may, of course, be subject to other assumptions which do not concern the values of 'a' and 'b'. Presumably, the only assumption we shall need to make is that $\vec{a} \neq \vec{0}$.] Since (*) follows from (1) and Definition 6–2, we can show that (2) follows from (1) if we can show that (**) follows from (1) and various theorems about translations. Presumably the only part of (1) we shall need is '$\vec{a} \neq \vec{0}$'.

At this point, we can outline the proof of Theorem 6–13 as follows:

$$\cfrac{\cfrac{\cfrac{[\text{Def. } 6\text{–}2] \quad \cfrac{\overset{*}{(1)}}{(\vec{a}, \vec{b}) \text{ is linearly dependent}}}{(*)} \qquad \cfrac{\cfrac{\overset{*}{(1)}}{\vec{a} \neq \vec{0}} \quad [\text{various theorems}]}{(**)}}{\vec{b} \in [\vec{a}]}\ *}{[\text{Theorem } 6\text{–}13]}$$

Here, the last inference is by the deduction rule and the one just preceding it is by the elimination rule for '∃'. The left side of the upper part of the scheme is clear. In fact, it is an example of the valid scheme:

$$\cfrac{p \longleftrightarrow r \quad \cfrac{p \text{ and } q}{p}}{r}$$

[Explain. What sentences should be taken for 'p', 'q', and 'r'?]

All that is left to do is to complete the right side of the scheme by deriving (**) from the assumption '$\vec{a} \neq \vec{0}$' and whatever theorems about translations turn out to be helpful. Since you have already done this in working Exercise 4 of Part H on page 232, we shall merely sketch the derivation, leaving the details for you to fill in. Before going on to this, however, it is worth noticing that the preceding discussion and outline applies to proving any theorem of the form:

If $(\vec{a}_1, \ldots)$ is linearly dependent and ___ then ___.

[Explain. Will a similar argument work if the 'and ___' is missing?]

To prove (**) under the assumption that $\vec{a} \neq \vec{0}$ it is natural to start with the additional assumption that

(3) $\vec{a}a + \vec{b}b = \vec{0}$ and not $(a = 0$ and $b = 0)$.

and attempt to derive (2). As you may recall, the first step in deriving (2) is to show that, under these two assumptions, $b \neq 0$. The next step is to use this result together with the first part of (3) to show that

(2′) $\vec{b} = \vec{a} \cdot -(a/b)$.

From (2′) it follows [by the introduction rule for '∃'] that

(2″) $\exists_x \vec{b} = \vec{a}x$.

Finally, (2) follows from (2″) and Definition 5-1. Using the deduction rule we find that (**) is a consequence of the assumption '$\vec{a} \neq \vec{0}$' and whatever theorems were called on in arriving at (2′).

The argument outlined in the preceding paragraph can be diagrammed as follows:

$$\dfrac{\dfrac{[\text{Def. } 5\text{-}1]}{\vec{b} \in [\vec{a}] \Longleftrightarrow \exists_x\, \vec{b} = \vec{a}x} \qquad \dfrac{\dfrac{\dfrac{\vec{a} \neq \vec{0} \quad (3) \quad [\text{various theorems}]}{b \neq 0}\,{}^{*} \quad (3) \quad [\text{various theorems}]}{\vec{b} = \vec{a} \cdot -(a/b)}}{\exists_x\, \vec{b} = \vec{a}x}}{\vec{b} \in [\vec{a}]}\,{}^{*}$$

$$(**)$$

Once the role of the rules for '∃' in such proofs is clear it is not difficult to write something like the following in proof of Theorem 6-13:

Suppose that

(1) $(\vec{a}, \vec{b})$ is linearly dependent and that $\vec{a} \neq \vec{0}$.

Since $(\vec{a}, \vec{b})$ is linearly dependent it follows, by definition, that there are numbers—say, a and b—such that

(3) $\vec{a}a + \vec{b}b = \vec{0}$ and not $(a = 0$ and $b = 0)$.

Suppose that $b = 0$. It follows that $\vec{a}a + \vec{b}b = \vec{a}a + \vec{b}0 = \vec{a}a + \vec{0} = \vec{a}a$. So, since $\vec{a}a + \vec{b}b = \vec{0}$, $\vec{a}a = \vec{0}$ and, since $\vec{a} \neq \vec{0}$, $a = 0$. Hence, if $b = 0$ then *both* a and b are zero. Since a and b are not both zero it follows that $b \neq 0$.

Since $\vec{a}a + \vec{b}b = \vec{0}$ and $b \neq 0$ it follows that $\vec{b} = \vec{a} \cdot -(a/b)$ and, so, that $\exists_x\, \vec{b} = \vec{a}x$. It follows, by definition, that $\vec{b} \in [\vec{a}]$. Since this is the case whatever the numbers a and b are, assuming only that they satisfy (3) [and that $\vec{a} \neq \vec{0}$], and since, by (1), there are such numbers [and $\vec{a} \neq \vec{0}$] it follows from (1) that $\vec{b} \in [\vec{a}]$. Hence, the theorem.

Note how, in this form of proof, the use of the elimination rule for '∃' is referred to implicitly in the second sentence ["there are numbers—say, a and b—such that "] and in the next-to-last sentence ["Since this is the case whatever the numbers a and b are, assuming only that they satisfy (3), and since there are such numbers"].

Exercises

Part A

1. Theorem 6-13 on page 258 is the first of a sequence of similar theorems. The second is:

TC 259

In the small derivation scheme on page 259 the first inference is valid by the elimination rule for 'and'; the second is valid by the replacement rule for biconditional sentences. For 'p', take '$(\vec{a}, \vec{b})$ is linearly dependent'; for 'q' take '$\vec{a} \neq \vec{0}$'; for 'r' take the sentence (☆) on page 258. With these replacements 'p and q' becomes sentence (1) and 'p ⟺ r' becomes an instance of Definition 6-2.

The discussion in the text so far sets forth a scheme which is likely to be useful in proving any theorem which is a conditional sentence whose antecedent implies a sentence of the form:

$(\vec{a}_1, \ldots)$ is linearly dependent

The left side of the scheme [the part analyzed in the preceding paragraph] leads to the appropriate conclusion of the form:

(☆′) $\exists_{x_1} \ldots (\vec{a}_1 x_1 + \ldots = \vec{0}$ and not $(x_1 = 0$ and $\ldots))$

The essential part of the proof — that is, the only part which requires ingenuity on the part of the prover — is to show that the corresponding sentence:

(☆☆′) $(\vec{a}_1 a_1 + \ldots = \vec{0}$ and not $(a_1 = 0$ and $\ldots)) \Longrightarrow r$

[where 'r' is to be replaced by the consequent of the conditional sentence which is to be proved] is a consequence of known theorems and the antecedent of the theorem to be proved. The conditions for the validity of the (E∃)-inference with (☆′) and (☆☆′) as premisses will then be satisfied if one has had the forethought to choose the variables 'a_1', ... which are introduced in the antecedent of (☆☆′) from among those which do not occur in its consequent. The final inference, by the deduction rule will be valid unless, in deriving (☆☆′), one has used an inference which "blocks" one of the variables — '$\vec{a}_1$', for example — in the antecedent of the sentence he now wishes to discharge. An example, in the context of the proof of Theorem 6-13, of such an error would be deriving '$\vec{b} \neq \vec{0}$' or '$\forall_{\vec{x}}\, \vec{x} \neq \vec{0}$' from '$\vec{a} \neq \vec{0}$'. Such errors are rather difficult to commit unless one is really trying to make them — or is paying no attention at all to what he is writing.

As illustrated in the remainder of the analysis, in the text, of the proof of Theorem 6-13, the normal procedure for showing that (☆☆′) is a consequence of known theorems and the antecedent of the theorem to be proved is to take the antecedent of (☆☆′) as an additional assumption and attempt to derive the common consequent of (☆☆′) and this theorem. Again, since an application of the deduction rule is in prospect, one must be careful not to "block" any of the variables which occur in this additional assumption. As before, once one has recognized this as an assumption, such care as is needed is likely to be automatic.

The technique of obtaining (☆☆′) described in the preceding paragraph is built into the kind of paragraph proof which we have been using previously and which is illustrated in the proof on page 260. How this is done is pointed out in the paragraph following that proof.

TC 260 (1)

Students may be curious as to the inference:

$$\dfrac{[\text{Def. } 5\text{-}1]}{\vec{b} \in [\vec{a}] \Longleftrightarrow \exists_x\, \vec{b} = \vec{a}x}$$

This is actually an example of the replacement rule for equations in which the second premiss is a valid sentence [see TC 249(2)]:

$$\frac{[\vec{a}] = \{\vec{x}\colon \exists_x\ \vec{x} = \vec{a}x\} \qquad \vec{b} \in \{\vec{x}\colon \exists_x\ \vec{x} = \vec{a}x\} \Longleftrightarrow \exists_x\ \vec{b} = \vec{a}x}{\vec{b} \in [\vec{a}] \Longleftrightarrow \exists_x\ \vec{b} = \vec{a}x}$$

The biconditional conclusion of this inference might have been adopted as a "contextual definition" of '$[\vec{a}]$' in place of adopting the "explicit definition" Definition 5-1. It is almost as easy to infer the explicit definition from the same valid sentence and the contextual definition:

$$\cfrac{\vec{b} \in \{\vec{x}\colon \exists_x\ \vec{x} = \vec{a}x\} \Longleftrightarrow \exists_x\ \vec{b} = \vec{a}x \qquad \vec{b} \in [\vec{a}] \Longleftrightarrow \exists_x\ \vec{b} = \vec{a}x}{\cfrac{\vec{b} \in [\vec{a}] \Longleftrightarrow \vec{b} \in \{\vec{x}\colon \exists_x\ \vec{x} = \vec{a}x\}}{[\vec{a}] = \{\vec{x}\colon \exists_x\ \vec{x} = \vec{a}x\}}}$$

Here, the first inference is valid by the replacement rule for biconditional sentences and the second is valid by the extensionality principle [see TC 249(3)].

Since corresponding explicit and contextual definitions are derivable from one another and valid sentences, we shall often refer to either one by writing 'definition'. This practice is illustrated in the third paragraph of the paragraph proof of Theorem 6-13.

The discussion on pages 254 - 260 presents several rather involved matters of logic. This discussion is included to clarify some of the problems that arise in applying such properties as Definitions 6-1 and 6-2. Unfortunately, this discussion tends to focus one's attention on details that normally are not apparent in practice. It is sometimes difficult to motivate students to consider details that do not "appear to be relevant" or that are intended to "promote a better understanding" of the material.

Our most important goal is to equip students to understand and present paragraph-type arguments like that at the bottom of page 260. We recommend that as often as possible you arrange the activity of your class so that complicated tree-form derivations are done as a class project. We also recommend that exercises for individual consideration be those that provide experience with paragraph-type arguments involving existential quantification.

Stress should also be placed on the proof strategy exhibited in the second paragraph [beginning with 'Suppose that b = 0'.] on page 260. This strategy will be very valuable in subsequent work both in this course and in advanced mathematics. In simple terms, one wishes to make an assertion ['b ≠ 0' in this case] so he, in effect, does so, and quickly adds "because if not..." [in this case, the 'if not" is the opening sentence of the second paragraph]. He then proceeds to show that based upon this "if not" there are "unfortunate consequences" [in this case, the contradiction that "if b = 0 then both a and b are zero, but a and b cannot both be 0"]. Too much stress on the tree-form versions of such arguments may obscure this important strategy for students encountering such ideas for the first time.

Parts A - G are not recommended as a single assignment. One means of handling these exercises is:

First day

(a) Part A as an in-class demonstration to emphasize the application of the various rules introduced in this section.

(b) Part B and Exercises 1 - 3 of Part C as homework.

Second day

(a) Discuss Exercise 4 of Part C in class.

(b) Exercises 5 - 8 of Part C and Part D as homework.

Third day

(a) Discuss Part E in class.

(b) Part F for homework.

Fourth day

(a) Discuss the proof of Theorem 6-7 given in the text.

(b) Part G for homework.

Answers for Part A

1. Suppose that

   $(\vec{a}, \vec{b}, \vec{c})$ is linearly dependent and
   that $(\vec{a}, \vec{b})$ is linearly independent.

   Since $(\vec{a}, \vec{b}, \vec{c})$ is linearly dependent it follows, by definition, that there are numbers — say, a, b, and c — such that

   (*) $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$ and not ($a = 0$, $b = 0$, and $c = 0$).

   Suppose that $c = 0$. It follows that $\vec{a}a + \vec{b}b + \vec{c}c = \vec{a}a + \vec{b}b + \vec{c}0 = \vec{a}a + \vec{b}b + \vec{0} = \vec{a}a + \vec{b}b$. So, since $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$, $\vec{a}a + \vec{b}b = \vec{0}$ and, since $(\vec{a}, \vec{b})$ is linearly independent, $a = 0$ and $b = 0$. Hence, if $c = 0$ then a, b, and c are all zero. Since a, b, and c are not all zero it follows that $c \neq 0$.

   Since $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$ and $c \neq 0$ it follows that $\vec{c} = \vec{a} \cdot -(a/c) + \vec{b} \cdot -(b/c)$ and, so, $\exists_x \exists_y\ \vec{c} = \vec{a}x + \vec{b}y$. It follows, by definition, that $\vec{c} \in [\vec{a}, \vec{b}]$. Since this is the case whatever the numbers a and b are, subject only to the assumption (*) and since, by the earlier assumption, there are such numbers it follows, subject only to the earlier assumption, that $\vec{c} \in [\vec{a}, \vec{b}]$. Hence, the theorem.

2. (a) $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly dependent and $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent

   $$\Longrightarrow$$

   $$\vec{d} \in [\vec{a}, \vec{b}, \vec{c}]$$

   [Definition: $[\vec{a}, \vec{b}, \vec{c}] = \{\vec{x}\colon \exists_x \exists_y \exists_z\ \vec{x} = \vec{a}x + \vec{b}y + \vec{c}z\}$]

   (b) Yes. [A student who has answered Exercise 1 after the pattern given in the text should have no difficulty — other than writer's cramp — in giving a proof of the theorem in part (a).]

$$((\vec{a}, \vec{b}, \vec{c}) \text{ is linearly dependent and } (\vec{a}, \vec{b}) \text{ is linearly independent})$$
$$\longrightarrow$$
$$\vec{c} \in [\vec{a}, \vec{b}]$$

Prove this theorem. [*Hint*: The proof should be very easy if you use the paragraph proof on page 260 as a model.]

2. (a) Write the third theorem in the sequence which begins with Theorem 6-13 [*Hint*: You should have no difficulty doing this. But, to make sense of what you write you will need to give a definition for '$[\vec{a}, \vec{b}, \vec{c}]$'. If you need a hint, look at Definitions 5-1 and 5-2.]
   (b) Do you see how to prove the theorem you wrote for part (a)?
3. Complete the following definition and theorem:
   (a) $[\vec{a}_1, \ldots, \vec{a}_n] = \{\vec{x}: \exists_{x_1} \ldots$ ________ $\}$
   (b) $((\vec{a}_1, \ldots, \vec{a}_{n+1})$ is linearly dependent and ________ $)$

$$\longrightarrow$$
$$\vec{a}_{n+1} \in \text{________}$$

[In later chapters we shall refer to this definition and theorem as 'Definition 6-4' and 'Theorem 6-13', respectively.]

Part B

1. Complete the following proof of the theorem:

$$\vec{a} \in [\vec{b}] \longrightarrow [\vec{a}] \subseteq [\vec{b}]$$

Suppose that $\vec{a} = \vec{b}a$ and $\vec{c} = \vec{a}c$. Since ________ it follows that $\vec{c} = \vec{b}(ac)$ and, so, that $\exists_x$ ________. Hence, for any number c,

$$\vec{c} = \vec{a}c \longrightarrow \exists_x \vec{c} = \vec{b}x$$

and so

$$\exists_x \vec{c} = \vec{a}x \longrightarrow \exists_x \vec{c} = \vec{b}x.$$

In other words, by Definition 5-1,

$$\text{________} \longrightarrow \text{________}.$$

Since this is the case for any $\vec{c}$ it follows that

$$\text{____} \subseteq \text{____}.$$

Hence, for any number a,

$$\vec{a} = \vec{b}a \longrightarrow \text{________}$$

and so . . . .

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3. (a) $\exists_{x_n} \vec{x} = \vec{a}_1 x_1 + \ldots + \vec{a}_n x_n$

   (b) $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly independent; $[\vec{a}_1, \ldots, \vec{a}_n]$

*

When we have adopted dimensionality postulates the sentence:

$$(\vec{a}, \vec{b}, \vec{c}, \vec{d}) \text{ is linearly dependent}$$

will become a theorem. In consequence, the case $n > 3$ of Theorem 6-13 will be completely void of interest [since its antecedent can never be satisfied] and the case $n = 3$ will reduce [in view of the fact that, in any case, $[\vec{a}, \vec{b}, \vec{c}] \subseteq T$] to:

$$(\vec{a}, \vec{b}, \vec{c}) \text{ is linearly independent} \Longrightarrow [\vec{a}, \vec{b}, \vec{c}] = T$$

The converse of this sentence will also turn out to be a theorem.

Answers for Part B

1. $(\vec{b}a)c = \vec{b}(ac)$; $\vec{c} = \vec{b}x$; $\vec{c} \in [\vec{a}]$; $\vec{c} \in [\vec{b}]$; $[\vec{a}]$; $[\vec{b}]$; $[\vec{a}] \subseteq [\vec{b}]$;

   [final portion of proof:] and so

$$\exists_x \vec{a} = \vec{b}x \Longrightarrow [\vec{a}] \subseteq [\vec{b}].$$

   Hence, by Definition 5-1, if $\vec{a} \in [\vec{b}]$ then $[\vec{a}] \subseteq [\vec{b}]$.

   [This proof illustrates (twice) the use of the rule according to which an inference of the form:

$$\frac{Fc \Longrightarrow q}{\exists_x Fx \Longrightarrow q}$$

   is valid provided that its premiss is treated as an assertion about all values of 'c' and that 'c' does not occur in the sentence taken for 'q'. The first restriction is satisfied in the first of the two applications of the rule because the premiss, which is:

$$\vec{c} = \vec{a}c \Longrightarrow \exists_x \vec{c} = \vec{b}x$$

   depends on no assumption in which 'c' occurs. (The only such assumption has just been discharged.) The second restriction is obviously satisfied. A later step depends on the validity of the inference:

$$\frac{\vec{c} \in [\vec{a}] \Longrightarrow \vec{c} \in [\vec{b}]}{[\vec{a}] \subseteq [\vec{b}]}$$

   provided that its premiss is treated as an assertion about all values of '$\vec{c}$'. This restriction is satisfied because the only assumption in which '$\vec{c}$' occurs has already been discharged. The validity of the inference is a consequence of the usual definition of '$\subseteq$' — in this case:

$$S \subseteq T \Longleftrightarrow \forall_{\vec{x}} [\vec{x} \in S \Longrightarrow \vec{x} \in T]$$

   with 'S' and 'T' ranging over subsets of T — with which we assume students to be familiar.]

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2. Prove the theorem:

$$\vec{a} \in [\vec{b}] \longrightarrow (\vec{a}, \vec{b}) \text{ is linearly dependent}$$

[*Hint*: Suppose that $\vec{a} = \vec{b}a$. It follows that ____ + ____ $= \vec{0}$ and, since $1 \neq 0$, $\exists_x \exists_y (\vec{a}x + \vec{b}y = \vec{0}$ and not (________)) – that is, $(\vec{a}, \vec{b})$ is linearly dependent. Hence, for any number a, ...]

3. Prove the theorem:

$$\vec{a} \in [\vec{b}] \longrightarrow \vec{b} \in [\vec{a}] \quad [\vec{a} \neq \vec{0}]$$

[*Hint*: Recall how to "unabbreviate" a restricted sentence. Then, use two previously proved theorems. (This is an easy one.)]

4. Prove the theorem:

$$\vec{a} \in [\vec{b}] \longrightarrow [\vec{a}] = [\vec{b}] \quad [\vec{a} \neq \vec{0}]$$

Part C

1. (a) Draw arrows to describe two non-$\vec{0}$ translations $\vec{b}$ and $\vec{c}$ such that $(\vec{b}, \vec{c})$ is linearly dependent. Mark a point A and locate points B and C such that $B = A + \vec{b}$ and $C = A + \vec{c}$. Draw a line l through B and C.
   (b) What do you notice about the points A, B, and C?
   (c) Is the sequence $(B - A, C - A)$ linearly dependent? Explain.
   (d) Do you think that the sequence $(C - B, A - B)$ is linearly dependent? Explain.
   (e) Do you think that another choice of A, $\vec{b}$, and $\vec{c}$ satisfying the conditions in part (a) would have led to different answers for parts (b), (c), and (d)?
   (f) Would you give different answers for (b) – (d) if either [or both] of $\vec{b}$ and $\vec{c}$ were $\vec{0}$?
2. Repeat Exercise 1(a) – (e) with 'linearly dependent' replaced by 'linearly independent'.

* 3. [Point to ponder] On the basis of what you have noticed in Exercises 1 and 2 try to formulate a definition which describes, in terms of linear dependence or independence, when an arbitrary triple of points is collinear.

4. From Exercises 1 and 2 it is easy to guess that:

Theorem 6–14 $(B - A, C - A)$ is linearly dependent

$$\Longleftrightarrow$$

$(C - B, A - B)$ is linearly dependent

is a theorem. In fact, Exercise 1 suggests the ______-part of this biconditional sentence and Exercise 2 suggests the ______-part. [Complete, and explain.] Below is a proof of the only if-part of Theorem 6–14 in which some of the algebra has been taken for granted. In studying any proof you should do three things:

---

2. Suppose that $\vec{a} = \vec{b}a$. It follows that $\vec{a}1 + \vec{b} \cdot -a = \vec{0}$ and, since $1 \neq 0$, $\exists_x \exists_y (\vec{a}x + \vec{b}y = \vec{0}$ and not $(x = 0$ and $y = 0))$ — that is, $(\vec{a}, \vec{b})$ is linearly dependent. Hence, for any number a,

$$\vec{a} = \vec{b}a \Longrightarrow (\vec{a}, \vec{b}) \text{ is linearly dependent}$$

and so

$$\exists_x \vec{a} = \vec{b}x \Longrightarrow (\vec{a}, \vec{b}) \text{ is linearly dependent.}$$

Hence, by Definition 5-1, the theorem.

3. Suppose that $\vec{a} \in [\vec{b}]$. It follows by Exercise 2 that $(\vec{a}, \vec{b})$ is linearly dependent. So, by Exercise 1 of Part A, it follows, for $\vec{a} \neq \vec{0}$, that $\vec{b} \in [\vec{a}]$. Hence, for $\vec{a} \neq \vec{0}$, if $\vec{a} \in [\vec{b}]$ then $\vec{b} \in [\vec{a}]$.

4. Suppose that $\vec{a} \in [\vec{b}]$. It follows by Exercise 3 that, for $\vec{a} \neq \vec{0}$, $\vec{b} \in [\vec{a}]$. By Exercise 1, since $\vec{a} \in [\vec{b}]$, $[\vec{a}] \subseteq [\vec{b}]$; and, since [for $\vec{a} \neq \vec{0}$] $\vec{b} \in [\vec{a}]$, $[\vec{b}] \subseteq [\vec{a}]$. Since $[\vec{a}] \subseteq [\vec{b}]$ and [for $\vec{a} \neq \vec{0}$] $[\vec{b}] \subseteq [\vec{a}]$ it follows that $[\vec{a}] = [\vec{b}]$. Hence, for $\vec{a} \neq \vec{0}$, if $\vec{a} \in [\vec{b}]$ then $[\vec{a}] = [\vec{b}]$.

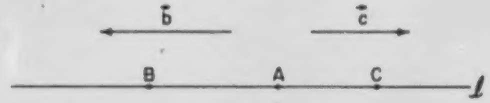
The first two exercises are exploratory for Theorem 6-14. This theorem will be required as an adjunct to the definition:

$$\{A, B, C\} \text{ is collinear} \Longleftrightarrow (B - A, C - A) \text{ is linearly dependent}$$

Students should be able to guess this definition upon pondering Exercise 3. Make an effort to see that they do.

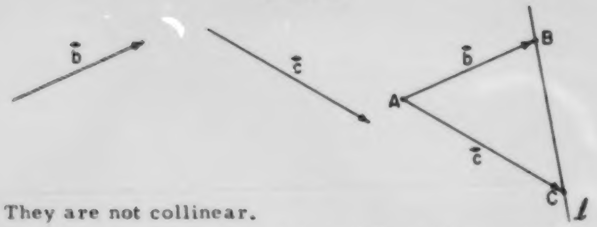
Answers for Part C

1. (a) [Any two parallel arrows will do. If the arrows drawn describe the same translation (i.e., if $\vec{b} = \vec{c}$) then any line containing B will do for a line containing B and C.] Here is a typical picture for the given conditions.



   (b) They seem collinear.
   (c) Yes. $B - A = \vec{b}$, $C - A = \vec{c}$, and $(\vec{b}, \vec{c})$ is linearly dependent.
   (d) Yes. Assuming that A, B, and C are collinear, I believe that $(C - B, A - B)$ must be linearly dependent.
   (e) No.
   (f) No. [Except that if $\vec{b}$, say, were $\vec{0}$ then, since B would be the same point as A, I'd be absolutely certain that A, B, and C were collinear and that $(C - B, A - B)$ was linearly dependent.]

2. (a)



   (b) They are not collinear.

   (c) Yes. Same reason as before. [With 'irdependent' in place of 'dependent'!]

   (d) Yes. Since A, B, and C are not collinear, I don't see how (C - B, A - B) could be linearly dependent.

   (e) No.

3. [See discussion preceding answers.]

4. only if; if; That Exercise 1 suggests the only if-part is clear. Exercise 2 suggests that if (B - A, C - A) is linearly independent then so is (C - B, A - B). So, by contraposition, it suggests the if-part of Theorem 6-14.

Answers for Part C [cont.]

4. First explanation of detail in proof: From the assumption and Postulate 3 it follows that

$$(B - A)a + [(B - A) + (C - B)]b = \vec{0}.$$

So, by Postulate 4 [and Postulate 1(b)], it follows that

$$(C - B)b + (B - A)(a + b) = \vec{0}.$$

Since $B - A = -(A - B)$ and $-\vec{a}c = \vec{a} \cdot -c$ it follows that

$$(C - B)b + (A - B) \cdot -(a + b) = \vec{0}.$$

Second explanation: Suppose that $b = 0$ and $-(a + b) = 0$. It follows that $-(a + 0) = 0$ and, so, that $-a = 0$ and, finally, that $a = 0$. Hence, if $b = 0$ and $-(a + b) = 0$ then $a = 0$ and [of course] $b = 0$.

Third explanation: From the preceding conditional sentence it follows, by contraposition, that

$$\text{not } (a = 0 \text{ and } b = 0) \Longrightarrow \text{not } (b = 0 \text{ and } -(a + b) = 0).$$

Earlier it has been shown that

$$(B - A)a + (C - A)b = \vec{0} \Longrightarrow (C - B)b + (A - B) \cdot -(a + b) = \vec{0}.$$

The required conclusion follows from these since, as is easily seen, any inference of the form:

$$\frac{p \Longrightarrow r \qquad q \Longrightarrow s}{(p \text{ and } q) \Longrightarrow (r \text{ and } s)}$$

is valid. Here's why:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \overset{*}{\text{and}} q}{p} \qquad p \Longrightarrow r}{r} \qquad \dfrac{\dfrac{p \overset{*}{\text{and}} q}{q} \qquad q \Longrightarrow s}{s}}{r \text{ and } s}}{(p \text{ and } q) \Longrightarrow (r \text{ and } s)}\,{}^{*}$$

Fourth explanation: To save writing, let's take the conditional sentence we've just proved for '$Fab \Longrightarrow Gbt$', taking '$-(a + b)$' for t'. Now, consider the scheme:

$$\dfrac{\dfrac{Fab \Longrightarrow Gbt}{Fab \Longrightarrow \exists_x \exists_y\, Gxy}}{\exists_x \exists_y\, Fxy \Longrightarrow \exists_x \exists_y\, Gxy}$$

The first inference is valid by two applications of a rule which is practically $(\mathrm{I}\exists)$. [If you want details, the first inference can be validated by modus ponens, two $(\mathrm{I}\exists)$s, and the deduction

(i) Read through the proof, taking for granted details which look reasonable and can be checked later. Satisfy yourself that, if these details check out, the argument is a proof of the theorem.
(ii) Check the details.
(iii) Try to identify the key ideas which suggested the proof to the person who wrote it.

[The last is the really important point. If you can do this then you will have greater success in thinking up your own proofs.] Study the following proof from these three points of view – understanding, checking, and "seeing through". The details you should check are indicated by '[Explain.]'. Write out your explanations and then try to explain the key idea.

Suppose that a and b are any real numbers such that

$$(B - A)a + (C - A)b = \vec{0}.$$

Since, by Postulate 3,

$$C - A = (B - A) + (C - B)$$

it follows that

$$(C - B)b + (A - B) \cdot -(a + b) = \vec{0}. \text{ [Explain.]}$$

Moreover,

$$(b = 0 \text{ and } -(a + b) = 0) \longrightarrow (a = 0 \text{ and } b = 0). \text{ [Explain.]}$$

Hence, for any a and b,

$$(B - A)a + (C - B)b = \vec{0} \text{ and not } (a = 0 \text{ and } b = 0)$$

$$\longrightarrow \qquad \text{[Explain.]}$$

$$(C - B)b + (A - B) \cdot -(a + b) = \vec{0}$$
$$\text{and not } (b = 0 \text{ and } -(a + b) = 0).$$

Consequently, if $(B - A, C - A)$ is linearly dependent then so is $(C - B, A - B)$. [Explain.]

5. Prove the if-part of Theorem 6–14.

6. Here is an instance of the only if-part of Theorem 6–14.

$$(C - B, A - B) \text{ is linearly dependent}$$
$$\longrightarrow$$
$$(A - C, B - C) \text{ is linearly dependent}$$

(a) Check that this is an instance of the only if-part of Theorem 6–14.

Answers for Part C [cont.]

rule. The second inference is valid by two applications of the rule you get from (E∃) and the deduction rule. It is all right to use this because the premiss of the inference is a consequence [by the first inference] of the sentence we started with, and that sentence is a theorem. Also, our replacement for '$\exists_x \exists_y$ Gxy' certainly doesn't contain either 'a' or 'b'. Now that that's settled, all you need to notice is that the only if-part of Theorem 6-14 follows from the last sentence we've proved and the definition of linear dependence. To see that this is so, all you need to know is the replacement rule for biconditional sentences.

Key idea: This was, evidently, to express the 'B - A' and 'C - A' in the assumption in terms of the 'C - B' and 'A - B' that were needed in the conclusion. Since it worked here, it will probably work in proving the other part of the theorem.

5. Suppose that a and b are any real numbers such that

$$(C - B)a + (A - B)b = \vec{0}.$$

Since, by Postulate 3,

$$C - B = (A - B) + (C - A)$$

it follows that

$$(B - A) \cdot -(a + b) + (C - A)a = \vec{0}.$$

Moreover,

$$(-(a + b) = 0 \text{ and } a = 0) \Longrightarrow (a = 0 \text{ and } b = 0).$$

Hence, for any a and b,

$$(C - B)a + (A - B)b = \vec{0} \text{ and not } (a = 0 \text{ and } b = 0)$$
$$\Longrightarrow$$
$$(B - A) \cdot -(a + b) + (C - A)a = \vec{0} \text{ and not } (-(a + b) = 0 \text{ and } a = 0).$$

Consequently, if (C - B, A - B) is linearly dependent then so is (B - A, C - A).

6. (a) In the only if-part of Theorem 6-14, subsitute [simultaneously] 'C' for 'B', 'B' for 'A', and 'A' for 'C'.

(b) Show that the if-part of Theorem 6–14 is a consequence of two instances of its only if-part.

7. If, in Theorem 6–14, you replace 'linearly dependent' by 'linearly independent', is the result a theorem? Explain.

8. There are at least two ways of guessing what sentences may be theorems. One way is illustrated by the way Exercises 1 and 2 suggested Theorem 6–14. Another way which sometimes leads to interesting theorems is to notice that the key idea of a proof of one theorem can be used to prove another. Either of these ways might lead you to guess:

$$(B - A, C - A, D - A) \text{ is linearly dependent}$$
$$\Longleftrightarrow$$
$$(C - B, D - B, A - B) \text{ is linearly dependent}$$

(a) Use the ideas in the proof given in Exercise 4 to prove the only if-part of this theorem.

(b) Use the method of Exercise 6 to show that the only if-part of this theorem implies its if-part.

(c) Make up, and work, exercises like Exercises 1–3 to discover what the theorem means.

[You may prefer to do part (c) before doing parts (a) and (b). That's fine.]

Part D

An existential generalization is analogous to an alternation sentence. For, intuitively, any sentence of the following form seems "reasonable":

$$\exists_x Fx \text{ if and only if } (F1 \text{ or } F0 \text{ or } F\sqrt{2} \text{ or } F\pi \text{ or } \ldots)$$

[Of course, there are too many real numbers for it to be possible to complete this sentence as the '. . .' suggests.] Similarly, universal generalizations appear to be analogous to conjunction sentences:

$$\forall_x Fx \text{ if and only if } (F1 \text{ and } F0 \text{ and } F\sqrt{2} \text{ and } F\pi \text{ and } \ldots)$$

If we think of "treating Fa as an assertion" as being analogous to using all of the instances $F1, F0, F\sqrt{2}, F\pi, \ldots$ at once as premisses, there is an analogy between inferences of the form:

$$\frac{Fa}{\forall_x Fx}$$

and those of the form:

$$\frac{p \qquad q}{p \text{ and } q}$$

---

6. (b) In addition to the given instance we need the sentence:

$$(A - C, B - C) \text{ is linearly dependent}$$
$$\Longrightarrow$$
$$(B - A, C - A) \text{ is linearly dependent}$$

This is obtained from the only if-part of Theorem 6-14 by the simultaneous substitutions of 'A' for 'B', 'C' for 'A', and 'B' for 'C'. [There are other ways of using the only if-part of this theorem in proving its if-part. For example, another instance of the only if-part is:

$$(A - B, C - B) \text{ is linearly dependent}$$
$$\Longrightarrow$$
$$(C - A, B - A) \text{ is linearly dependent}$$

The if-part follows readily from this and two instances of the theorem:

$$(\vec{a}, \vec{b}) \text{ is linearly dependent}$$
$$\Longleftrightarrow$$
$$(\vec{b}, \vec{a}) \text{ is linearly dependent} \qquad]$$

7. The result is a theorem. This is because of the fact that any sentence of the form '$p \Longleftrightarrow q$' implies the corresponding sentence of the form 'not $p \Longleftrightarrow$ not q'. [See Exercise 4 of Part A on page 225.]

8. (a) Suppose that a, b, and c are any real numbers such that

$$(B - A)a + (C - A)b + (D - A)c = \vec{0}.$$

Since, by Postulate 3,

$$C - A = (B - A) + (C - B)$$

and

$$D - A = (B - A) + (D - B)$$

it follows that

$$(C - B)b + (D - B)c + (A - B) \cdot -(a + b + c) = \vec{0}.$$

Moreover,

$$(b = 0,\ c = 0, \text{ and } -(a + b + c) = 0)$$
$$\Longrightarrow$$
$$(a = 0,\ b = 0, \text{ and } c = 0).$$

Hence, for any a, b, and c,

$$(B - A)a + (C - A)b + (D - A)c = \vec{0} \text{ and}$$
$$\text{not } (a = 0,\ b = 0, \text{ and } c = 0)$$
$$\Longrightarrow$$
$$(C - B)b + (D - B)c + (A - B) \cdot -(a + b + c) = \vec{0}$$
$$\text{and not } (b = 0,\ c = 0, \text{ and } -(a + b + c) = 0)$$

Consequently, if (B - A, C - A, D - A) is linearly dependent then so is (C - B, D - B, A - B).

8. (b) By the theorem just proved,

(C - B, D - B, A - B) is linearly dependent

$\Longrightarrow$

(D - C, A - C, B - C) is linearly dependent,

(D - C, A - C, B - C) is linearly dependent

$\Longrightarrow$

(A - D, B - D, C - D) is linearly dependent,

and

(A - D, B - D, C - D) is linearly dependent

$\Longrightarrow$

(B - A, C - A, D - A) is linearly dependent.

The if-part of the theorem of part (a) is a consequence of these three instances of the only if-part.

(c) [Answers will be various. But students should be able to guess at the definition:

{A, B, C, D} is coplanar

$\Longleftrightarrow$

(B - A, C - A, D - A) is linearly dependent]

Parts D, E, and F explore an important analogy which is useful in suggesting valid inferences involving open sentences and quantified sentences, as well as in making such inferences seem "reasonable". The analogy may be set forth as follows:

'Fa', when used as a premiss asserting something about all values of 'a',

is analogous to a set of "representative" instances, one for each value of 'a',

'$\forall_x$ Fx' is analogous to a "conjunction" of the [infinitely many] members of such a set, and

'$\exists_x$ Fx' is analogous to an "alternation" of the members of such a set of instances.

So, for example, the following are pairs of analogous kinds of inference:

$$\frac{\forall_x\ Fx}{Ft},\quad \frac{p \text{ and } q}{p}\ \text{[or: } \frac{p \text{ and } q}{q}\text{]};\quad \frac{Fa}{\forall_x\ Fx},\quad \frac{p \qquad q}{p \text{ and } q}$$

$$\frac{\exists_x\ Fx \qquad Fa \Longrightarrow r}{r},\quad \frac{p \text{ or } q \qquad p \Longrightarrow r \qquad q \Longrightarrow r}{r};$$

$$\frac{Ft}{\exists_x\ Fx},\quad \frac{p}{p \text{ or } q}\ \text{[or: } \frac{q}{p \text{ or } q}\text{]}$$

These and other examples of the analogy are discussed in the exercises and the related commentary. Here are two which are not:

$$\frac{p \text{ and } \exists_x\ Fx}{\exists_x\ (p \text{ and } Fx)},\quad \frac{p \text{ and } (q \text{ or } r)}{(p \text{ and } q) \text{ or } (p \text{ and } r)};$$

$$\frac{p \text{ or } Fa}{p \text{ or } \forall_x\ Fx},\quad \frac{p \text{ or } q \qquad p \text{ or } r}{p \text{ or } (q \text{ and } r)}$$

There are also analogies between valid sentences. A very important example furnishes motivation for Part F. Here is a less important but still notable one:

$$\forall_x\ Fx \Longrightarrow \exists_x\ Fx;\quad (p \text{ and } q) \Longrightarrow (p \text{ or } q)$$

[The easily-established fact that sentences of the first form are valid indicates that our rules of logic are appropriate only in case the domains of our variables are non-empty. This, of course, is not an important restriction — we shall never wish to discuss nothing! It is, however, well to note that when we agree to quantify over points we are tacitly assuming that $\mathcal{E}$ is non-empty. Note that none of the foregoing puts any restriction on the discussion of the empty set. The set $\emptyset$ exists and is, for example, a subset of $\mathcal{E}$. The only restriction is against taking the domain of any of our variables to *be* $\emptyset$.]

1. To the same extent as indicated above, each of our other three basic rules for '∀' or '∃' is analogous to a previously adopted basic rule for 'and' or 'or'. Pair up the analogous rules.
2. What kind of inferences involving generalization sentences are analogous to inferences of the form:

$$\frac{p \longrightarrow q \quad p \longrightarrow r}{p \longrightarrow (q \text{ and } r)} \quad ?$$

To inferences of the form:

$$\frac{p \longrightarrow r \quad q \longrightarrow r}{(p \text{ or } q) \longrightarrow r} \quad ?$$

[Note that the analogues of these two kinds of inferences require restrictions. Similar "restrictions" are implicit in the forms displayed above. Do you see how?]

Part E

In the following exercises we shall explore further the analogy between generalization sentences and conjunction and alternation sentences. As a starting point, you have already seen that elimination inferences for '∃' are analogous to elimination inferences for 'or':

$$\frac{\exists_x Fx \quad Fa \longrightarrow r}{r} \qquad \frac{p \text{ or } q \quad p \longrightarrow r \quad q \longrightarrow r}{r}$$

The requirement on the first of these that the sentence used for '$Fa \longrightarrow r$' must be treated as an assertion about all values of 'a' corresponds with the need, in the second kind of inference, to have *both* conditional premisses. The requirement that, in inferences of the first kind, 'a' must not occur in the conclusion corresponds with the requirement that both conditional premisses in the second have the same consequent. As you may recall, we can get around this last requirement, as is illustrated in the following derivation:

$$\dfrac{p \text{ or } q \quad \dfrac{\dfrac{\dfrac{\overset{*}{p} \quad p \longrightarrow r}{r}}{r \text{ or } s}}{p \longrightarrow (r \text{ or } s)}{}^{*} \quad \dfrac{\dfrac{\dfrac{\overset{*}{q} \quad q \longrightarrow s}{s}}{r \text{ or } s}}{q \longrightarrow (r \text{ or } s)}{}^{*}}{r \text{ or } s}$$

1. Give a scheme analogous to the preceding to show that an inference of the form:

$$\frac{\exists_x Fx \quad Fa \longrightarrow Ga}{\exists_x Gx}$$

Answers for Part D

1. [The basic rules have been paired up in the preceding discussion. The question as to which of the two parts of the elimination rule for 'and' is "the analogue" of the elimination rule for '∀' is best answered by saying 'Both' and by pointing out that it is our notation which suggests that one kind of '∀'-inference corresponds with two kinds of 'and'-inference. To say, for a given replacement for 'Fa', that any inference of the form '$\forall_x\ Fx/Ft$' is valid is to assert the validity of many different inferences which are obtainable by making many different choices for 't'. On the other hand, the elimination rule for 'and' could be made more like that for '∀' if we introduced a subscript i whose domain is {1, 2} and made the claim that any inference of the form '$(p_1 \text{ and } p_2)/p_i$' is valid. Similar remarks apply to the introduction rules for '∃' and 'or'.]

2. $\dfrac{p \Longrightarrow Fa}{p \Longrightarrow \forall_x\ Fx}$; $\dfrac{Fa \Longrightarrow r}{\exists_x\ Fx \Longrightarrow r}$; The restrictions are (i) that the premisses of these inferences be treated as assertions about all values of 'a' and (ii) that 'a' not occur in the sentences which replace 'p' and 'r'. In the analogous 'and' and 'or' inference, (i) corresponds to the fact that each has two premisses — one for each of the two sentences which are connected by 'and' or 'or' in its conclusion. The restriction (ii) corresponds to the fact that both premisses in the 'and' inference have the same antecedent and both premisses in the 'or' inference have the same consequent.

[The first of the forms of inference displayed in the exercise has been treated previously in Exercise 3 on page 230. Students should see that the second follows at once from the elimination rule for 'or' and the deduction rule. In connection with this exercise you might give students one of each of the following pairs of analogous inferences, and ask for the other:

$$\frac{p \Longrightarrow \forall_x\ Fx}{p \Longrightarrow Ft}, \quad \frac{p \Longrightarrow (q \text{ and } r)}{p \Longrightarrow q}; \quad \frac{\exists_x\ Fx \Longrightarrow r}{Ft \Longrightarrow r}, \quad \frac{(p \text{ or } q) \Longrightarrow r}{p \Longrightarrow r}$$

Or, a better question may be to investigate the converses of the inferences discussed in this exercise.]

As is brought out in Part E, the analogy between the basic rules of logic for quantifiers and the basic rules for 'and' and 'or' makes it possible to transform a scheme which justifies one of a pair of analogous kinds of inference into a scheme which justifies the other. So, if either of a pair is a valid kind of inference then so is the other.

Answers for Part E

1.

$$\dfrac{\exists_x\ Fx \quad \dfrac{\dfrac{\dfrac{\overset{*}{Fa} \quad Fa \Longrightarrow Ga}{Ga}}{\exists_x\ Gx}}{Fa \Longrightarrow \exists_x\ Gx}{}^{*}}{\exists_x\ Gx}$$

[If, as suggested in the discussion of Exercise 1 of Part D, page 265, we had introduced a subscript i with values 1 and 2, the is valid provided that its conditional premiss is taken as an assertion about all values of the indicated variable. [*Hint*: What kind of inference is analogous to the 'or'-introduction inferences '*r*/(*r* or *s*)' and '*s*/(*r* or *s*)' used in the given scheme?]

2. Use the result in Exercise 1 to show that, with the same provision, any inference of the form:

$$\frac{Fa \longrightarrow Ga}{\exists_x Fx \longrightarrow \exists_x Gx}$$

is valid.

3. The same procedure you used in Exercise 2 will suffice to show that, with the same provision, any inference of the form:

$$\frac{Fa \longrightarrow Gt}{\exists_x Fx \longrightarrow \exists_x Gx}$$

is valid. Show this. [This kind of inference was used (twice) at the end of the proof given in Exercise 4 of Part C.]

4. Inferences of the kind referred to in Exercise 2 are analogous to inferences of the form:

$$\frac{p \longrightarrow r \qquad q \longrightarrow s}{(p \text{ or } q) \longrightarrow (r \text{ or } s)}$$

   (a) Show that the result of replacing both 'or's, above, by 'and's is a valid form of inference.

   (b) Devise an analogous scheme to show that any inference of the form:

$$\frac{Fa \longrightarrow Ga}{\forall_x Fx \longrightarrow \forall_x Gx}$$

   is valid provided that its premiss is taken as an assertion about all values of '*a*'.

Part F

1. One rule which we have found to be very useful is that any sentence of the form:

$$\text{not } (p \text{ or } q) \longleftrightarrow (\text{not } p \text{ and not } q)$$

is valid. State an analogous rule concerning generalization sentences. Do you think that this analogous rule seems reasonable?

2. Recall that in the proof given on page 255 of the if-part of Theorem 6-7 we used the fact that any inference of the form:

$$\frac{\text{not } Fa}{\text{not } \exists_x Fx}$$

justification of the "complex dilemma" displayed in the text might have been simplified to a form more like the answer given above:

$$\left.\dfrac{\dfrac{\dfrac{\overset{*}{p_i} \qquad p_i \Longrightarrow r_i}{r_i}}{r_1 \text{ or } r_2}}{p_i \Longrightarrow (r_1 \text{ or } r_2)}{}^{*}\right\} \text{[for i = 1 \underline{and} for i = 2]}$$

$$\dfrac{p_1 \text{ or } p_2 \qquad p_i \Longrightarrow (r_1 \text{ or } r_2)}{r_1 \text{ or } r_2}$$

However, the note would be needed to warn the reader that both — rather than a chosen one — of the cases must be dealt with. Alternatively, one might adopt a general rule that an 'i' in a premiss means that one is to consider both such premisses, while an 'i' in a conclusion means that one may consider either conclusion.]

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2. [All that is necessary is to add to the answer for Exercise 1:

$$\dfrac{}{\exists_x Fx \Longrightarrow \exists_x Gx}{}^{\dagger}$$

and place a '†' over the premiss '$\exists_x$ Fx'.]

3. [Merely repeat the answer for Exercise 2, but replace the two occurrences of 'Ga' (see answer for Exercise 1, above) by 'Gt's.]

4. (a)

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{*}{p \text{ and } q}}{p} \qquad p \Longrightarrow r}{r} \qquad \dfrac{\dfrac{\overset{*}{p \text{ and } q}}{q} \qquad q \Longrightarrow s}{s}}{r \text{ and } s}}{(p \text{ and } q) \Longrightarrow (r \text{ and } s)}{}^{*}$$

(b)

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{*}{\forall_x Fx}}{Fa} \qquad Fa \Longrightarrow Ga}{Ga}}{\forall_x Gx}}{\forall_x Fx \Longrightarrow \forall_x Gx}{}^{*}$$

[Note that the validity of the ($I\forall$)-inference requires that its premiss be taken as an assertion about all values of 'a' and, so, that each premiss on which this depends and which contains 'a' must be so taken. Since, however, '$\forall_x$ Fx' does not contain 'a', this premiss may be taken as an assumption.]

is valid provided its premiss is taken as an assertion about all values of '*a*'. We also know [how?] that any inference of the form:

$$\frac{\forall_x \text{ not } Fx}{\text{not } Fa}$$

is valid. Since '*a*' does not occur in the premiss of this inference, we may certainly take its conclusion as an assertion about all values of '*a*'. Now, show that the if-part of any sentence of the form you gave in answering Exercise 1 is valid.

3. The only if-part of a sentence of the form you gave in answering Exercise 1 is analogous to a sentence of the form:

$$\text{not } (p \text{ or } q) \longrightarrow (\text{not } p \text{ and not } q)$$

Here is a scheme which shows that any sentence of this form is valid:

$$\frac{\dfrac{p \longrightarrow (\overset{*}{p} \text{ or } q)}{\text{not } (p \text{ or } q) \longrightarrow \text{not } p} \qquad \dfrac{q \longrightarrow (\overset{*}{p} \text{ or } q)}{\text{not } (p \text{ or } q) \longrightarrow \text{not } q}}{\text{not } (p \text{ or } q) \longrightarrow (\text{not } p \text{ and not } q)}$$

[The '*'s are meant to indicate that any sentence of the form of either premiss is valid.] The analogues of sentences of the form of the premisses are sentences of the form:

$$Fa \longrightarrow \exists_x Fx$$

and, as you know, such sentences are valid. Now, using these hints, show that any sentence of the form:

$$\text{not } \exists_x Fx \longrightarrow \forall_x \text{ not } Fx$$

is valid.

*

We can use what we have learned in Parts E and F in giving a very simple proof of Theorem 6-7. To begin with, since we know that any sentence of the form:

$$\text{not } \exists_x Fx \longleftrightarrow \forall_x \text{ not } Fx$$

is valid, and since we know from Part D on pages 252 and 253 that blocks of *similar* quantifiers can be handled just like single quantifiers, we know that any sentence of the form:

$$\text{not } \exists_x \exists_y Fxy \longleftrightarrow \forall_x \forall_y \text{ not } Fxy$$

Answers for Part F

1. Any sentence of the form '$\text{not } \exists_x Fx \Longleftrightarrow \forall_x \text{ not } Fx$' is valid.

   This rule seems reasonable since to say that nothing exists which behaves in a certain way amounts to the same thing as saying that each thing fails to behave in this way.

2. $$\cfrac{\cfrac{\cfrac{\overset{*}{\forall_x \text{ not } Fx}}{\text{not } Fa}}{\text{not } \exists_x Fx}}{\forall_x \text{ not } Fx \Longrightarrow \text{not } \exists_x Fx} \; *$$

   The first of the three inferences is valid by the elimination rule for '∀', the second by the rule referred to in the exercise. [The latter rule is justified on page 257. This rule is, incidentally, analogous to a rule stating the validity of inferences of the form:

   $$\frac{\text{not } p \qquad \text{not } q}{\text{not } (p \text{ or } q)}$$

   ]

3. $$\cfrac{\cfrac{Fa \Longrightarrow \exists_x Fx}{\text{not } \exists_x Fx \Longrightarrow \text{not } Fa}}{\text{not } \exists_x Fx \Longrightarrow \forall_x \text{ not } Fx}$$

   The first inference is, of course, by contraposition. The second is an example of a kind referred to in Exercise 2 of Part D. [See first answer for this Exercise on TC 265(1).]

is valid. Even more, for any positive integer n, any sentence of the form:

$$(*)\quad \text{not } \exists_{x_1} \ldots \exists_{x_n} Fx_1 \ldots x_n \Longleftrightarrow \forall_{x_1} \ldots \forall_{x_n} \text{ not } Fx_1 \ldots x_n$$

is valid.

Now, let's recall the definition of linear dependence:

$$(\vec{a}_1, \ldots, \vec{a}_n) \text{ is linearly dependent}$$
$$\Longleftrightarrow$$
$$\exists_{x_1} \ldots \exists_{x_n} (\vec{a}_1 x_1 + \ldots + \vec{a}_n x_n = \vec{0} \text{ and not } (x_1 = 0, \ldots, \text{ and } x_n = 0))$$

From this, by a rule about biconditional sentences, it follows at once that

$$(\vec{a}_1, \ldots, \vec{a}_n) \text{ is } \textit{not} \text{ linearly dependent}$$
$$\Longleftrightarrow$$
$$\text{not } \exists_{x_1} \ldots \exists_{x_n} (\vec{a}_1 x_1 + \ldots + \vec{a}_n x_n = \vec{0} \text{ and not } (x_1 = 0, \ldots, \text{ and } x_n = 0)).$$

So, using (*) and biconditional replacement, it follows that

$$(\vec{a}_1, \ldots, \vec{a}_n) \text{ is not linearly dependent}$$
$$\Longleftrightarrow$$
$$\forall_{x_1} \ldots \forall_{x_n} \text{ not } (\vec{a}_1 x_1 \ldots + \vec{a}_n x_n = \vec{0} \text{ and not } (x_1 = 0, \ldots, \text{ and } x_n = 0)).$$

With this we have almost arrived at Theorem 6-7:

$$(\vec{a}_1, \ldots, \vec{a}_n) \text{ is linearly independent}$$
$$\Longleftrightarrow$$
$$\forall_{x_1} \ldots \forall_{x_n} [\vec{a}_1 x_1 + \ldots + \vec{a}_n x_n = \vec{0} \Longrightarrow (x_1 = 0, \ldots, \text{ and } x_n = 0)]$$

To obtain this from the preceding result we need to use biconditional replacement and the definition of linear independence:

$$(\vec{a}_1, \ldots, \vec{a}_n) \text{ is linearly independent}$$
$$\Longleftrightarrow$$
$$(\vec{a}_1, \ldots, \vec{a}_n) \text{ is not linearly dependent}$$

and we need to use the fact that any sentence of the form:

$$(**)\qquad \text{not } (p \text{ and not } q) \Longleftrightarrow [p \Longrightarrow q]$$

is valid. For, from this last it would follow that the sentence:

$$\text{not } (\vec{a}_1 a_1 + \ldots + \vec{a}_n a_n = \vec{0} \text{ and not } (a_1 = 0, \ldots, \text{ and } a_n = 0))$$
$$\Longleftrightarrow$$
$$[(\vec{a}_1 a_1 + \ldots + \vec{a}_n a_n = \vec{0} \Longrightarrow (a_1 = 0, \ldots, \text{ and } a_n = 0)]$$

is valid. So, by Exercise 4(b) of Part E, we would know that the sentence:

$$\forall_{x_1} \ldots \forall_{x_n} \text{ not } (\vec{a}_1 x_1 + \ldots + \vec{a}_n x_n = \vec{0} \text{ and not } (x_1 = 0, \ldots, \text{ and } x_n = 0))$$
$$\Longleftrightarrow$$
$$\forall_{x_1} \ldots \forall_{x_n} [\vec{a}_1 x_1 + \ldots + \vec{a}_n x_n = \vec{0} \Longrightarrow (x_1 = 0, \ldots, \text{ and } x_n = 0)]$$

is valid. With this, we could complete the proof of Theorem 6-7 by using biconditional replacement.

Part G

All that remains for the proof of Theorem 6-7 is the proof that any sentence of the form (**) on page 268 is valid. We have carried out half of this proof already, on page 256, when we showed that any inference of the form:

$$\frac{p \Longrightarrow q}{\text{not } (p \text{ and not } q)}$$

is valid. You can complete the job by showing that any inference of the form:

$$\frac{\text{not } (p \text{ and not } q)}{p \Longrightarrow q}$$

is valid. [*Hint:* Use what you know (page 171) about 'not', 'and' and 'or' to transform the premiss into an alternation sentence. Then, recall the rule for denying an alternative (page 172).]

6.09 Chapter Summary

Vocabulary Summary

linear combination
sequence
linearly dependent
linearly independent
subsequence
permutation

universal quantifier
existential quantifier
index
universal generalization
existential generalization
instance; general instance

Definitions

6-1. $\vec{a}$ is a linear combination of $(\vec{a}_1, \ldots, \vec{a}_n)$
$\exists_{x_1} \ldots \exists_{x_n}\ \vec{a} = \vec{a}_1 x_1 + \ldots + \vec{a}_n x_n$

6-2. $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly dependent
$\exists_{x_1} \ldots \exists_{x_n} (\vec{a}_1 x_1 + \ldots + \vec{a}_n x_n = \vec{0}$ and not $(x_1 = 0, \ldots,$ and $x_n = 0))$

6-3. A sequence is linearly independent if and only if it is not linearly dependent.

6-4. $[a_1, \ldots, \vec{a}_n] = \{\vec{x}: \exists_{x_1} \ldots \exists_{x_n}\ \vec{x} = \vec{a}_1 x_1 + \ldots + \vec{a}_n x_n\}$

Other Theorems

6-1. $(\vec{a})$ is linearly dependent $\longleftrightarrow \vec{a} = \vec{0}$.

6-2. For $n \geq 2$, $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly dependent $\longleftrightarrow$ one of the vectors $\vec{a}_1, \ldots, \vec{a}_n$ is a linear combination of the others.

6-3. (a) A sequence one of whose terms is $\vec{0}$ is linearly dependent.
(b) A sequence two of whose terms are equal is linearly dependent.

6-4. If any subsequence of a given sequence is linearly dependent then the given sequence is linearly dependent.

6-5. If any term of a given sequence is a multiple of another term then the given sequence is linearly dependent.

6-6. A permutation of a given sequence is linearly dependent if and only if the given sequence is linearly dependent.

6-7. $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly independent
$\longleftrightarrow \forall_{x_1} \ldots \forall_{x_n} [\vec{a}_1 x_1 + \ldots + \vec{a}_n x_n = \vec{0} \longrightarrow (x_1 = 0, \ldots,$ and $x_n = 0)]$

6-8. $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly independent
$\longrightarrow (\vec{a}_1 a_1 + \ldots + \vec{a}_n a_n = \vec{a}_1 b_1 + \ldots + \vec{a}_n b_n$
$\longrightarrow (a_1 = b_1, \ldots,$ and $a_n = b_n))$

6-9. If a sequence is linearly independent then any of its subsequences is linearly independent.

6-10. Any permutation of a linearly independent sequence is linearly independent.

The proof given, beginning here, for Theorem 6-7 is simple in concept and even obvious to one with some proficiency in the use of the appropriate rules of logic. With 'p' as an abbreviation for '$(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly dependent', '$Fx_1 \ldots x_n$' for '$\vec{a}_1 x_1 + \ldots + \vec{a}_n x_n = \vec{0}$', and '$Gx_1 \ldots x_n$' for '$(x_1 = 0, \ldots,$ and $x_n = 0)$', such a person would argue as follows. The sentence:

$$p \Longleftrightarrow \exists_{x_1} \ldots \exists_{x_n} (Fx_1 \ldots x_n \text{ and not } Gx_1 \ldots x_n)$$

is equivalent to:

$$\text{not } p \Longleftrightarrow \text{not } \exists_{x_1} \ldots \exists_{x_n} (Fx_1 \ldots x_n \text{ and not } Gx_1 \ldots x_n)$$

which is equivalent to:

$$\text{not } p \Longleftrightarrow \forall_{x_1} \ldots \forall_{x_n} \text{ not } (Fx_1 \ldots x_n \text{ and not } Gx_1 \ldots x_n)$$

which is equivalent to:

$$\text{not } p \Longleftrightarrow \forall_{x_1} \ldots \forall_{x_n} [Fx_1 \ldots x_n \Longrightarrow Gx_1 \ldots x_n]$$

Q. E. D.

The rules used are that a sentence of the form '$p \Longleftrightarrow q$' is equivalent to one of the form 'not $p \Longleftrightarrow$ not q', one of the form 'not $\exists_x \ldots$' to one of the form '$\forall_x$ not ...', and one of the form 'not (p and not q)' to one of the form '$p \Longrightarrow q$'. [Actually, as shown on page 269, the final transformation uses the last of the three equivalences just mentioned, the replacement rule for biconditional sentences and the kind of inference discussed in Exercise 4(b) of Part E. Biconditional replacement does not of itself justify the transformation of the third of the sentences displayed above into the fourth, for the expressions being abbreviated by:

$$\text{not } (Fx_1 \ldots x_n \text{ and not } Gx_1 \ldots x_n)$$

and:

$$Fx_1 \ldots x_n \Longrightarrow Gx_1 \ldots x_n$$

are not sentences. The sentences one needs to deal with are those obtained from these expressions by replacing the 'x's by 'a's.]

TC 269

Answers for Part G

$$\cfrac{\cfrac{\cfrac{\cfrac{\text{not (p and not q)} \overset{*}{\Longleftrightarrow} \text{(not p or not not q)} \qquad \text{not (p and not q)}}{\text{not p or not not q}} \qquad \cfrac{\overset{\dagger}{\text{p}}}{\text{not not p}}}{\text{not not q}}}{\text{q}}}{\text{p} \Longrightarrow \text{q}}\ \dagger$$

6-11. Any linearly independent sequence is a sequence of distinct, non-$\vec{0}$, terms.

6-12. $((\vec{a}, \vec{b})$ is linearly independent and $\vec{a} + \vec{b} + \vec{c} = \vec{0})$
$\Longrightarrow (\vec{a}a + \vec{b}b + \vec{c}c = \vec{0} \Longleftrightarrow a = b = c)$

6-13. $((\vec{a}_1, \ldots, \vec{a}_{n+1})$ is linearly dependent and $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly independent$) \Longrightarrow \vec{a}_{n+1} \in [\vec{a}_1, \ldots, \vec{a}_n]$

6-14. $(B - A, C - A)$ is linearly dependent $\Longleftrightarrow (C - B, A - B)$ is linearly dependent

Other Basic Rules of Logic

Dealing with generalization sentences [See pages 240 and 241 for notation.]

Any inference of any of the following forms is valid:

$$\frac{\forall_x\, Fx}{Ft}$$ [Elimination rule for '∀', (E∀)]

$$\frac{Fa}{\forall_x\, Fx}$$ [Introduction rule for '∀', (I∀)]

$$\frac{\exists_x\, Fx \quad Fa \Longrightarrow q}{q}$$ [Elimination rule for '∃', (E∃)]

$$\frac{Ft}{\exists_x\, Fx}$$ [Introduction rule for '∃', (I∃)]

In the (I∀) and (E∃) inferences the premiss in which the indicated variable occurs must be taken as an assertion about all values of this variable. Also, in (E∃) this variable must not occur in the sentence taken for 'q'.

Other Rules of Logic

An inference of any of the following forms is valid:

$$\frac{p \Longleftrightarrow q}{\text{not } p \Longleftrightarrow \text{not } q}$$ [page 225]

$$\frac{\text{not } p \Longleftrightarrow \text{not } q}{p \Longleftrightarrow q}$$ [page 225]

$$\frac{p \Longrightarrow q \quad p \Longrightarrow r}{p \Longrightarrow (q \text{ and } r)}$$ [page 230]

$$\frac{p \Longrightarrow r \quad q \Longrightarrow r}{(p \text{ or } q) \Longrightarrow r}$$ [page 265]

$$\frac{p \Longrightarrow q \quad p \Longrightarrow \text{not } q}{\text{not } p}$$ [page 256]

$$\frac{p \Longrightarrow q}{\text{not } (p \text{ and not } q)}$$ [page 256]

$$\frac{\text{not } (p \text{ and not } q)}{p \Longrightarrow q}$$ [page 269]

$$\frac{p \Longrightarrow \forall_x\, Fx}{p \Longrightarrow Ft}$$ [page 250]

$$\frac{\exists_x\, Fx \Longrightarrow q}{Ft \Longrightarrow q}$$ [page 251]

$$\frac{p \Longrightarrow Fa}{p \Longrightarrow \forall_x\, Fx}$$ [page 251]

$$\frac{Fa \Longrightarrow q}{\exists_x\, Fx \Longrightarrow q}$$ [page 249]

provided, in each of these last two cases, that the premiss be taken as an assertion about all values of 'a' and that 'a' occurs only as indicated;

$$\frac{\text{not } Fa}{\text{not } \exists_x\, Fx}$$ [page 257]

$$\frac{Fa \Longrightarrow Ga}{\forall_x\, Fx \Longrightarrow \forall_x\, Gx}$$ [page 266]

$$\frac{Fa \Longrightarrow Gt}{\exists_x\, Fx \Longrightarrow \exists_x\, Gx}$$ [page 266]

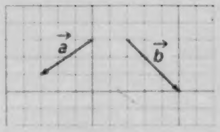
provided, in each of these last three cases, that the premiss be taken as an assertion about all values of 'a'.

A sentence of any of the following forms is valid:

$\forall_x\, Fx \Longrightarrow Ft$ [page 250]
$Ft \Longrightarrow \exists_x\, Fx$ [page 250]
$\forall_x\forall_y\, Fxy \Longleftrightarrow \forall_y\forall_x\, Fxy$ [page 252]
$\exists_x\exists_y\, Fxy \Longleftrightarrow \exists_y\exists_x\, Fxy$ [page 253]
$\text{not } \exists_x\, Fx \Longleftrightarrow \forall_x \text{ not } Fx$ [page 267]

Chapter Test

1. At the right are arrows which describe $\vec{a}$ and $\vec{b}$, where $(\vec{a}, \vec{b})$ is linearly independent. Use graph paper to do the following.



   (a) Draw an arrow to describe $\vec{a}2 + \vec{b}$.
   (b) Draw an arrow to describe $\vec{a} + \vec{b}2$.
   (c) Draw an arrow to describe $\vec{a} + \vec{b}$.
   (d) Given the sequence $(\vec{a}2 + \vec{b}, \vec{a} + \vec{b}2)$ whose terms you described in (a) and (b), tell whether this sequence is linearly dependent or independent.
   (e) Prove that your answer in (d) is correct.

2. Suppose that $\vec{a}$ is a linear combination of $\vec{b}$, $\vec{c}$, and $\vec{d}$. Show that the sequence $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly dependent.

3. Assume that $(B - A, C - B)$ is linearly independent and consider the equation:
(*) $(B - A) \cdot (a - 3) + (C - B) \cdot (b + 2) + (A - C) \cdot (c + 1) = \vec{0}$.
Find three ordered triples (a, b, c) such that (*) is satisfied.

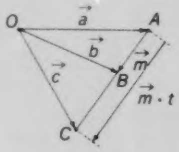
4. Here are four logical inference schemes:

(I) $\dfrac{p \quad p \Longrightarrow q}{q}$
(II) $\dfrac{p}{\text{not not } p}$
(III) $\dfrac{p \Longrightarrow q \quad \text{not } q}{\text{not } p}$
(IV) $\dfrac{\text{not } q \Longrightarrow \text{not } p}{p \Longrightarrow q}$

Which of the given inference schemes is:
(a) a double denial rule?
(b) modus ponens?
(c) a rule of contraposition?
(d) modus tollens?

5. Recall that $[\vec{a}, \vec{b}]$ is the set of all linear combinations of $\vec{a}$ and $\vec{b}$.
   - (a) Prove that if $\vec{c}$ and $\vec{d}$ are in $[\vec{a}, \vec{b}]$ then $\vec{c} + \vec{d}$ is in $[\vec{a}, \vec{b}]$. [*Note*: By our agreement, to say that $\vec{c}$ is in $[\vec{a}, \vec{b}]$ is to say that $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$.]
   - (b) Prove that if $\vec{c}$ is in $[\vec{a}, \vec{b}]$ then, for any p, $\vec{c} \cdot p$ is in $[\vec{a}, \vec{b}]$.

6. The diagram at the right describes translations $\vec{a}$, $\vec{b}$, and $\vec{c}$ from O to the *three* points A, B, and C, respectively. Also, from the diagram, we see that $B - A = \vec{m}$ and $C - A = \vec{m} \cdot t$, where $t \neq 0$.



   - (a) Tell whether $(\vec{a}, \vec{b}, \vec{c})$ is linearly dependent or linearly independent.
   - (b) Prove your answer in (a).

*

Background Topic

In the exercises on pages 173–175 you reviewed systems of two linear equations:

$$(*) \qquad \begin{cases} a_1a + b_1b = c_1 \\ a_2a + b_2b = c_2 \end{cases}$$

You learned that whether or not such a system has a unique solution-pair (a, b) depends on whether or not the number $a_1b_2 - a_2b_1$ is different from 0. This number, you should recall, is called the *determinant* of $((a_1, b_1), (a_2, b_2))$ [or, sometimes, the determinant of the system $(*)$]:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

The system $(*)$ has a unique solution if and only if the determinant of $(*)$ is not 0. [If the determinant of $(*)$ is 0 then $(*)$ either has no solution or has infinitely many solutions.]

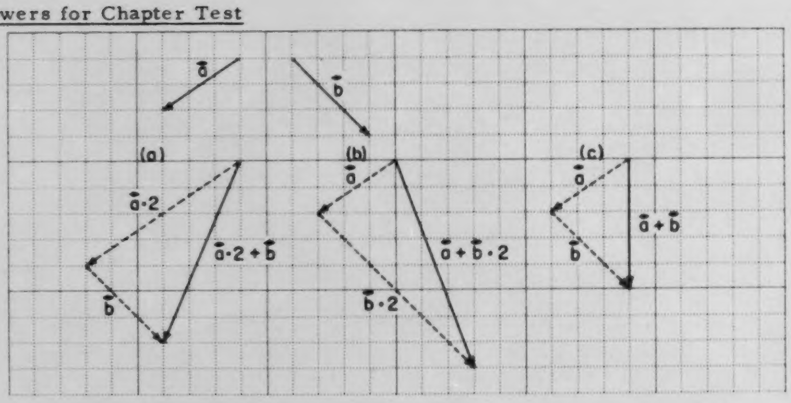
Now, let's consider the following problem. Suppose that $(\vec{c}, \vec{d})$ is linearly independent and that, for given numbers a_1, b_1, and b_2,

$$(1) \qquad \vec{a} = \vec{c}a_1 + \vec{d}a_2 \text{ and } \vec{b} = \vec{c}b_1 + \vec{d}b_2.$$

What we wish to know is whether or not $(\vec{a}, \vec{b})$ is linearly independent.

Answers for Chapter Test

1.



(d) Linearly independent.

(e) Let a and b be numbers such that $(\vec{a} \cdot 2 + \vec{b}) \cdot a + (\vec{a} + \vec{b} \cdot 2) \cdot b = \vec{0}$. Then, $\vec{a} \cdot (2a + b) + \vec{b} \cdot (a + 2b) = \vec{0}$. Since $(\vec{a}, \vec{b})$ is linearly independent $2a + b = 0$ and $a + 2b = 0$. So, $a = 0$ and $b = 0$. Hence $(\vec{a} \cdot 2 + \vec{b}, \vec{a} + \vec{b} \cdot 2)$ is linearly independent.

2. Since $\vec{a}$ is a linear combination of $\vec{b}$, $\vec{c}$, and $\vec{d}$, $\vec{a} = \vec{b} \cdot b + \vec{c} \cdot c + \vec{d} \cdot d$, for some b, c, d. So, $\vec{a} \cdot 1 + \vec{b} \cdot -b + \vec{c} \cdot -c + \vec{d} \cdot -d = \vec{0}$. Since $1 \neq 0$, this last result means that $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly dependent.

3. Notice that $(B - A) + (C - B) + (A - C) = \vec{0}$ and $(B - A, C - B)$ is linearly independent. So, a, b, and c are such that $a - 3 = b + 2 = c + 1$. In other words, $b = a - 5$ and $c = a - 4$. Hence, for any a, the triple $(a, a - 5, a - 4)$ satisfies (✱). Examples of such triples are $(0, -5, -4)$, $(5, 0, 1)$, $(4, -1, 0)$, $(3, -2, -1)$.

4. (a) II; (b) I; (c) IV; (d) III

5. (a) Suppose that $\vec{c}$ and $\vec{d}$ are in $[\vec{a}, \vec{b}]$. Then, $\vec{c} = \vec{a} \cdot c_1 + \vec{b} \cdot c_2$ and $\vec{d} = \vec{a} \cdot d_1 + \vec{b} \cdot d_2$, for some c_1, c_2, d_1, d_2. So, $\vec{c} + \vec{d} = (\vec{a} \cdot c_1 + \vec{b} \cdot c_2) + (\vec{a} \cdot d_1 + \vec{b} \cdot d_2) = (\vec{a} \cdot c_1 + \vec{a} \cdot d_1) + (\vec{b} \cdot c_2 + \vec{b} \cdot d_2) = \vec{a} \cdot (c_1 + d_1) + \vec{b} \cdot (c_2 + d_2)$. Hence $\vec{c} + \vec{d}$ is in $[\vec{a}, \vec{b}]$, for $\vec{c} + \vec{d}$ is a linear combination of $\vec{a}$ and $\vec{b}$. Thus, if $\vec{c}$ and $\vec{d}$ are in $[\vec{a}, \vec{b}]$ then $\vec{c} + \vec{d}$ is in $[\vec{a}, \vec{b}]$.

   (b) Suppose that $\vec{c}$ is in $[\vec{a}, \vec{b}]$. Then, $\vec{c} = \vec{a} \cdot c + \vec{b} \cdot d$, for some c, d. So, for any p, $\vec{c} \cdot p = (\vec{a} \cdot c + \vec{b} \cdot d)p = (\vec{a}c)p + (\vec{b}d)p = \vec{a}(cp) + \vec{b}(dp)$. This means that, for any p, $\vec{c} \cdot p$ is in $[\vec{a}, \vec{b}]$, for $\vec{c} \cdot p$ is a linear combination of $\vec{a}$ and $\vec{b}$.

6. (a) $(\vec{a}, \vec{b}, \vec{c})$ is linearly dependent.

   (b) From the information given, $\vec{m} = B - A = \vec{b} - \vec{a}$ and $\vec{m} \cdot t = C - A = \vec{c} - \vec{a}$. So, $(B - A) \cdot t = C - A$ so that $(\vec{b} - \vec{a}) \cdot t = \vec{c} - \vec{a}$. Thus, $\vec{a} \cdot (1 - t) + \vec{b} \cdot t + \vec{c} \cdot -1 = \vec{0}$ and, since $-1 \neq 0$, it follows that $(\vec{a}, \vec{b}, \vec{c})$ is linearly dependent.

This Background Topic introduces an important application of second-order determinants to the investigation of linear dependence and independence of 2-termed sequences of vectors. In brief, if $(\vec{c}, \vec{d})$ is linearly independent then

$$(\vec{c}a_1 + \vec{d}a_2,\ \vec{c}b_1 + \vec{d}b_2) \text{ is linearly independent}$$
$$\Longleftrightarrow$$
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

This is the second in an infinite sequence of theorems. The first is:

$$(\vec{c}) \text{ is linearly independent}$$
$$\Longrightarrow$$
$$[(\vec{c}a) \text{ is linearly independent} \Longleftrightarrow a \neq 0]$$

We mention this rather trivial theorem in order to point out that it is related to the statement that, for $(\vec{c})$ linearly independent, $(\vec{c})$ and $(\vec{c}a)$ determine the same direction if and only if $a \neq 0$. More precisely [still for $(\vec{c})$ linearly independent] $(\vec{c})$ and $(\vec{c}a)$ determine the same sense if and only if $a > 0$ [and determine opposite senses if and only if $a < 0$]. The two-dimensional case treated in these exercises can be extended and refined along similar lines. Briefly, a linearly independent sequence $(\vec{c}, \vec{d})$ determines a bidirection — the set of linear combinations $[\vec{c}, \vec{d}]$. This is related to the common "direction" of a family of parallel planes in $\mathcal{E}$ in the same way as a direction — say, $[\vec{c}]$ — is related to the common direction of a family of parallel lines. Just as a line can be oriented in either of two ways by choosing one of the two senses contained in its directions, so a plane can be oriented by choosing one of the two senses contained in its bidirection. [These two senses correspond with the two possible senses of rotation — clockwise or counterclockwise — in the plane.] Now, as we shall prove later, for $(\vec{c}, \vec{d})$ linearly independent, $(\vec{c}, \vec{d})$ and $(\vec{c}a_1 + \vec{d}a_2, \vec{c}b_1 + \vec{d}b_2)$ determine the same bidirection if and only if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$

and, in this bidirection, determine the same sense if and only if this determinant is positive.

Returning to more mundane matters, your students are likely to discover — by obtaining correct answers by incorrect methods — that:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0$$

is as good a criterion as the one given. Whether they do or not, you might take this occasion to point out that

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2 = a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

That is "one can interchange the rows and the columns of a determinant without changing its value".

You may also point out to students that, for any n, it is possible to define 'nth order determinants' and apply such to similar independence problems for n-termed sequences [and to the solution of systems of n linear equations.

Answers for Exercises

1. linearly independent

$$\begin{vmatrix} 5 & 5 \\ 1 & -1 \end{vmatrix} = -5 - 5 = -10 \neq 0$$

2. linearly independent

$$\begin{vmatrix} 3 & 3 \\ 0 & -4 \end{vmatrix} = -12 \neq 0$$

3. linearly dependent

$$\begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix} = 6 - 6 = 0$$

4. linearly independent

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1 \neq 0$$

5. linearly independent

$$\begin{vmatrix} 0 & 4 \\ 9 & -1 \end{vmatrix} = 0 - 36 = -36 \neq 0$$

6. linearly independent

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2 \neq 0$$

7. linearly dependent

$$\begin{vmatrix} 5 & -5 \\ 7 & -7 \end{vmatrix} = -35 - -35 = 0$$

8. linearly independent

$$\begin{vmatrix} -2 & -2 \\ -2 & 2 \end{vmatrix} = -4 - 4 = -8 \neq 0$$

9. linearly dependent

$$\begin{vmatrix} 9 & 9 \\ 11 & 11 \end{vmatrix} = 99 - 99 = 0$$

10. linearly dependent

$$\begin{vmatrix} 0 & 61 \\ 0 & -125 \end{vmatrix} = 0 - 0 = 0$$

In other words, what we wish to know is whether or not the equation:

$$\vec{a}a + \vec{b}b = \vec{0}$$

has any solution other than (0, 0). [Explain.] In view of (1), this amounts to asking whether or not the equation:

$$\vec{c}(a_1a + b_1b) + \vec{d}(a_2a + b_2b) = \vec{0}$$

has any solution other than (0, 0). [Explain.] Since $(\vec{c}, \vec{d})$ is linearly independent, this amounts to asking whether or not the system:

$$\begin{cases} a_1a + b_1b = 0 \\ a_2a + b_2b = 0 \end{cases}$$

has a solution other than (0, 0). Since (0, 0) is [obviously] a solution of this system, what we wish to know is merely whether or not this system has a unique solution. What is the answer?

Exercises

Assuming that $(\vec{c}, \vec{d})$ is linearly independent, use determinants to determine which of the given pairs are linearly independent.

Sample. $(\vec{c}1 + \vec{d}2, \vec{c}3 + \vec{d}4)$

Solution. $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$

Answer. ? ['linearly independent' or 'linearly dependent']

1. $(\vec{c}5 + \vec{d}, \vec{c}5 + \vec{d} \cdot -1)$

2. $(\vec{c}3, \vec{c}3 - \vec{d}4)$

3. $(\vec{c}2 + \vec{d}6, \vec{c} + \vec{d}3)$

4. $(\vec{d}, \vec{c})$ [i.e.: $(\vec{c}0 + \vec{d}1, \vec{c}1 + \vec{d}0)$]

5. $(\vec{d}9, \vec{c}4 - \vec{d})$

6. $(\vec{c} + \vec{d}, \vec{c} - \vec{d})$

7. $(\vec{c}5 + \vec{d}7, \vec{c} \cdot -5 - \vec{d}7)$

8. $(-\vec{c}2 - \vec{d}2, -\vec{c}2 + \vec{d}2)$

9. $(\vec{c}9 + \vec{d}11, \vec{c}9 + \vec{d}11)$

10. $(\vec{0}, \vec{c} \cdot 61 + \vec{d} \cdot -125)$

Chapter Seven

Lines in $\mathscr{E}$

7.01 Collinear Points

So far, our postulates have had to do, in the main, with translations and with how translations act on points. In order to get to the more familiar notions concerning geometric figures in $\mathscr{E}$, we need postulates which link geometric notions with those we have been studying. One such notion is that of a line.

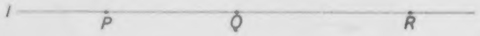
In our intuitive consideration of geometry in Chapter 1 it was convenient to describe a line as being a set of points which is the union of two opposite rays. Now we wish to define 'line', 'ray', and so on, in terms of 'point' and 'translation'. Our earlier description will turn out to be a theorem. So will the other results which we reached intuitively in Chapter 1. This will be an important test of the adequacy of our definitions. If it turned out that they didn't enable us to reach the conclusions which seem intuitively sound, we might suspect that something was wrong with them.

Some of the exercises in the preceding chapter have probably given you a hint as to how we shall define the word 'line'. To begin with, we shall decide what it means, in terms of translations, to say that three points "line up" or are *collinear*. Then we can say that a line is a set of points which consists of all the points which line up with any two of its members.

As is often the case, the intuitive line of thought which leads us to adopt a definition is reversed once the definition has been adopted. This is natural enough, since we wish to be sure that, starting from the definition, we can recover the intuitive notions which led us to it. In the present case we have an intuitive notion of what a line is and our notion of collinearity is that collinear points are points which belong to some line. We also have intuitive notions as to how translations affect lines. In the following exercises you will review the arguments, based on these intuitions, which suggest how 'A, B, and C are collinear' can be expressed by means of terms which refer to translations.

Exercises

1. Suppose that, as shown in the figure below, P, Q, and R are points which belong to a line l.



   - (a) What can you say about the translations $Q - P$ and $R - P$? About the sequence $(Q - P, R - P)$?
   - (b) From the figure, guess nonzero real numbers q and r such that $(Q - P)q + (R - P)r = \vec{0}$.
   - (c) Is the sequence $(R - Q, P - Q)$ linearly dependent?
   - (d) Suppose that S is any point such that $\{P, Q, S\}$ is a set of collinear points. Where must S be, and what can you say about $(Q - P, S - P)$?
   - (e) Suppose that T is any point such that $\{P, Q, T\}$ is a set of noncollinear points. Where must T not be, and what can you say about $(Q - P, T - P)$? About the sequence $(P - T, Q - T)$?

2. Instead of saying that points A, B, and C are collinear we shall sometimes say that $\{A, B, C\}$ is a set of collinear points. More often we shall say that $\{A, B, C\}$ is a collinear set or merely that $\{A, B, C\}$ is collinear.
   - (a) If $C = B$, can $\{A, B, C\}$ be noncollinear?
   - (b) Is there a noncollinear set which consists of just two points? Of a single point?

The preceding exercises suggest that the following definition agrees with our intuitive notions concerning the relationship between collinearity of points and linear dependence of translations:

> Definition 7-1 $\{A, B, C\}$ is collinear
>
> $\Longleftrightarrow$
>
> $(B - A, C - A)$ is linearly dependent

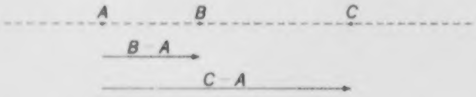


Fig. 7–1

As we have pointed out in the past, it is only by choice that we call this a definition rather than a postulate. In general, we choose to list as definitions those postulates which introduce new terms into our system. Of course, since a definition is a postulate, it naturally plays the same kind of role in proving theorems as does any postulate.

Answers for Exercises

1. (a) Each is a multiple of the other.; It is linearly dependent.
   - (b) Choose q = 7 and r = −3. [Any choice of q and r such that q/r = −7/3 is correct.]
   - (c) Yes. [There are various ways to establish this. For one thing, the linear dependence of (R - Q, P - Q) follows from that of (Q - P, R - P) by Theorem 6-14. Students should be reminded, if necessary, of this theorem which will be referred to again on page 277. On the other hand, the linear dependence of (R - Q, P - Q) can be established "by inspection" as was that of (Q - P, R - P): $(R - Q)3 + (P - Q)4 = \vec{0}$ and $3 \neq 0$.]
   - (d) S must be a point of ℓ, and (Q - P, S - P) is linearly dependent.
   - (e) T must not be a point of ℓ, and (Q - P, T - P) is linearly independent. (P - T, Q - T) is linearly independent. [The latter follows, by Theorem 6-14, from the linear independence of (Q - P, T - P).]

2. (a) No. [For, then, any line containing A and B also contains A, B, and C.]
   - (b) No.; No. [For any two points, and any single point, are contained in some line.]

The connection between collinearity and linear dependence which is explored in Exercise 1, and the explanations given for the answers for Exercise 2, are based on an intuitive notion of what a line is and a notion of collinearity according to which given points are collinear if and only if all are contained in one line. As indicated on page 275, we now take the intuitive connection between collinearity and linear dependence as the basis for a formal definition of the word 'collinear' in contexts of the form '{A, B, C} is collinear'. The word 'line' is defined on page 279. In Exercises 3 and 4 on page 280 we investigate the question as to whether, under these definitions, it can be proved that {A, B, C} is collinear if and only if it is a subset of some line. The fact that this can be proved shows that our definitions are not inappropriate.

Definition 7-1 defines 'collinear' in a rather special context. It could be extended by agreeing that an arbitrary set of points is collinear if and only if each of its 3-point subsets is collinear. However, once 'line' has been defined we shall revert to the ordinary use of 'collinear' according to which a set is collinear if and only if it is a subset of a line.

There is one point which needs to be noted before adopting a definition like Definition 7–1. As you know,

$$\{A, B, C\} = \{B, C, A\} = \{C, A, B\}$$
$$= \{A, C, B\} = \{B, A, C\} = \{C, B, A\}.$$

So, adopting Definition 7–1 would get us into trouble if it were not the case that, for example,

$$(1)\qquad \begin{array}{c}(B - A, C - A)\text{ is linearly dependent}\\ \Longleftrightarrow\\ (C - B, A - B)\text{ is linearly dependent}\end{array}$$

and

$$(2)\qquad \begin{array}{c}(B - A, C - A)\text{ is linearly dependent}\\ \Longleftrightarrow\\ (C - A, B - A)\text{ is linearly dependent.}\end{array}$$

[Explain why Definition 7–1 would cause trouble if (1) or (2) were not true. Are (1) and (2) true?] Since we have already proved Theorem 6–14 and Theorem 6–6, we can be sure of (1) and (2). But, there are other similar statements which we need to be sure of. Do we need other theorems as a check on these statements?

Exercises

1. Draw a picture showing three collinear points—K, L, and M.
   - **(a)** What can you say about the sequence $(L - K, M - K)$?
   - **(b)** Name five other 2-termed sequences of non-$\vec{0}$ translations about which you can say the same thing.
   - **(c)** Add to your picture two arrows to represent $L - K$ and $M - K$ and estimate values of 'a' and 'b' such that $(L - K)a + (M - K)b = \vec{0}$.
2. Draw a picture showing two points—R and S.
   - **(a)** Mark a point P such that $\{R, S, P\}$ is collinear and a point $Q \neq P$ such that $\{R, S, Q\}$ is collinear. Is it the case that $\{P, Q, R\}$ is collinear?
   - **(b)** According to Definition 7-1, what must you show in order to establish that $\{P, Q, R\}$ is collinear? If you were going to show this, what does part (a) give you to work with?
   - **(c)** Since $\{R, S, P\}$ is collinear you know, by definition, that $(S - R, P - R)$ is linearly dependent. Since $R \neq S$ you know that $S - R \neq \vec{0}$. So, by a previous theorem you know that $P - R \in$ ________. [State the theorem.]
   - **(d)** Prove that $Q - R \in [S - R]$.
   - **(e)** You now have shown that there are numbers—say, p and q—such that $P - R = (S - R)p$ and $Q - R = (S - R)q$.

By Definition 7-1 we have both:

$$\{A, B, C\}\text{ is collinear} \Longleftrightarrow (B - A, C - A)\text{ is linearly dependent}$$

and:

$$\{B, C, A\}\text{ is collinear} \Longleftrightarrow (C - B, A - B)\text{ is linearly dependent}$$

So, since $\{A, B, C\} = \{B, C, A\}$ it follows from Definition 7-1 that

$$(1)\qquad \begin{array}{c}(B - A, C - A)\text{ is linearly dependent}\\ \Longleftrightarrow\\ (C - B, A - B)\text{ is linearly dependent.}\end{array}$$

If there were a counter-example to this generalization then the adoption of Definition 7-1 would make our theory contradictory. [By the rule of contradiction (see page 156) each sentence would then be a theorem.] Fortunately, (1) is already a theorem — Theorem 6-14 — and so are the other theorems which follow in a similar manner from Definition 7-1. Hence, at this stage of our development, Definition 7-1 is not "creative" and may safely be adopted.

As to what theorems are needed to justify the adoption of Definition 7-1, note that the instance (1) of Theorem 6-14 "takes care of" the fact that $\{A, B, C\} = \{B, C, A\}$ and that the instance of (1) obtained by substituting 'B' for 'A', 'C' for 'B', and 'A' for 'C' takes care of the fact that $\{B, C, A\} = \{C, A, B\}$. The instance (2) of Theorem 6-6 takes care of the fact that $\{A, B, C\} = \{A, C, B\}$ and two similar instances take care of the facts that $\{B, C, A\} = \{B, A, C\}$ and $\{C, A, B\} = \{C, B, A\}$. The preceding, together with a "transitivity argument", takes care of all cases. So, Theorems 6-6 and 6-14 are sufficient to the purpose.

The necessity for justifying a definition like Definition 7-1 may be brought out by considering the ad hoc "definition":

$$(*)\qquad \{A, B, C\}\text{ is coterminal} \Longleftrightarrow A = C \neq B$$

From (*) and its instance:

$$\{A, C, B\}\text{ is coterminal} \Longleftrightarrow A = B \neq C.$$

together with '$\{A, B, C\} = \{A, C, B\}$', we can infer:

$$A = C \neq B \Longleftrightarrow A = B \neq C$$

and, substituting 'A' for 'C', arrive at the conclusion:

$$A = B$$

[To do so, note that 'not $(A = B \neq A)$' and '$A = A$' are valid sentences.] Obviously, in any nontrivial theory, (*) is so creative as to be almost totally destructive!

Finally, note that, instead of Definition 7-1, we might have adopted a definition of collinearity for 3-termed sequences of points:

$$(**)\qquad (A, B, C)\text{ is collinear} \Longleftrightarrow (B - A, C - A)\text{ is linearly dependent}$$

In this case no justification like that given for Definition 7-1 would be required. However, as a matter of convenience we would wish to make sure that, for example:

$$(A, B, C)\text{ is collinear} \Longleftrightarrow (B, C, A)\text{ is collinear}$$

is a theorem. And, in view of (**), this amounts to proving (1).

Answers for Exercises

1. (a) Linearly dependent.

   (b) $(L - M, K - M)$, $(M - L, K - L)$, $(M - K, L - K)$, $(K - L, M - L)$, and $(K - M, L - M)$. [For each of the pairs listed there are 3 others obtainable by replacing terms by their opposites. Others, which students are unlikely to suggest, may be obtained by replacing each term of a listed sequence by a non-$\vec{0}$ linear combination of its terms. For, as is easy to show, if $(\vec{c}, \vec{d})$ is linearly dependent then so is $(\vec{c}a_1 + \vec{d}a_2, \vec{c}b_1 + \vec{d}b_2)$.]

   (c) [Here is a spot where various students can be asked to demonstrate their own solutions to the class. On the other hand, you may wish to save yourself paper-work by giving students work sheets on which the points in question are marked. If so, note that there are two "essentially different" cases — that in which K is between L and M $(ab < 0)$ and that in which it is not $(ab > 0)$.]

2. (a) Yes. [This answer should seem intuitively correct. The remaining parts of this exercise show how to justify it on the basis of Definition 7-1. The result to be proved may be thought of as a start toward a proof that the line through R and S is determined by any two of its points. Due to the definition of 'line' which we shall adopt the argument will not be used in this way, but the techniques illustrated in the solutions will be of value.]

   (b) It must be established that $(Q - P, R - P)$ is linearly dependent. From part (a), we know that $(S - R, P - R)$ is linearly dependent, that $Q \neq P$, and that $(S - R, Q - R)$ is linearly dependent.

   (c) $[S - R]$. The theorem is: If $(\vec{a}, \vec{b})$ is linearly dependent and $\vec{a} \neq \vec{0}$ then $\vec{b} \in [\vec{a}]$. [This — Theorem 6-13 — will be of much use, and it may be well to review its proof: Assuming $(\vec{a}, \vec{b})$ to be linearly dependent there are numbers — say, a and b — such that $\vec{a}a + \vec{b}b = \vec{0}$ and a and b are not both 0. If $b = 0$ then $\vec{a}a = \vec{0}$ and, assuming that $\vec{a} \neq \vec{0}$, $a = 0$. So, $b \neq 0$ and $\vec{b} = \vec{a}(a/b)$. Hence, $\exists_x\ \vec{b} = \vec{a}x$ and, by definition, $\vec{b} \in [\vec{a}]$.]

   (d) Since $\{R, S, Q\}$ is collinear and $R \neq S$ it follows that $(S - R, Q - R)$ is linearly dependent and $S - R \neq \vec{0}$. So, as in part (c), $Q - R \in [S - R]$. [Students should realize that the results obtained in (c) and (d) are instances of the only if-part of a potentially useful lemma:

   $$\{A, B, C\} \text{ is collinear} \Longleftrightarrow C - A \in [B - A] \quad [A \neq B]$$

   The if-part is easily established.]

Answers for Exercises [cont.]

   (e) $Q - P = (S - R)\cdot\underline{(q - p)}$ and $R - P = (S - R)\cdot\underline{-p}$

   (f) By (e), $(Q - P)\cdot p + (R - P)\cdot(q - p) = \vec{0}$. Since $Q \neq P$, $Q - P \neq \vec{0}$ and, again by (e), $q - p \neq 0$. So, by definition, $(Q - P, R - P)$ is linearly dependent. [Students may find it more natural to solve the first equation in (e) for '$S - R$' and substitute into the second. This is a perfectly satisfactory way of finding numbers r and s such that $(q - p)r + -ps = 0$, but it is much more elegant to note that p and $q - p$ are such numbers. The realization of this — i.e. recognition that '$ar + bs = 0$' has a solution given by '$(r, s) = (-b, a)$' — is the basis for numerous short cuts. Students may recall Exercise 7(b) on page 221 according to which

   $$(\vec{b} \in [\vec{a}] \text{ and } \vec{c} \in [\vec{a}]) \Longrightarrow (\vec{b}, \vec{c}) \text{ is linearly dependent.}$$

   From this and the results in part (e) it follows at once that $(Q - P, R - P)$ is linearly dependent.]

3. (a) Infinitely many.; Infinitely many.; [Exactly] one.

   (b) By definition, $\{G, H, H\}$ is collinear if and only if $(H - G, H - G)$ is linearly dependent. Since $(H - G, H - G)$ is linearly dependent by Theorem 6-3(b), it follows that $\{G, H, H\}$ is collinear. But, $\{G, H, H\} = \{G, H\}$. So, $\{G, H\}$ is collinear.

4. (a) Collinear. (b) S is point of ℓ.

Condition (ii) is, intuitively, satisfied by a set ℓ if and only if ℓ is a subset of some line. Intuitively, a set ℓ satisfies condition (iii) if and only if it contains each line which contains two of its points. For example, any line, any plane, and $\mathcal{E}$ itself are examples of sets of points which satisfy (iii). So are $\emptyset$ and any set which consists of a single point. The purpose of (i) is to get rid of such degenerate sets. [By definition, a set is degenerate if and only if it has fewer than two members.] It follows that a set ℓ which satisfies (i), (ii), and (iii) is a subset of a line and, also, contains a line. Since, intuitively, no line contains another line, it follows that a set which satisfies (i)-(iii) is a line. This suggests Definition 7-2. Note that Definition 7-2 might be reformulated:

$$\ell \text{ is a line}$$

$$\Longleftrightarrow$$

(a) ℓ is a subset of $\mathcal{E}$ which contains at least two points, and

(b) $\forall_X \forall_Y\ [(\{X, Y\} \subseteq \ell \text{ and } X \neq Y) \Longleftrightarrow \ell = \{Z : \{X, Y, Z\} \text{ is collinear}\}]$

* * *

Complete:

$Q - P = (S - R) \cdot$ _______ and $R - P = (S - R) \cdot$ _______

(f) Use your result from part (e) and the fact that $Q \neq P$ to prove $(Q - P, R - P)$ is linearly dependent.

3. Let G and H be two points.
   - **(a)** How many lines contain G? Contain H? Contain both G and H?
   - **(b)** Prove that G and H are collinear. [Use Definition 7 - 1. $\{G, H\} = \{G, H, H\}$]
4. Suppose that l is a line, and that l contains two points P and Q.
   - **(a)** Given that R is a point of line l, what can be said about $\{P, Q, R\}$?
   - **(b)** Given that S is a point such that $\{P, Q, S\}$ is collinear, what can be said about S and the line l?

7.02 Lines

Now that we have a formalized notion of collinear points, we are in a position to make use of this notion, together with our intuitive ideas about what lines are (or, ought to be), in formulating a definition of the term *line*.

In order to agree with our intuitive notions, we want to be sure that, among other things, any line l is such that

(i) l is a subset of $\mathscr{E}$ and l contains at least two points.

Certainly, this condition (i) is not enough to pin down exactly what we mean by *line*, for we expect a line to contain many more than two points. And, furthermore, you probably can think of many sets of more than two points that aren't anything like what you think of as a line.

The results in the previous set of exercises suggest that the notion of collinear points can be used to help us define the term *line*. One way to use this notion is as follows:

(ii) Given that P, Q, and R are points of l, then P, Q, and R are collinear.

We saw that (i) wasn't enough to fully describe what is generally thought of as a line. So, a natural question to ask at this stage is:

Are properties (i) and (ii) enough to fully describe what we think of as a line?

Describe, and draw pictures of, at least three sets of points each of which satisfies the conditions (i) and (ii) but which are *not* what you consider to be lines.

There are two basic theorems about lines:

(1) Any two points are contained in at least one line.

(2) Any two points are contained in at most one line.

The second of these follows readily from Definition 7-2. For, if ℓ is a line which contains two given points A and B it follows, using part (b) of the definition, that $\ell = \{Z: \{A, B, Z\}$ is collinear$\}$. [This result is established in a slightly different way in Exercise 1 of Part B on page 280.] In view of this, proving (1) amounts to showing that, for $A \neq B$, $\{Z: \{A, B, Z\}$ is collinear$\}$ is a line which contains A and B. The proof, in section 7.03, of Theorem 7-1 amounts to a proof of this. The proof of (1) is more complicated than the proof of (2) because Definition 7-2 requires a good deal of a set if it is to be considered a line.

One can give an alternative definition which requires less:

$$\text{(3)} \qquad \ell \text{ is a line} \iff \exists_X \exists_Y (X \neq Y \text{ and } \ell = \{Z: \{X, Y, Z\} \text{ is collinear}\})$$

If we were to adopt this definition, it would be easy to prove (1). For, according to (3), for $A \neq B$, $\{Z: \{A, B, Z\}$ is collinear$\}$ is a line, and it is easy to show that this line contains A and B. With (3) as definition, however, the proof of (2) becomes complicated.

Whether one chooses Definition 7-2 or (3) as definition for 'line', essentially the same complications occur — either in the proof of (1) or in the proof of (2).

For completeness [and to justify the preceding remark] we shall now sketch a derivation of (2) from (3). Precisely, we shall show that if ℓ is a line, according to definition (3), which contains two given points P and Q, then $\ell = \{Z: \{P, Q, Z\}$ is collinear$\}$.

Suppose that ℓ is such a line. By (3) there are two points — say, R and S — such that $\ell = \{Z: \{R, S, Z\}$ is collinear$\}$. So, since P and Q belong to ℓ, both $\{R, S, P\}$ and $\{R, S, Q\}$ are collinear. It follows, as in Exercise 2 on page 277, that $\{P, Q, R\}$ is collinear and, similarly, that $\{P, Q, S\}$ is collinear. More conveniently [see the remark concerning the solution for Exercise 2(d)],

$$(*) \qquad R - P \in [Q - P] \text{ and } S - P \in [Q - P].$$

Since $S - R = (S - P) - (R - P)$ it follows that $S - R \in [Q - P]$ and, so, that $[S - R] \subseteq [Q - P]$. Suppose, now, that $\{R, S, T\}$ is collinear — that is, that $T - R \in [S - R]$. We shall show that $\{P, Q, T\}$ is also collinear. To do so, note that $T - P = (R - P) + (T - R)$. Since, by (*), $R - P \in [Q - P]$ and, by assumption, $T - R \in [S - R] \subseteq [Q - P]$ it follows that $T - P \in [Q - P]$ and, so, $\{P, Q, T\}$ is collinear. Hence, if $\{R, S, T\}$ is collinear then $\{P, Q, T\}$ is collinear. The converse follows by a similar argument. [Since, by assumption, $\{R, S, P\}$ and $\{R, S, Q\}$ are collinear it follows that

$$(**) \qquad P - R \in [S - R] \text{ and } Q - R \in [S - R].$$

Proceeding from here just as we did from (*) leads to the result that if $\{P, Q, T\}$ is collinear then so is $\{R, S, T\}$.]

It has been shown that, if P and Q are two points which belong to $\{Z: \{R, S, Z\}$ is collinear$\}$ then this latter set is $\{Z: \{P, Q, Z\}$ is collinear$\}$. Consequently, if ℓ is a line according to definition (3) which contains the two points P and Q then $\ell = \{Z: \{P, Q, Z\}$ is collinear$\}$. There cannot, then, be two such lines.

Here is another insight into the relationship between collinearity and being a line:

(iii) Given that P and Q are *two* points of l and that $\{P, Q, R\}$ is collinear, then R is a point of l.

Recalling the direct link between collinear points and linearly dependent translations and between linearly dependent translations and multiples of a translation should suggest at least one way to determine a point (in fact, many points) R such that R and two given points P and Q of l are collinear. [The diagram below illustrates this.]

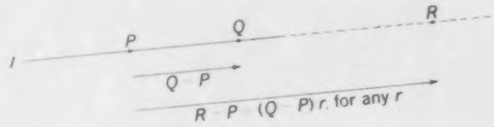


Fig. 7–2

This discussion suggests the following definition:

> Definition 7–2 l is a line if and only if
> (a) l is a subset of $\mathscr{E}$ which contains at least two points,
>
> and
>
> (b) $\forall_X\forall_Y\,[(\{X, Y\} \subseteq l \text{ and } X \neq Y) \Longrightarrow \forall_Z\,[Z \in l \Longleftrightarrow \{X, Y, Z\} \text{ is collinear}]]$

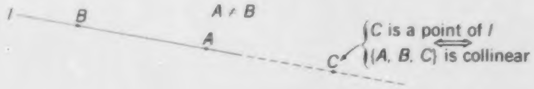


Fig. 7–3

Notice that (a) of this definition says what (i) says, and that part (b) of this definition says what (ii) *and* (iii) say. [In words, part (b) says: For each *two* points of l, each third point belongs to l if and only if it and the two given points are collinear.]

From now on, we shall use 'l', 'm', and 'n', with or without subscripts, as variables whose domain is the set of lines of $\mathscr{E}$. [Since, otherwise, we should soon run out of letters, we shall also use 'l', 'm', and 'n' as indices on quantifiers. Read '$\forall_l$' as 'for each line l', etc.]

Looked at from another point of view, it is not difficult to see that the preceding argument shows that, for R ≠ S, {Z: {R, S, Z} is collinear} satisfies condition (b) of Definition 7-2. Since it is easily seen that the set in question contains the two points R and S, it follows that it is a line according to Definition 7-2. Consequently, any set which is a line according to (3) is a line according to Definition 7-2. The converse being obvious, (3) and Definition 7-2 are equally satisfactory as definitions of 'line'.

From still a third point of view, the argument in question shows that, for R ≠ S, $\overleftrightarrow{RS}$ [see Definition 7-3 on page 281] is a line. That this is the case is also shown in the proof, in section 7.03, of Theorem 7-1. The argument given there might be replaced by the one presented above [with 'R' and 'S' replaced by 'A' and 'B']. Actually, the two arguments are essentially one.

TC 280 (1)

To be sure that students understand the application of Definition 7-1, we recommend Part A as in-class exercises.

Answers for Part A

1. (☆) is equivalent to Definition 7-2 because, by Definition 7-1, {A, B, C} is collinear if and only if (B - A, C - A) is linearly dependent. [It follows from this and various rules of logic that the following are pairs of equivalent sentences:

$$\left\{\begin{array}{l} C \in \ell \Longleftrightarrow \{A,\ B,\ C\} \text{ is collinear} \\ C \in \ell \Longleftrightarrow (B - A,\ C - A) \text{ is linearly dependent} \end{array}\right.$$

$$\left\{\begin{array}{l} \forall_Z\ [Z \in \ell \Longleftrightarrow \{A,\ B,\ Z\} \text{ is collinear}] \\ \forall_Z\ [Z \in \ell \Longleftrightarrow (B - A,\ Z - A) \text{ is linearly dependent}] \end{array}\right.$$

$$\left\{\begin{array}{l} (\{A, B\} \subseteq \ell \text{ and } A \neq B) \Longrightarrow \forall_Z\ [Z \in \ell \Longleftrightarrow \{A,\ B,\ Z\} \text{ is collinear}] \\ (\{A, B\} \subseteq \ell \text{ and } A \neq B) \Longrightarrow \forall_Z\ [Z \in \ell \Longleftrightarrow (B - A,\ Z - A) \text{ is linearly dependent}] \end{array}\right.$$

$$\left\{\begin{array}{l} [\text{Sentence (b) of Definition 7-2}] \\ [\text{Sentence (b) of } (\star)] \end{array}\right.$$

For "logic buffs" [only] it is noteworthy that the equivalence of the sentences of the first pair follows, by the replacement rule for biconditional sentences, from Definition 7-1 and that the equivalence of Definition 7-2 and (☆) follows, by a second application of the same rule, from the equivalence of the sentences of the fourth pair. [In all, there are three applications of the replacement rule and one application of a rule for universal quantifiers.] The process cannot be reduced to the mere replacement of '{X, Y, Z} is collinear' by '(Y - X, Z - X) is linearly dependent' because these are not sentences and, so, not directly subject to the replacement rule.]

2. Yes.; No. [Both answers can be justified formally. For the latter, let P = K + (L - K)2 and Q = K + (L - K)3. Since, by hypothesis, K ≠ L, it follows that P ≠ Q and that neither of them is either L or K. Certainly at least one of P and Q is not M. Finally, by definition, both {K, L, P} and {K, L, Q} are collinear.]

Exercises

Part A

1. Notice that we might just as well have defined a line l as follows:
   l is a line if and only if (a) l is a subset of $\mathcal{E}$ which contains at
   (*) least two points, and (b) $\forall_X \forall_Y [(\{X, Y\} \subseteq l \text{ and } X \neq Y) \longrightarrow \forall_Z [Z \in l \longleftrightarrow (Y - X, Z - X) \text{ is linearly dependent}]]$
   Explain why (*) is equivalent to Definition 7-2.
2. Consider a set $\{K, L, M\}$ where $(L - K, M - K)$ is a linearly dependent sequence of distinct, non-$\vec{0}$, translations. Is it the case that, for any point C of this set, $\{K, L, C\}$ is collinear? Is it the case that, for any point C such that $\{K, L, C\}$ is collinear, $C \in \{K, L, M\}$?
3. Now that we have defined both 'line' and 'collinear' it is only common decency to check to see whether, according to these definitions, if $\{A, B, C\}$ is a subset of a line l then $\{A, B, C\}$ is collinear. Do this. [*Hint*: Consider two cases, according as $A = B$ or $A \neq B$.]
4. We should also check to see that if $\{A, B, C\}$ is collinear then there is a line l such that $\{A, B, C\} \subseteq l$. In case $A \neq B$ this is easy to do once we have proved:

$$A \neq B \longrightarrow \exists_l \{A, B\} \subseteq l$$

We shall prove this theorem in the next section.

(a) Explain how this theorem would enable us to deal with the case $A \neq B$. With the case $A \neq C$.

(b) Oddly enough, our postulates do not say enough to make it possible to prove, now, that, given any point, there is a line which contains it. Try to guess why.

Part B

1. Suppose that $A \neq B$ and that both A and B do belong to some line — say, l. [From our work in Chapter 1 we know, intuitively, that there is exactly one line which contains both A and B; but, our problem now is whether this follows from our definition of 'line' and our other postulates.]

(a) From our definition of 'line' we know that

$C \in l$ if and only if ________________.

(b) From our definition of 'collinear' we know that

$\{A, B, C\}$ is collinear if and only if ________________.

(c) We also know that

$(B - A, C - A)$ is linearly dependent if
$C - A \in [B - A]$. [Explain.]

(d) Since $B - A \neq \vec{0}$ [Why?] we also know that

$C - A \in [B - A]$ if ________________. [By what theorem?]

3. Suppose that $\{A, B, C\} \subseteq \ell$. For $A = B$, $(B - A, C - A)$ is $(\vec{0}, C - A)$ and, so is linearly dependent. For $A \neq B$, since $\{A, B\} \subseteq \ell$ and $C \in \ell$ it follows by (☆) that $(B - A, C - A)$ is linearly dependent. Hence, in either case, $\{A, B, C\}$ is, by Definition 7-1, collinear. Hence, if $\{A, B, C\} \subseteq \ell$ then $\{A, B, C\}$ is collinear.

4. (a) Suppose that $A \neq B$. It follows from the assumed theorem that there is a line — say ℓ — such that $\{A, B\} \subseteq \ell$. Assuming that $\{A, B, C\}$ is collinear it follows from Definition 7-2 that $C \in \ell$, also. Hence, for $A \neq B$, if $\{A, B, C\}$ is collinear then $\exists_\ell \{A, B, C\} \subseteq \ell$. [Substituting 'C' for 'B' in the preceding one obtains the same conclusion under the restriction '$A \neq C$'.] The theorem is of no help in the remaining case, in which $A = B$ and $A = C$.

   (b) Using the theorem vouched for in part (a) we could prove '$\exists_\ell A \in \ell$' if we could prove '$\exists_Y A \neq Y$'. Our problem seems, then, to be that of proving that, given any point A, there is another point. This would be solved if we could prove that $\mathcal{E}$ contains at least two points. [Then, one of them at least would be different from the given point.] A check of our

Answers for Part A [cont.]

postulates suggests no way of proving that there are any points at all — let alone that there are at least two! If, as seems consistent with our postulates, there were exactly one point then there would, by Definition 7-2(a), be no lines. [This question is taken up again in Exercise 3 on page 287. As pointed out in the commentary on that exercise, we can at this stage prove that $\mathcal{E} \neq \emptyset$ but not that $\mathcal{E}$ is nondegenerate. In fact, it is almost obvious that all our postulates are satisfied if $\mathcal{E}$ contains a single point and $\mathcal{T}$ contains the single translation $\vec{0}$.]

Answers for Part B

1. (a) $\{A, B, C\}$ is collinear

   (b) $(B - A, C - A)$ is linearly dependent

   (c) In general, if $\vec{c} \in [\vec{b}]$ then, by definition, there is a number — say, c — such that $\vec{c} = \vec{b}c$. It follows that $\vec{b}c + \vec{c} \cdot -1 = \vec{0}$ and, since $-1 \neq 0$, that $(\vec{b}, \vec{c})$ is linearly dependent. Hence, if $\vec{c} \in [\vec{b}]$ then $(\vec{b}, \vec{c})$ is linearly dependent. In particular, $(B - A, C - A)$ is linearly dependent if $C - A \in [B - A]$.

   (d) $B - A \neq \vec{0}$ because $A \neq B$. So, $C - A \in [B - A]$ if $(B - A, C - A)$ is linearly dependent. [Theorem 6-14]

(e) By definition,

$C - A \in [B - A]$ if and only if ______________.

(f) From (a) through (e) it follows that

$C \in l \longleftrightarrow \exists_x\, C = A + (B - A)x$. [Explain.]

2. In Exercise 1 you have seen that if $A \neq B$, and if l is a line which contains A and B, then

$$l = \{X: \exists_x\, X = A + (B - A)x\}.$$

(a) Does it follow that, given two points, there is a line which contains them?

(b) Does it follow that there are not two lines, each of which contains two given points?

Part C

Although we have not yet shown that the set described in Exercise 2 of Part B is a line, we are optimistic enough to adopt a definition:

Definition 7-3 $\overleftrightarrow{AB} = \{X: \exists_x\, X = A + (B - A)x\}$

[Read '$\overleftrightarrow{AB}$' as 'double arrow $A\,B$'.] Notice that Definition 7-3 is not restricted to the case $A \neq B$. [Restricted definitions are a nuisance, and we shall avoid them unless we have very good reasons not to.] Our intuitions tell us that, for $A \neq B$, $\overleftrightarrow{AB}$ is a line which contains both A and B. We shall check on this shortly; also, in the next section, we shall see that we can prove a theorem to this effect. It is not really important what $\overleftrightarrow{AB}$ is when $A = B$. But, to get the question out of the way, what is $\overleftrightarrow{AA}$? Is it a line?

1. (a) Draw a picture to show two points P and Q and the translation $Q - P$.

(b) In your picture, locate the points C, D, E, F, G, H, I and J such that

$C = P + (Q - P)0$, $D = P + (Q - P)\frac{3}{4}$,
$E = P + (Q - P)1$, $F = P + (Q - P)2$,
$G = P + (Q - P)3$, $H = P + (Q - P) \cdot -\frac{1}{2}$,
$I = P + (Q - P) \cdot -1$, $J = P + (Q - P) \cdot -2$.

(c) Why is each point described in part (b) a point in $\overleftrightarrow{PQ}$?

(d) Draw a line l through the points P and Q. Do you expect that each point of $\overleftrightarrow{PQ}$ is also a point of l? Do you expect that each point of l is also a point of $\overleftrightarrow{PQ}$?

2. Prove each of the following.

(a) $\{A, B\} \subseteq \overleftrightarrow{AB}$ (b) $A = B \longrightarrow \overleftrightarrow{AB} = \{A\}$

(c) $\overleftrightarrow{AB}$ is a line $\longrightarrow A \neq B$ [*Hint*: Can you use (b)?]

(d) $C \in \overleftrightarrow{AB} \longleftrightarrow C - A \in [B - A]$

(e) $\exists_x\, C - A = (B - A)x$

(f) From (a) and (b),

$C \in \ell \Longleftrightarrow (B - A,\ C - A)$ is linearly dependent

and, from (c) and (d),

$(B - A,\ C - A)$ is linearly dependent $\Longleftrightarrow C - A \in [B - A]$.

So, from these results and (e),

$$C \in \ell \Longleftrightarrow \exists_x\, C - A = (B - A)x.$$

Equivalently [by Theorem 2-1],

$$C \in \ell \Longleftrightarrow \exists_x\, C = A + (B - A)x.$$

2. (a) No. [That there is such a line remains to be proved. See Theorem 7-1. However, intuitively, your students may argue that given two points A and B, there is a line containing them and it, in fact, is the line consisting of the images of A under all multiples of B - A. Don't stifle this conviction.]

(b) Yes. For if ℓ_1 and ℓ_2 are lines containing the two points A and B, then $\ell_1 = \{X: \exists_x\, X = A + (B - A)x\} = \ell_2$.

Note that Definition 7-3 is equivalent to:

$\overleftrightarrow{AB} = \{X: X - A \in [B - A]\}$ [see Exercise 2(d) of Part C]

and that in case $A \neq B$ [but only in this case]

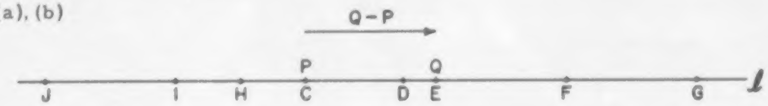
$$\overleftrightarrow{AB} = \{X: \{A, B, X\} \text{ is collinear}\}.$$

[The alternative to Definition 7-2 which is discussed in the commentary for page 278 is equivalent to 'ℓ is a line $\Longleftrightarrow \exists_X \exists_Y\, (X \neq Y$ and $\ell = \overleftrightarrow{XY})$'.] In view of Theorem 7-1, it is appropriate to read '$\overleftrightarrow{AB}$' as 'line AB' in case $A \neq B$. $\overleftrightarrow{AA}$ is $\{A\}$ and, of course, is not a line.

Answers for Part C

Answer to questions: $\overleftrightarrow{AA}$ is the singleton $\{A\}$. $\overleftrightarrow{AA}$ is not a line, for it doesn't contain two points.

1. (a), (b)



(c) Let S be any of the points described in (b). Then, from the data given in (b), it follows that $\exists_x\, S = P + (Q - P)x$. So, $S \in \overleftrightarrow{PQ}$.

(d) Yes.; Yes.

7.03 The Line Containing Two Given Points

From Part B of the preceding exercises we know that there is at most one line containing two given points. Also, if there is such a line, we know what it is. In fact, we can go a step further and make a shrewd [?] guess:

> **Theorem 7-1** For $A \neq B$,
> $\overleftrightarrow{AB}$ is the line which contains A and B.

This theorem is a short way of saying that, for $A \neq B$,

(1) $\overleftrightarrow{AB}$ is a line and $\{A, B\} \subseteq \overleftrightarrow{AB}$;

(2) $\{A, B\} \subseteq l \longrightarrow l = \overleftrightarrow{AB}$, for any line l.

Since you have already proved (2) and the second part of (1), all that remains to be proved is:

(*) $A \neq B \longrightarrow \overleftrightarrow{AB}$ is a line

To prove (*), the natural procedure is to assume that $A \neq B$ and show that, under this assumption, $\overleftrightarrow{AB}$ satisfies our definition of 'line'. To do this we must show that

(a) $\overleftrightarrow{AB}$ is a subset of $\mathscr{E}$ which contains at least two points, and

(b) $\forall_X \forall_Y\, [(\{X, Y\} \subseteq \overleftrightarrow{AB}$ and $X \neq Y)$
$\longrightarrow \forall_Z\, [Z \in \overleftrightarrow{AB} \longleftrightarrow (Y - X,\ Z - X)$ is linearly dependent]].

As to (a), $\overleftrightarrow{AB}$ is certainly a subset of $\mathscr{E}$ and you have already proved a theorem which, under our assumption about A and B, implies that $\overleftrightarrow{AB}$ contains at least two points. [Explain.] So, all that we need to consider is (b).

We can establish (b) if, assuming that

(*) $\{P, Q\} \subseteq \overleftrightarrow{AB}$ and $P \neq Q$,

we can show that

$$\forall_Z\, [Z \in \overleftrightarrow{AB} \longleftrightarrow (Q - P, Z - P) \text{ is linearly dependent}].$$

[Explain.] Finally, to show this, it will be sufficient to show that

(i) $(Q - P, C - P)$ is linearly dependent $\longrightarrow C \in \overleftrightarrow{AB}$

and

(ii) $C \in \overleftrightarrow{AB} \longrightarrow (Q - P, C - P)$ is linearly dependent.

TC 281 (2)

2. (a) Since $A = A + (B - A)0$ and $B = A + (B - A)1$ it follows that $\exists_x\ A = A + (B - A)x$ and $\exists_x\ B = A + (B - A)x$. So, $A \in \overleftrightarrow{AB}$ and $B \in \overleftrightarrow{AB}$. By definition, then, $\{A, B\} \subseteq \overleftrightarrow{AB}$.

   (b) Suppose that $A = B$. Then $\overleftrightarrow{AB} = \overleftrightarrow{AA} = \{X\colon \exists_x\ X = A + (A - A)x\}$. Since $A - A = \vec{0}$, $\vec{0}a = \vec{0}$, and $A + \vec{0} = A$ it follows that $\overleftrightarrow{AA} = \{X\colon X = A\} = \{A\}$.

   (c) By Definition 7-2, if $\overleftrightarrow{AB}$ is a line then $\overleftrightarrow{AB}$ contains at least two points and, so, is not $\{A\}$. By (b), if $\overleftrightarrow{AB}$ is not $\{A\}$ then $A \neq B$. Hence, if $\overleftrightarrow{AB}$ is a line then $A \neq B$.

   (d) By Definition 7-3,

   $$C \in \overleftrightarrow{AB} \Longleftrightarrow \exists_x\ C = A + (B - A)x.$$

   Since, by Theorem 2-1,

   $$C = A + (B - A)c \Longleftrightarrow C - A = (B - A)c$$

   it follows that

   $$\exists_x\ C = A + (B - A)x \Longleftrightarrow \exists_x\ C - A = (B - A)x.$$

   By definition,

   $$\exists_x\ C - A = (B - A)x \Longleftrightarrow C - A \in [B - A].$$

   Hence, $C \in \overleftrightarrow{AB} \Longleftrightarrow C - A \in [B - A]$.

TC 282

The proof of (2) is accomplished in solving Exercise 1 of Part B on page 280 — at least, all that remains is to transform the result, as stated in Exercise 2, by applying Definition 7-3. [The phrase 'for any line l' which follows (2) is redundant in view of the convention adopted at the foot of page 279.] The second part of (1) is Exercise 2(a) of Part C.

By definition, $\overleftrightarrow{AB}$ is a set of points [$\overleftrightarrow{AB} = \{X\colon \ldots\}$]. Since $\{A, B\} \subseteq \overleftrightarrow{AB}$ it follows that, for $A \neq B$, $\overleftrightarrow{AB}$ contains at least two points. This takes care of Definition 7-2(a).

First explanation: Use the deduction rule to discharge (✲); apply introduction rule for '∀' to the resulting conditional sentence to infer (b).

Second explanation: From (i) and (ii) we can infer:

$$C \in \overleftrightarrow{AB} \Longleftrightarrow (Q - P,\ C - P) \text{ is linearly dependent}$$

and from this we can infer:

$$\forall_Z\ [Z \in \overleftrightarrow{AB} \Longleftrightarrow (Q - P,\ Z - P) \text{ is linearly dependent}]$$

Doing so will not preclude the use of the deduction rule mentioned previously because 'C' does not occur in (✲).

[Explain.]

If we are to use the assumption (*) in establishing (i) and (ii), we need to be sure what it means. To say that $\{P, Q\} \subseteq \overleftrightarrow{AB}$ means, of course, merely that $P \in \overleftrightarrow{AB}$ and $Q \in \overleftrightarrow{AB}$ and so, by Definition 7–3, that

$$\exists_x\, P = A + (B - A)x \text{ and } \exists_x\, Q = A + (B - A)x.$$

So, we can replace (*) by the assumption that

$$(**)\quad P = A + (B - A)p,\ Q = A + (B - A)q \text{ and } P \neq Q.$$

Since, in (i) and (ii), we are interested in $Q - P$, we note that, by (**),

$$Q - P = (B - A)(q - p) \text{ and } q - p \neq 0.$$

[Explain.] We are now ready to tackle (i) and (ii). Before reading further, try to derive (i). It may be a help to recall that, as you have proved earlier, $C \in \overleftrightarrow{AB} \Longleftrightarrow C - A \in [B - A]$. Then, try to derive (ii).

Derivation of (i). Suppose that $(Q - P, C - P)$ is linearly dependent. Since $P \neq Q$ it follows that $C - P \in [Q - P]$. [Why?] Since $Q - P \in [B - A]$ [Why?] it follows that $[Q - P] \subseteq [B - A]$. So, $C - P \in [B - A]$. Since $P - A \in [B - A]$ [Why?] and $C - A = (P - A) + (C - P)$ it follows that $C - A \in [B - A]$ [Why?]. So, $C \in \overleftrightarrow{AB}$. Hence, (i).

Derivation of (ii). Suppose that $C \in \overleftrightarrow{AB}$. By definition, there is a number—say, c—such that $C = A + (B - A)c$. Since $P = A + (B - A)p$ it follows that $C - P = (B - A)(c - p)$. Since $Q - P = (B - A)(q - p)$ it follows that

$$(Q - P)(p - c) + (C - P)(q - p) = \vec{0}. \text{ [Explain.]}$$

Since $q - p \neq 0$ it follows that $(Q - P, C - P)$ is linearly dependent. Hence, (ii).

Proof of (b). It follows from (i) and (ii) that, for any C,

$$C \in \overleftrightarrow{AB} \Longleftrightarrow (Q - P, C - P) \text{ is linearly dependent.}$$

Hence [discharging the assumption (**)],

$$\begin{aligned}&\text{if } P = A + (B - A)p,\ Q = A + (B - A)q, \text{ and } P \neq Q,\\ &\text{then } C \in \overleftrightarrow{AB} \Longleftrightarrow (Q - P, C - P) \text{ is linearly dependent.}\end{aligned}$$

Consequently,

$$\begin{aligned}&\text{if } \exists_x\, P = A + (B - A)x,\ \exists_x\, Q = A + (B - A)x, \text{ and } P \neq Q,\\ &\text{then } \forall_Z\, [Z \in \overleftrightarrow{AB} \Longleftrightarrow (Q - P, Z - P) \text{ is linearly dependent}].\end{aligned}$$

The 'Why?'s sprinkled through the derivation of (i) are easily answered. For the first, 'Theorem 6-14'; for the second, '$Q - P = (B - A)(q - p)$'; for the third, '$P - A = (B - A)p$, by (**)'; for the third, '[B - A] is a vector space [Part A on page 192] and, so, is closed with respect to addition.'.

The explanation asked for in the derivation of (ii) boils down to the fact that

$$(q - p)(p - c) + (c - p)(q - p) = 0$$

[and $(B - A)0 = \vec{0}$]. This is a third illustration of the trick previously exemplified in the solution of Exercise 2(f) on page 278 and of Exercise 7(b) on page 221. We emphasize this trick because it is a handy one and seems not readily mastered.

It should perhaps be pointed out that, while the assumption that $P \neq Q$ is used in both derivations, this assumption is essential only in the case of (i). Its use — in the guise '$q - p \neq 0$' — in the derivation of (ii) merely simplifies the derivation.

Be sure that your students understand the transition from:

$$\exists_x\, P = A + (B - A)x \text{ and } \exists_x\, Q = A + (B - A)x$$

to the sentence:

$$P = A + (B - A)p \text{ and } Q = A + (B - A)q \text{ and } P \neq Q.$$

The use of 'x' as an index in both existential generalizations in no way implies any connection between the values of the corresponding variables. This is seen more clearly by our use of 'p' and 'q' in the second sentence.

The 'Proof of (b)' merely reiterates the explanations asked for on page 282 and already given in the corresponding commentary. The 'Proof of Theorem 7-1' reiterates remarks previously made on page 282. In a very reasonable sense, the proof of Theorem 7-1 is completed at the foot of page 283. This gives occasion for repeating and extending earlier remarks we have made on the function of proof.

(1) The purpose of studying formal rules of logic and formal organizations for proofs is to teach one to dispense with formality while maintaining logical coherence.

(2) There are many ways of giving a logically coherent argument. One is illustrated on pages 282 and 283. This amounts to showing that you could reach the desired conclusion if you could establish certain results, and then going on to establish these results. Recapitulation, as on page 284, is unnecessary, time-consuming, and often boring.

(3) Insistence on formal proofs whenever proofs are asked for is one of the better ways to kill student interest in the content of the course. Be happy — and show it — when students succeed in exposing the main lines of an argument in a short paragraph. One painless way to check on the ability to fill in details is to ask one student to present his argument and to ask other students questions like 'How do you suppose he knew that?'.

Hence, if $P \in \overleftrightarrow{AB}$, $Q \in \overleftrightarrow{AB}$, and $P \neq Q$,

$$\text{then } \forall_Z\, [Z \in \overleftrightarrow{AB} \longleftrightarrow (Q - P, Z - P) \text{ is linearly dependent}].$$

Consequently,

$$\forall_X \forall_Y\, [(\{X, Y\} \subseteq \overleftrightarrow{AB} \text{ and } X \neq Y) \Longrightarrow \forall_Z\, [Z \in \overleftrightarrow{AB} \longleftrightarrow (Y - X, Z - X) \text{ is linearly dependent}]]$$

Proof of Theorem 7-1. Since we have established (a) and (b) under the assumption that $A \neq B$, and since $\{A, B\} \subseteq \overleftrightarrow{AB}$, it follows that, for $A \neq B$,

$\overleftrightarrow{AB}$ is *a* line which contains A and B.

Theorem 7-1 follows from this and the fact that, for $A \neq B$,

if l is a line which contains A and B then $l = \overleftrightarrow{AB}$

—that is, nothing other than $\overleftrightarrow{AB}$ is a line which contains A and B. [You proved this in Part B on page 280.]

As a corollary to Theorem 7-1 we have:

There is one and only one line which contains two given points.

[Instead of 'contains' one sometimes says 'passes through'.] Here is another way in which this corollary is sometimes stated:

Two points determine a [unique] line.

Exercises

Part A

After having proved that, for $A \neq B$, $\overleftrightarrow{AB}$ is a line which contains A and B, we could have completed the proof of Theorem 7-1 by proving that

(*) there are not two lines which contain two given points.

Instead of this, we used the previously proved theorem:

(1) $(A \neq B \text{ and } \{A, B\} \subseteq l) \longrightarrow l = \overleftrightarrow{AB}$

1. Show that (*) follows from (1). [*Hint*: Given two points, A and B, such that both belong to a line l and both belong to a line m it follows from (1) that $l =$ ____ and that ____ $=$ ____. So, $l = m$. Hence, if $A \neq B$ then there do not exist . . . .]

Answers for Part A

1. Suppose that A and B are two points which belong to a line ℓ and to a line m. Then, by (1), $\ell = \overleftrightarrow{AB}$ and $m = \overleftrightarrow{AB}$. So, $\ell = m$. Hence, if $A \neq B$ there are not two lines which contain both A and B.

TC 285 (1)

2. $A \neq B \Longrightarrow \text{not } \exists_\ell \exists_m (\ell \neq m \text{ and } \{A, B\} \subseteq \ell \cap m)$ [Students may have '$\{A, B\} \subseteq \ell$ and $\{A, B\} \subseteq m$' rather than '$\{A, B\} \subseteq \ell \cap m$', but the latter has the advantage of brevity.]

3. $\exists_\ell \exists_m (\ell \neq m \text{ and } \{A, B\} \subseteq \ell \cap m) \Longrightarrow A = B$

4. From our work on page 249 we know that inferences of the form:

$$\frac{Fa \Longrightarrow q}{\exists_x\, Fx \Longrightarrow q} \qquad \text{['a' not in the consequent]}$$

are valid. So, [by two applications of this rule] the sentence of Exercise 3 follows from:

$$(\ell \neq m \text{ and } \{A, B\} \subseteq \ell \cap m) \Longrightarrow A = B$$

5. The sentence of Exercise 4 is actually equivalent with (2). [For a proof, see Exercise 8, below.] Suppose that $A \neq B$ and $\{A, B\} \subseteq \ell \cap m$. Since $\{A, B\} \subseteq \ell \cap m$ it follows that $\{A, B\} \subseteq \ell$ and that $\{A, B\} \subseteq m$. So, by hypothesis, $A \neq B$ and $\{A, B\} \subseteq \ell$, and $A \neq B$ and $\{A, B\} \subseteq m$. Hence, by (1) [and an instance of it obtained by substituting 'm' for 'ℓ'.], it follows that $\ell = \overleftrightarrow{AB}$ and that $m = \overleftrightarrow{AB}$. So, [by the replacement rule for equations] $\ell = m$. Hence,

$$(A \neq B \text{ and } \{A, B\} \subseteq \ell \cap m) \Longrightarrow \ell = m.$$

6. [Taking for granted the validity of sentences of the forms '(p and q) $\Longleftrightarrow$ (q and p)' and '[not p $\Longrightarrow$ q] $\Longleftrightarrow$ [not q $\Longrightarrow$ p]' the derivation can be abbreviated to:

$$\begin{array}{c} (A \neq B \text{ and } \{A, B\} \subseteq \ell \cap m) \Longrightarrow \ell = m \\ \hline \{A, B\} \subseteq \ell \cap m \Longrightarrow (A \neq B \Longrightarrow \ell = m) \\ \hline \{A, B\} \subseteq \ell \cap m \Longrightarrow (\ell \neq m \Longrightarrow A = B) \\ \hline (\ell \neq m \text{ and } \{A, B\} \subseteq \ell \cap m) \Longrightarrow A = B \end{array}$$

7. Sentence (1) is a previously proved theorem. By Exercise 5, (1) implies (2). By Exercise 6, (2) implies (3). By Exercise 4, (3) implies the sentence of Exercise 3 which, in turn, implies its contrapositive. The latter is a formal restatement of (✩).

2. Although a paragraph proof of (*) like that suggested in the hint for Exercise 1 is perfectly adequate, it will be a help in constructing similar proofs if we put this paragraph proof more formally. To begin with, translate (*) into "formal language". [*Hint*: $A \neq B \longrightarrow \text{not } \exists_l \exists_m (\ldots)$.]
3. Your answer for Exercise 2 is a conditional sentence of the form 'not $q \longrightarrow$ not p', and this suggests that it might be easier to prove the sentence of which it is the contrapositive. Write this sentence.
4. Your answer for Exercise 3 is of the form '$\exists_l \exists_m Flm \longrightarrow q$'. As you know, you can derive this from a sentence which does not contain quantifiers. Explain how, and write this sentence.
5. From your previous work with the logic of 'not', 'and', and '$\longrightarrow$', you should see that the sentence you wrote in answer to Exercise 4 is a consequence of:

(2) $(A \neq B \text{ and } \{A, B\} \subseteq l \cap m) \longrightarrow l = m$

Derive (2) from (1).

6. From (2), derive:

(3) $(l \neq m \text{ and } \{A, B\} \subseteq l \cap m) \longrightarrow A = B$

7. By retracing your steps in Exercises 4 and 3, complete the proof of (*).
8. Your work in deriving (3) from (2) can be generalized to show that any inference of the form:

$$\frac{(\text{not } p \text{ and } q) \longrightarrow r}{(\text{not } r \text{ and } q) \longrightarrow p}$$

is valid. Give a scheme which shows this.

Part B

Your work in Exercise 8 of Part A shows that the sentences (2) and (3) say the same thing. Your work in Exercise 7 showed that, because (3) is a theorem, so is:

(4) $A \neq B \longrightarrow \text{not } \exists_l \exists_m (l \neq m \text{ and } \{A, B\} \subseteq l \cap m)$

This is a formal statement of what (*) says, and can also be translated as:

(A) Two points are contained in at most one line.

Here is another theorem which appears to say something quite different:

(B) Two lines have at most one point in common.

8. There are many possible schemes. We give one which goes back to first principles:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{*}{\text{not}}\ p \qquad \dfrac{\text{not } \overset{\dagger}{r} \text{ and } q}{q}}{\text{not } p \text{ and } q} \qquad (\text{not } p \text{ and } q) \Longrightarrow r}{r}}{\text{not } p \Longrightarrow r}{}^{*} \qquad \dfrac{\text{not } \overset{\dagger}{r} \text{ and } q}{\text{not } r}}{\text{not not } p}}{p}}{(\text{not } r \text{ and } q) \Longrightarrow p}{}^{\dagger}$$

A shorter, but more sophisticated, scheme makes use of exportation, an inference scheme given on page 188, and importation:

$$\dfrac{\dfrac{\dfrac{(\text{not } p \text{ and } q) \Longrightarrow r}{\text{not } p \Longrightarrow [q \Longrightarrow r]}}{\text{not } r \Longrightarrow [q \Longrightarrow p]}}{(\text{not } r \text{ and } q) \Longrightarrow p}$$

Here is a third scheme to the same purpose:

$$\dfrac{\dfrac{\text{not } (\text{not } p \text{ and } q) \Longleftrightarrow [q \Longrightarrow p] \qquad \dfrac{(\text{not } p \text{ and } q) \Longrightarrow r}{\text{not } r \Longrightarrow \text{not } (\text{not } p \text{ and } q)}}{\text{not } r \Longrightarrow [q \Longrightarrow p]}}{(\text{not } r \text{ and } q) \Longrightarrow p}$$

1. **(a)** Mark two points, A and B, so that you can draw a line l which contains both points. Draw *another* line, m, through A. The preceding instructions leave you considerable freedom in choosing A, B, and m. Can you follow these instructions *and* choose A, B, and m so that $B \in m$?
   (b) Draw two lines, l and m, so that you can mark a point A contained in both. Mark *another* point, B, on l. The preceding instructions leave you considerable freedom in choosing l, m, and A. Can you follow these instructions *and* choose l, m, and A so that $B \in m$?
2. Exercise 1 may have suggested to you that the claims made in sentences (A) and (B) are not as different as they appear to be. This is the case. Just as (A) is a translation of (4) which you derived from (3), (B) is a translation of a sentence:

   (5) $\qquad l \neq m \longrightarrow \text{not } \exists_X \ldots$

   which you can derive from (2). Complete (5) and derive it from (2).
3. Tidy things up by showing that (2) is a consequence of (5) and that (3) is a consequence of (4). What is your final conclusion about (2), (3), (4), (5), (A), and (B)?
4. Notice that (A) does *not* say that any two points are contained in a line; and (B) does *not* say that any two lines have a common point.
   (a) Are there two points which are not contained in any line?
   (b) Are there two lines which do not have a common point?
   (c) If your answer to either (a) or (b) is 'Yes.', draw a picture to justify your answer.

*Part C

1. In Exercise 3 of Part B you probably concluded that the six sentences (2), (3), (4), (5), (A), and (B) are different ways of saying the same thing. Write a seventh sentence (**) which is related to (B) as (*) is related to (A).
2. In Exercise 5 of Part A you showed that the sentence

   (2) $\qquad (A \neq B \text{ and } \{A, B\} \subseteq l \cap m) \longrightarrow l = m$

   is a theorem by deriving it from the previously proved theorem (1). There is another way of proving (2) which shows rather clearly what part (b) of Definition 7–2 [of 'line'] "really means". To begin with, note that it would be easy to prove (2) if we could prove:

   (2′) $(A \neq B \text{ and } \{A, B\} \subseteq l_1 \cap l_2) \longrightarrow l_1 \subseteq l_2$ [Explain.]

   Derive (2′) from Definition 7–2. [*Hint*: Suppose that l_1 and l_2 are lines such that $A \neq B$ and $\{A, B\} \subseteq l_1 \cap l_2$. It follows that $\{A, B\} \subseteq l_1$ and that $\{A, B\}$ __________. Suppose, now that

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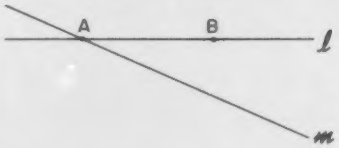
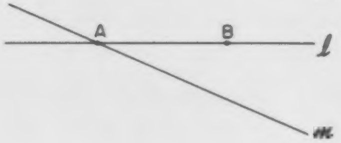
Answers for Part B

1. (a)



; No.

(b)



; No.

2. (5) $l \neq m \Longrightarrow \text{not } \exists_X \exists_Y (X \neq Y \text{ and } \{X, Y\} \subseteq l \cap m)$

   As in Exercise 4 of Part A, (2) implies:

   (*) $\exists_X \exists_Y (X \neq Y \text{ and } \{X, Y\} \subseteq l \cap m) \Longrightarrow l = m$

   and this implies its contrapositive, (5). So, (2) implies (5). [Note that this is the same form of argument by which it was shown that (3) implies (4).]
3. (5) implies (*), above, and (*) implies (2) [by the rule given on page 250]; precisely similarly, (4) implies a sentence similar to (*), and this implies (3). So, each of (2), (3), (4), and (5) implies any of them, and since (A) and (B) are conventional ways of saying what (4) and (5) say, respectively, all six sentences are just different ways of saying the same thing.
4. (a) No. [Theorem 7-1]

   (b) Yes.; but we are not yet in a position to prove this.

   (c) [Students may draw two parallel lines or two skew lines as intuitive justification for the answer for part (b).]

Answers for Part C

1. (☆☆) There are not two points which are contained in two given lines.
2. [(2) follows from (2′): Suppose that $A \neq B$ and $\{A, B\} \subseteq l \cap m$. It follows from (2′) that $l \subseteq m$. Also, since $l \cap m = m \cap l$, it follows that $m \subseteq l$. Since $l \subseteq m$ and $m \subseteq l$, $l = m$. Hence, if $A \neq B$ and $\{A, B\} \subseteq l \cap m$ then $l = m$.]

   [Proof of (2′) is completed by fill-ins: $\subseteq l_2$; A, B, C} is collinear; A, B, C} is collinear; Definition 7-2 that $C \in l_2$; Similarly, if $C \in l_2$ then $C \in l_1$. So, $l_1 = l_2$. Hence, if $A \neq B$ and $\{A, B\} \subseteq l_1 \cap l_2$ then $l_1 = l_2$.]

$C \in l_1$. Since $\{A, B\} \subseteq l_1$ and $A \neq B$ it follows from Definition 7-2 that $\{$__________. Since $\{A, B\} \subseteq l_2$, $A \neq B$, and $\{$__________ it follows from . . . . Hence, if $C \in l_1$ then $C \in l_2$ . . . .]

3. (a) Reread Exercise 4 of Part A on page 280. We have now proved that any two points are contained in a line. So, as suggested in Exercise 4(a), we can prove:

$$\{A, B, C\} \text{ is collinear} \Longrightarrow \exists_l \{A, B, C\} \subseteq l$$

under the assumption that $\{A, B, C\}$ contains at least two points. To get rid of this assumption we need to take care of the case $A = B = C$. That is, we need to prove that any point A is contained in some line. The obvious argument for this is:

> Let B be any point such that $B \neq A$. Then, by Theorem 7-1, $\overleftrightarrow{AB}$ is a line and $A \in \overleftrightarrow{AB}$.

This is a rather short argument and you should be able to figure out what its weak point is. Do this.

(b) Later in the course we shall adopt a postulate which will furnish a foundation for the weak step in the preceding argument. Suggest a postulate about translations which would do the job needed here. [*Hint*: Do our postulates guarantee that there are any translations? If so, how many?]

Part D

Prove:

> Theorem 7-2 $(\{C, D\} \subseteq \overleftrightarrow{AB}$ and $C \neq D)$
> $$\Longrightarrow \overleftrightarrow{AB} = \overleftrightarrow{CD}$$

[*Hint:* If $\overleftrightarrow{AB}$ contains two points, can it be the case that $A = B$?]

Part E

Suppose that A, B, C, and D are points of a line l and that $A \neq B$ and $C \neq D$.

1. What kind of a sequence is $(B - A, C - A)$? What kind of a translation is $B - A$? What can you say, in consequence, about $C - A$? [*Hint*: You've done all this before in Exercise 2(c) on page 277.]
2. Prove that $D - A \in [B - A]$.
3. Prove that $D - C \in [B - A]$. [*Hint*: Can you express '$D - C$' in terms of '$D - A$' and '$C - A$'?]
4. Prove that $[B - A] = [D - C]$. [*Hint*: What more do you need after Exercise 3? When you have figured this out, look again at the hypothesis preceding these exercises. If, in this, you interchanged 'A' and 'C' and interchanged 'B' and 'D', would you be saying anything different?]

Answers for Part C [cont.]

3. (a) The weak point is the assumption. To "discharge" it we need the premiss '$\exists_Y\ Y \neq A$' and this we have no way of proving.

(b) $\exists_{\vec{x}}\ \vec{x} \neq \vec{0}$; Given this and the easily proved theorem '$A + \vec{a} = A \Longrightarrow \vec{a} = \vec{0}$' we can derive, in succession '$\vec{a} \neq \vec{0} \Longrightarrow A + \vec{a} \neq A$', '$\vec{a} \neq \vec{0} \Longrightarrow \exists_Y\ Y \neq A$', '$\exists_{\vec{x}}\ \vec{x} \neq \vec{0} \Longrightarrow \exists_Y\ Y \neq A$', and '$\exists_Y\ Y \neq A$'. [The suggested postulate can be reformulated as 'There exists a linearly independent 1-termed sequence.'. This is the first of a sequence of sentences of the form:

> There exists a linearly independent n-termed sequence.

Any such sentence is, by definition, equivalent to the corresponding sentence of the form:

> The dimension of $\mathcal{T}$ is at least n.

It should perhaps be noted that, given that '$\exists_Y\ Y \neq A$' is a theorem, it follows immediately that '$\exists_X \exists_Y\ Y \neq X$' is a theorem — in short, it follows that $\mathcal{E}$ is nondegenerate. Similarly, we can prove that $\mathcal{E} \neq \emptyset$ without any addition to our postulates. For, '$\exists_X\ X \in \mathcal{E}$' is a consequence of the postulate '$A + \vec{a} \in \mathcal{E}$'. The moral is that in adopting our rules for quantification we have implicitly assumed that the domains of our variables are nonempty. This may seem sneaky; but it allows for simpler rules than would otherwise be required, and foregoing the privilege of talking about nothing is no great price to pay for simplicity.]

Answer for Part D

Suppose that $\{C, D\} \subseteq \overleftrightarrow{AB}$ and $C \neq D$. Since $\overleftrightarrow{AB}$ contains two points it follows that $\overleftrightarrow{AB} \neq \{A\}$ and, so [by Exercise 2(b) on page 281], $A \neq B$. Hence, by Theorem 7-1, $\overleftrightarrow{AB}$ is a line. Since $C \neq D$, $\overleftrightarrow{CD}$ is *the* line which contains C and D. So, since $\overleftrightarrow{AB}$ is a line and contains C and D, $\overleftrightarrow{AB} = \overleftrightarrow{CD}$.

The exercises of Part E should bear an air of familiarity. Bringing them up again at this point is intended to prepare students for the section which follows.

Answers for Part E

1. A linearly dependent sequence [since $\{A, B, C\}$ is collinear].; A non-$\vec{0}$ translation [since $A \neq B$].; That $C - A \in [B - A]$ by Theorem 6-14
2. Since $\{A, B\} \subseteq \ell$, $A \neq B$, and $D \in \ell$, it follows, by Definition 7-2, that $\{A, B, D\}$ is collinear and so, by Definition 7-1, that $(B - A, D - A)$ is linearly dependent. So, since $B - A \neq \vec{0}$, it follows that $D - A \in [B - A]$. [A lazy man's answer — with which you should be more than satisfied — is: Everything assumed about A, B, and C has also been assumed about A, B, and D. Since, by Exercise 1, $C - A \in [B - A]$, it is also the case that $D - A \in [B - A]$.]

3. Since [B - A] is a vector space, any linear combination of its members belongs to it. So, since C - A $\in$ [B - A], D - A $\in$ [B - A], and D - C = (D - A) - (C - A), it follows that D - C $\in$ [B - A].

4. By Exercise 3 [and the fact that [B - A] is a vector space], [D - C] $\subseteq$ [B - A]. So, to show that [B - A] = [D - C] it is sufficient to show that [B - A] $\subseteq$ [D - C]. Since the assumptions concerning A, B, C, and D which justified the first inclusion also apply to C, D, A, and B [in that order], the second inclusion is likewise justified. [An alternative answer is as follows: We proved somewhere that if $\vec{b} \neq \vec{0}$ and $\vec{b} \in [\vec{a}]$ then $[\vec{a}] = [\vec{b}]$. Since C $\neq$ D, D - C $\neq \vec{0}$ and so, by Exercise 3, [B - A] = [D - C]. (The result referred to was proved in Exercise 4(g), page 216 and in Exercise 4, page 262. In brief, if $\vec{0} \neq \vec{b} = \vec{a}b$ then $b \neq 0$, $\vec{a} = \vec{b} \cdot /b$, and $\vec{a} \in [\vec{b}]$. So, if $\vec{0} \neq \vec{b} \in [\vec{a}]$ then $[\vec{b}] \subseteq [\vec{a}]$ and $[\vec{a}] \subseteq [\vec{b}]$.)]

*

The results of Exercises 1 and 2 might be obtained in a different way: Under the assumptions made on A, B, C, and D it follows [either by Theorems 7-1 and 7-2 or by (2) on page 282] that $\overleftrightarrow{AB} = \overleftrightarrow{CD}$. So, by Definition 7-3,

$$P - A \in [B - A] \Longleftrightarrow P - C \in [D - C].$$

Since C - C and D - C belong to [D - C] it follows that C - A and D - A belong to [B - A].

7.04 Directions of Lines and of Translations

Intuitively, a translation $\vec{a}$ is "*in* the direction of" a given line l just if $\vec{a} = B - A$, where A and B are two points of l. This suggests defining

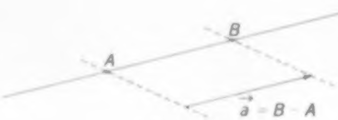


Fig. 7–4

the direction of l to be the set of all such translations. [This is a pun on the word 'in'.] It turns out to be somewhat simpler if we include $\vec{0}$ in this set and, so, define the direction of l to be

$$\{\vec{x}\colon \exists_Y \exists_Z\ (Y \in l \text{ and } Z \in l \text{ and } \vec{x} = Z - Y)\}.$$

Before adopting a formal definition, let's explore this notion further.

Suppose that A and B are two points of l. It follows by Exercise 3 of Part E that if C and D are any two points of l then $D - C \in [B - A]$. Since, also, $\vec{0} \in [B - A]$ it follows that if we adopt the proposed definition for the direction of l then each translation which belongs to the direction of l also belongs to $[B - A]$. On the other hand, suppose that $\vec{a} \in [B - A]$. Since $A \neq B$ it follows from Theorem 7–1 that $l = \overleftrightarrow{AB}$. So, by Definition 7–3, since $\vec{a} \in [B - A]$, $A + \vec{a} \in l$. Since $\vec{a} = (A + \vec{a}) - A$ it follows—using the proposed definition—that $\vec{a}$ belongs to the direction of l. Hence, each translation in $[B - A]$ belongs to the direction of l. Combining our two results we see that, under the proposed definition,

the direction of l is $[B - A]$, for any two points A and B of l.

This results suggests using the notation '$[l]$' as an abbreviation for 'the direction of l'. Doing so, our definition takes the form:

> Definition 7–4
> $[l] = \{\vec{x}\colon \exists_Y \exists_Z\ (Y \in l \text{ and } Z \in l \text{ and } \vec{x} = Z - Y)\}$

where we agree to read '$[l]$' as 'the direction of l'. In words, Definition 7–4 says that the direction of a line l is the set of all translations determined by points of l. Our main result is:

> Theorem 7–3 $(\{A, B\} \subseteq l \text{ and } A \neq B) \Longrightarrow [l] = [B - A]$

In deciding what to mean by 'the direction of ℓ' we need to choose something which is determined when ℓ is specified, and is such that the direction of ℓ is the direction of m if and only if ℓ and m are parallel. In the context of current mathematical thought and, in particular, of this course, it is also natural to choose this thing to be a set. If we had, at this point, a definition of 'parallel', it would be appropriate to take the direction of a line to be the set of all lines parallel to it [including the line itself]. Lacking such a definition we take, as the direction of ℓ, the set of all translations which map ℓ into itself. More simply described — and motivated by a pun — this is

(☆) $\{\vec{x}\colon \exists_Y \exists_Z\ (Y \in \ell \text{ and } Z \in \ell \text{ and } \vec{x} = Z - Y)\}$.

Having defined directions for lines we are able to define 'parallel'. Parallel lines are lines which have the same direction. That two lines are parallel if and only if they are coplanar and disjoint is a theorem which will be proved in a later chapter. The existence of a unique line through a given point and parallel to a given line is established in section 7.05.

Students should be prepared to understand (☆) by their experience with existential quantifiers and with Definition 7-3. However, you may encounter some who feel that the set consisting of all the point-differences in question should be described by using universal quantifiers. If so, point out that a translation $\vec{a}$ belongs to the set described by (☆) if and only if there are points of ℓ whose difference is $\vec{a}$ [which is what we wished to be the case]; and that there is no translation which is the difference of each pair of points of ℓ [so that (☆) with '$\exists$'s replaced by '$\forall$'s describes the empty set].

The exploration begun on page 288 motivates the introduction of the abbreviation '$[\ell]$' for (☆) by establishing Theorem 7-3.

Recapitulation of the proof of Theorem 7-3:

Suppose that $\{A, B\} \subseteq \ell$ and $A \neq B$.

Suppose that $\vec{a} \in [\ell]$. By Definition 7-4 there are points of ℓ — say, C and D, such that $\vec{a} = D - C$. From Exercise 3 of Part E it follows that if $C \neq D$ then $D - C \in [B - A]$. Also, if $C = D$ then $D - C = \vec{0} \in [B - A]$. Hence, if $\vec{a} \in [\ell]$ then $\vec{a} \in [B - A]$.

Suppose, on the other hand, that $\vec{a} \in [B - A]$. Since $A \neq B$ and $\{A, B\} \subseteq \ell$ it follows from Theorem 7-1 that $\ell = \overleftrightarrow{AB}$. Since $\vec{a} \in [B - A]$ it follows from Definition 7-3 that $A + \vec{a} \in \ell$. Since, by Postulate 2(b), $\vec{a} = (A + \vec{a}) - A$ it follows that there exist points Y and Z of ℓ such that $\vec{a} = Z - Y$. So, by Definition 7-4, $\vec{a} \in [\ell]$. Hence, if $\vec{a} \in [B - A]$ then $\vec{a} \in [\ell]$.

Since, as has been shown, $\vec{a} \in [B - A]$ if and only if $\vec{a} \in [\ell]$ it follows that $[\ell] = [B - A]$. Hence, if $\{A, B\} \subseteq \ell$ and $A \neq B$ then $[\ell] = [B - A]$.

Using this and reusing the second part of the preceding argument, we have:

‖ Theorem 7-4 $(A \in l \text{ and } \vec{a} \in [l]) \Longrightarrow A + \vec{a} \in l$

[Explain.]

When $\vec{a} \neq \vec{0}$ it is natural to say that the direction of $\vec{a}$ is the direction of the line through some given point A and the point $A + \vec{a}$. By Theorem 7-3, the direction of this line is $[(A + \vec{a}) - A]$; by Postulate 2(b), this is $[\vec{a}]$. So, for $\vec{a} \neq \vec{0}$ it is natural to say that the direction of $\vec{a}$ is $[\vec{a}]$. This being the case, we shall treat '$[\vec{a}]$' as though it were an abbreviation for 'the direction of $\vec{a}$'. To avoid having to make restrictions, we shall read '$[\vec{a}]$' as 'the direction of $\vec{a}$' even in case $\vec{a} = \vec{0}$. [What, then, is the direction of $\vec{0}$?] Sometimes when we wish to imply that $\vec{a} \neq \vec{0}$ we shall refer to $[\vec{a}]$ as a *proper* direction. So,

if $\vec{b} \in [\vec{a}]$, we shall say that $\vec{b}$ is in the direction of $\vec{a}$;
if $[\vec{a}] = [\vec{b}]$, we shall say that $\vec{a}$ and $\vec{b}$ are in the same direction, or that they have the same direction;
if $[l] = [\vec{a}]$, we shall say that the direction of l is that of $\vec{a}$, that l has the direction of $\vec{a}$, or [somewhat improperly] that l is in the direction of $\vec{a}$.

[Of course, the last case can occur only if $[\vec{a}]$ is a proper direction.] For example, Theorem 7-3 can be paraphrased as:

If A and B are two points of l then l has the direction of $B - A$.

Exercises

1. (a) Prove: $A \neq B \longrightarrow [\overleftrightarrow{AB}] = [B - A]$
   (b) Restate the theorem of part (a) using words ['direction', etc.] instead of brackets.
2. Restate each of the following theorems in terms of 'direction':
   (a) $\vec{a} \in [\vec{b}] \longrightarrow (\vec{a}, \vec{b})$ is linearly dependent
   (b) $((\vec{a}, \vec{b})$ is linearly dependent and $\vec{a} \neq \vec{0}) \longrightarrow \vec{b} \in [\vec{a}]$
   (c) $(\vec{a} \in [\vec{b}]$ and $\vec{b} \in [\vec{a}]) \longrightarrow [\vec{a}] = [\vec{b}]$
   (d) $((\vec{a}, \vec{b})$ is linearly dependent and $\vec{a} \neq \vec{0} \neq \vec{b}) \longrightarrow [\vec{a}] = [\vec{b}]$
3. Prove part (d) of Exercise 2. [*Hint*: (a), (b), and (c) have already been proved in Chapter 6.]
4. (a) Restate the result of Exercise 4 of Part E on page 287 in terms of 'direction'.
   (b) If you omit '$A \neq B$ and $C \neq D$' from your answer for part (a) is the resulting sentence a theorem? [Prove or give a counterexample.]
   (c) If you replace 'A, B, C, and D are points of a line l' by '$\{A, B, C\}$

Proof of Theorem 7-4:

Suppose that $A \in \ell$ and $\vec{a} \in [\ell]$. Since ℓ is a line it follows that ℓ contains at least two points and, so, contains a point — say, B — other than A. By Theorem 7-3, it follows that $[\ell] = [B - A]$ and, so, that $\vec{a} \in [B - A]$. By Theorem 7-1, $\ell = \overleftrightarrow{AB}$. So, by Definition 7-3, since $\vec{a} \in [B - A]$, $A + \vec{a} \in \ell$. Hence, if $A \in \ell$ and $\vec{a} \in [\ell]$ then $A + \vec{a} \in \ell$.

Another proof for Theorem 7-4:

Suppose that $A \in \ell$ and $\vec{a} \in [\ell]$. It follows by Definition 7-4 that there are points of ℓ — say, B and C — such that $\vec{a} = C - B$. What needs to be shown, then, is that if $\{A, B, C\} \subseteq \ell$ then $A + (C - B) \in \ell$.

Suppose, then, that $\{A, B, C\} \subseteq \ell$. If $B = C$ then $A + (C - B) = A + \vec{0} = A \in \ell$. If $B \neq C$ then $\ell = \overleftrightarrow{BC}$ and, since $A \in \ell$, $A - B \in [C - B]$. We wish to show that $A + (C - B) \in \ell$ — that is, that $(A + (C - B)) - B \in [C - B]$. But, $(A + (C - B)) - B = (A - B) + (C - B) \in [C - B]$ since $A - B \in [C - B]$.

Theorem 7-4 may be interpreted as saying that any translation which belongs to the direction of ℓ maps ℓ into itself. The converse is obvious since, if $\vec{a}$ maps ℓ into itself then, for any $A \in \ell$, $A + \vec{a} \in \ell$ and $\vec{a} = (A + \vec{a}) - A \in [\ell]$ by Definition 7-4. Thus, as mentioned earlier in this commentary, the direction of ℓ consists of those translations which map ℓ into itself.

Theorem 7-4 and Postulate 2(b) imply:

(☆☆) $(A \in \ell \text{ and } \vec{a} \in [\ell]) \Longrightarrow \exists_Z\, (Z \in \ell \text{ and } \vec{a} = Z - A)$

As to the question concerning the direction of $\vec{0}$, this is, by definition, $[\vec{0}]$, which is $\{\vec{0}\}$.

Somewhat later in the chapter we shall note, in passing, that it is convenient to abbreviate a sentence of the form:

$$\exists_x\, (x \in S \text{ and } Fx)$$

to a corresponding sentence of the form:

$$\exists_{x \in S}\, Fx$$

[Here, as in Chapter 6, although 'x' is a real number index, our remarks should be understood as applying to point indices — like 'X' — and translation indices — like '$\vec{x}$' — as well.] Informally, this is justified by remarking that 'there exists an x such that x belongs to S and ...' should mean the same as 'there exists an x in S such that ...'. This informal justification motivates adopting as a rule of logic the first part of:

> Any sentence of either of the forms:
>
> $$\exists_{x \in S}\, Fx \Longleftrightarrow \exists_x\, [x \in S \text{ and } Fx]$$
> $$\forall_{x \in S}\, Fx \Longleftrightarrow \forall_x\, [x \in S \Longrightarrow Fx]$$
>
> is valid.

If one adopts this rule as a basic rule of logic then it is not difficult to show that inferences rather similar to those which are validated by the introduction and elimination rules for '$\exists$' and '$\forall$' are valid. Specifically, inferences of any of the following forms are valid. [They should be compared with the basic rules as given on page 271, and are subject to the conditions given there].

$$\frac{t \in S \qquad \forall_{x \in S}\, Fx}{Ft}$$

$$\frac{a \in S \Longrightarrow Fa}{\forall_{x \in S}\, Fx}$$

$$\frac{\exists_{x \in S}\, Fx \qquad a \in S \Longrightarrow [Fa \Longrightarrow q]}{q}$$

$$\frac{t \in S \qquad Ft}{\exists_{x \in S}\, Fx}$$

Conversely, if one adopts as basic rules according to which inferences of any of these four kinds are valid then it is not difficult to justify the rule just given concerning the validity of two kinds of biconditionals.

By using such "restricted quantifiers", Definition 7-4 can be simplified to:

$$[\ell] = \{\vec{x}\colon \exists_{Y \in \ell} \exists_{Z \in \ell}\ \vec{x} = Z - Y\}$$

Also, (☆☆), above, can be transformed, successively, into:

$$\vec{a} \in [\ell] \Longrightarrow [A \in \ell \Longrightarrow \exists_{Z \in \ell}\ \vec{a} = Z - A]$$
$$\vec{a} \in [\ell] \Longrightarrow \forall_{Y \in \ell} \exists_{Z \in \ell}\ \vec{a} = Z - Y$$

Although we shall say very little concerning restricted quantifiers in the text, you may wish to anticipate the little we shall say by suggesting the abbreviations just mentioned for Definition 7-4 and (☆☆).

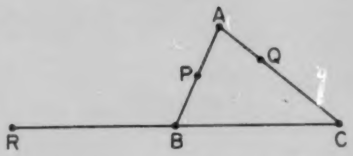
*

Sample Quiz

On your paper, draw a picture of three points — say, A, B, and C — such that A, B, and C are noncollinear.

1. In your picture, locate points P, Q, and R such that $P = A + (B - A)\frac{1}{2}$, $Q = A + (C - A)\frac{1}{3}$ and $R = B + (C - B)\cdot -1$.
2. Is P on $\overleftrightarrow{AB}$? Explain your answer.
3. Is Q on $\overleftrightarrow{AB}$? Explain your answer.
4. Express each of the translations $Q - P$ and $R - P$ as a linear combination of translations $B - A$, $C - B$, and $A - C$.
5. Tell what you must find out about the translations $Q - P$ and $R - P$ in order to determine that the points P, Q, and R are collinear.
6. Determine whether or not P, Q, and R are collinear.

Key to Sample Quiz



1. [Students should have a picture something like the one on the right.]
2. Yes. There is a number x such that $P = A + (B - A)x$.
3. No. There is no number x such that $Q = A + (B - A)x$. [To prove this, assume that there is a number — say, q — such that $Q = A + (B - A)q$. Then $(B - A)q + (C - A)\cdot -\frac{1}{3} = \vec{0}$. This last result implies that A, B, and C are collinear, a contradiction.]
4. $Q - P = (B - A)\cdot -\frac{1}{2} + (C - B)0 + (A - C)\cdot -\frac{1}{3}$

   [or: $(B - A)\cdot -\frac{1}{2} + (A - C)\cdot -\frac{1}{3}$]

   $R - P = (B - A)\frac{1}{2} + (C - B)\cdot -1 + (A - C)0$

   [or: $(B - A)\frac{1}{2} + (C - B)\cdot -1$]
5. That $Q - P$ and $R - P$ are linearly dependent.
6. From 4, $(Q - P)a + (R - P)b = (B - A)(b/2 - a/2) + (C - B)\cdot -b + (A - C)\cdot -a/3$. So, $(Q - P)a + (R - P)b = \vec{0}$ if and only if $(B - A)(b/2 - a/2) + (C - B)\cdot -b + (A - C)\cdot -a/3 = \vec{0}$. Since the latter is the case if and only if $b/2 - a/2 = -b = -a/3$, and that this equation is true for any pair of numbers a and b such that $a = 3b$, $Q - P$ and $R - P$ are linearly dependent and the points are collinear.

Answers for Exercises

1. (a) Suppose that $A \neq B$. Then, $\overleftrightarrow{AB}$ is a line, and $\overleftrightarrow{AB}$ contains the two points A and B. So, by Theorem 7-3, $[\overleftrightarrow{AB}] = [B - A]$. Hence, if $A \neq B$ then $[\overleftrightarrow{AB}] = [B - A]$.

   (b) If $A \neq B$ then the direction of $\overleftrightarrow{AB}$ is the direction of $B - A$.
2. (a) If $\vec{a}$ is in the direction of $\vec{b}$ then $(\vec{a}, \vec{b})$ is linearly dependent.

   (b) If $(\vec{a}, \vec{b})$ is linearly dependent and $\vec{a}$ is a proper translation then $\vec{b}$ is in the direction of $\vec{a}$.

   (c) If $\vec{a}$ is in the direction of $\vec{b}$ and $\vec{b}$ is in the direction of $\vec{a}$, then the direction of $\vec{a}$ is the direction of $\vec{b}$ [or: then $\vec{a}$ and $\vec{b}$ have the same direction].

   (d) If $(\vec{a}, \vec{b})$ is linearly dependent and both $\vec{a}$ and $\vec{b}$ are proper translations, then the direction of $\vec{a}$ is the direction of $\vec{b}$ [or: then $\vec{a}$ and $\vec{b}$ have the same direction].
3. Suppose that $(\vec{a}, \vec{b})$ is linearly dependent and that $\vec{a} \neq \vec{0} \neq \vec{b}$. By 2(b), it follows that $\vec{b} \in [\vec{a}]$ and that $\vec{a} \in [\vec{b}]$. So, $[\vec{b}] \subseteq [\vec{a}]$ and $[\vec{a}] \subseteq [\vec{b}]$. So, $[\vec{a}] = [\vec{b}]$. Hence, if $(\vec{a}, \vec{b})$ is linearly dependent and $\vec{a} \neq \vec{0} \neq \vec{b}$ then $[\vec{a}] = [\vec{b}]$.
4. (a) If A, B, C, D are points of a line ℓ and $A \neq B$ and $C \neq D$ then the direction of $B - A$ is the direction of $D - C$.

   (b) No. For if $B \neq A$ and $D = C$ then $[D - C] = \{\vec{0}\} \neq [B - A]$.

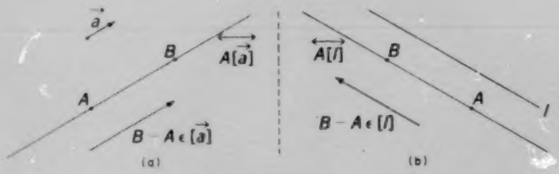
is collinear and $\{B, C, D\}$ is collinear' is the resulting sentence a theorem? [Prove or give a counter-example.]

5. (a) Suppose that $\vec{a} \neq \vec{0}$. Describe $\{X: \exists_l\ (X \in l \text{ and } [l] = [\vec{a}])\}$.
   (b) Is there a line whose direction is $[\vec{0}]$?
   (c) Does $\vec{0}$ belong to the direction of any line?
6. Draw a picture showing a point A and a proper translation $\vec{a}$. Describe, and picture,
   (a) $\{X: \exists_{\vec{x}}\ (\vec{x} \in [\vec{a}] \text{ and } X = A + \vec{x})\}$
   (b) $\{X: X - A \in [\vec{a}]\}$
   (c) $\{X: \exists_x\ X = A + \vec{a}x\}$
7. Prove: $\overleftrightarrow{AB} = \{X: X - A \in [B - A]\}$ [*Hint*: See Exercise 2(d) on page 281.]
8. Draw a line l and mark a point $A \notin l$. Describe, and picture, $\{X: X - A \in [l]\}$.

7.05 Lines in a Given Direction

The preceding Exercise 6 suggests that, for any point A and any proper direction $[\vec{a}]$ there is a unique line through A whose direction is $[\vec{a}]$. Exercise 8 suggests much the same thing. To investigate this likelihood we adopt:

Definition 7–5 (a) $\overleftrightarrow{A[\vec{a}]} = \{X: X - A \in [\vec{a}]\}$
(b) $\overleftrightarrow{A[l]} = \{X: X - A \in [l]\}$



$\vec{a}$
B
$\overleftrightarrow{A[\vec{a}]}$
A
$B - A \in [\vec{a}]$
(a)
$\overleftrightarrow{A[l]}$
B
A
l
$B - A \in [l]$
(b)

Fig. 7–5

What is $\overleftrightarrow{A[\vec{0}]}$? Exercise 6 suggests that, for $\vec{a} \neq \vec{0}$, $\overleftrightarrow{A[\vec{a}]}$ is the [unique] line through A "in the direction of $\vec{a}$". Exercise 8 suggests that $\overleftrightarrow{A[l]}$ is the [unique] line through A which has the same direction as l does. These suggestions sound rather like Theorem 7–1 which says that,

for $A \neq B$, $\overleftrightarrow{AB}$ is the [unique] line through A and B.

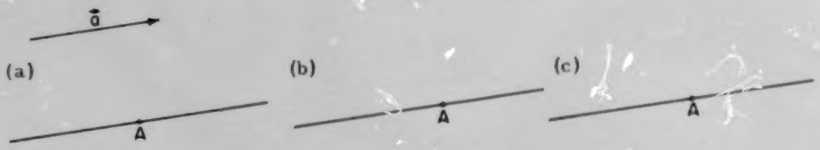
Comparing the definitions of '$\overleftrightarrow{A[\vec{a}]}$' and '$\overleftrightarrow{AB}$' it is easy to see that

$$\overleftrightarrow{A[\vec{a}]} = \overleftrightarrow{A(A + \vec{a})}. \qquad \text{[Explain.]}$$

(c) No. For if A, B, and C are distinct, and D = C, it is still the case that {A, B, C} is collinear and {B, C, D} is collinear and, yet, $[B - A] \neq \{\vec{0}\} = [D - C]$.

5. (a) $\mathcal{E}$, the set of all points. [Given that $\vec{a} \neq \vec{0}$ and A is any point, $A + \vec{a} \neq A$ so that $\overleftrightarrow{A(A + \vec{a})}$ is a line in the direction of $\vec{a}$.]
   (b) No.; all lines have proper directions.
   (c) Yes, $\vec{0}$ belongs to the direction of each line. [This answer assumes that there are lines — a fact which we are not yet in a position to establish formally.]

6.



In each case, the set which is pictured is the line through A in the direction of $\vec{a}$.

7. By the exercise referred to, $C \in \overleftrightarrow{AB}$ if and only if $C - A \in [B - A]$. So [immediately], $\overleftrightarrow{AB} = \{X: X \in \overleftrightarrow{AB}\} = \{X: X - A \in [B - A]\}$. [Without the exercise, proceed as follows:

$$C \in \overleftrightarrow{AB} \iff \exists_x\ C = A + (B - A)x \quad \text{[Def. 7-3]}$$
$$C \in \{X: X - A \in [B - A]\} \iff C - A \in [B - A]$$
$$C - A \in [B - A] \iff \exists_x\ C - A = (B - A)x.$$

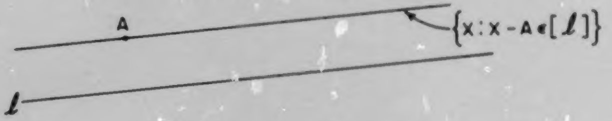
So, what needs to be proved is that

$$\exists_x\ C = A + (B - A)x \iff \exists_x\ C - A = (B - A)x.$$

But this follows at once from:

$$C = A + (B - A)c \iff C - A = (B - A)c \quad \text{[Theorem 2-1]}\ ]$$

8.



The set in question is the line containing A and in the direction of l. Alternately, it is the line through A parallel to l.

$\overleftrightarrow{A[\vec{0}]} = \{A\}$

By the result proved in Exercise 7, above,

$$\overleftrightarrow{A(A + \vec{a})} = \{X: X - A \in [(A + \vec{a}) - A]\}.$$

But, $(A + \vec{a}) - A = \vec{a}$.

Just as $\overleftrightarrow{AB}$ (for $A \neq B$) corresponds to "the line through two given points" of traditional high school geometry, so $\overleftrightarrow{A[l]}$ will correspond to "the line through a point parallel to a line l". This result is Theorem 7-6.

So, we can verify the suggestion of Exercise 6 by showing that, for $\vec{a} \neq \vec{0}$, a line through A has the direction of $\vec{a}$ if and only if it contains $A + \vec{a}$. [Explain.] Since this is an easy consequence of previous theorems [Which theorems?] we have proved part (a) of:

Theorem 7-5

(a) $\overleftrightarrow{A[\vec{a}]}$ is the line through A in the direction of $\vec{a}$ $[\vec{a} \neq \vec{0}]$.
(b) $\overleftrightarrow{A[l]}$ is the line through A in the direction of l.

Part (a) can now be used to prove part (b). Given a point A and a line l, there are points – say, C and D – such that $\{C, D\} \subseteq l$ and $C \neq D$. By Theorem 7-3, $[l] = [D - C]$. From this and Definition 7-5 it follows that

$$\overleftrightarrow{A[l]} = \{X: X - A \in [l]\} = \{X: X - A \in [D - C]\} = \overleftrightarrow{A[D - C]}.$$

Since $C \neq D$, $D - C \neq \vec{0}$ and so, by part (a), $\overleftrightarrow{A[D - C]}$ is the line through A whose direction is $[D - C]$. So, $\overleftrightarrow{A[l]}$ is the line through A whose direction is $[D - C]$ and, since $[D - C] = [l]$, it is the line through A whose direction is $[l]$.

As in the case of Theorem 7-1, each part of Theorem 7-5 is a short way of saying two things. Part (a) says that, for $\vec{a} \neq \vec{0}$,

(1a) $\overleftrightarrow{A[\vec{a}]}$ is a line, $A \in \overleftrightarrow{A[\vec{a}]}$, and $[\overleftrightarrow{A[\vec{a}]}] = [\vec{a}]$;

(2a) $(A \in m \text{ and } [m] = [\vec{a}]) \longrightarrow m = \overleftrightarrow{A[\vec{a}]}$, for any line m.

Part (b) says that

(1b) $\overleftrightarrow{A[l]}$ is a line, $A \in \overleftrightarrow{A[l]}$, and $[\overleftrightarrow{A[l]}] = [l]$;

(2b) $(A \in m \text{ and } [m] = [l]) \longrightarrow m = \overleftrightarrow{A[l]}$, for any line m.

A useful corollary of (2b) is:

$$A \in l \longrightarrow l = \overleftrightarrow{A[l]}$$

Using this it is easy to prove:

(☆) Two lines with the same direction have no common point.

[Compare (☆) with sentence (B) in Part B on page 285. Write a sentence like (5) on page 286 which has (☆) as one of its "English translations". Do you see how to transform the sentence you have written into one which follows easily from the corollary? If you need a hint, the procedure is about the same as that which you used in Part A on page 284.]

Suppose that $\vec{a} \neq \vec{0}$. If, under this assumption, we can show that

(*) $(A \in l \text{ and } [l] = [\vec{a}]) \Longleftrightarrow (A \in l \text{ and } A + \vec{a} \in l)$

then, since $\overleftrightarrow{A(A + \vec{a})}$ is the unique line which contains both A and $A + \vec{a}$, it will follow that $\overleftrightarrow{A(A + \vec{a})}$ is the unique line which contains A and has the direction $[\vec{a}]$. But, as noted previously, $\overleftrightarrow{A(A + \vec{a})} = \overleftrightarrow{A[\vec{a}]}$. So, it is enough to prove (*) [assuming $\vec{a} \neq \vec{0}$] in order to be justified in asserting Theorem 7-5(a).

The if-part of (*) follows by Theorem 7-3; the only if-part follows by Theorem 7-4.

If-part: Suppose that $A \in l$ and $A + \vec{a} \in l$. Since, for $\vec{a} \neq \vec{0}$, $A \neq A + \vec{a}$ it follows from Theorem 7-3 that $[l] = [(A + \vec{a}) - A] = [\vec{a}]$. Hence [for $\vec{a} \neq \vec{0}$], if $A \in l$ and $A + \vec{a} \in l$ then $A \in l$ and $[l] = [\vec{a}]$.

Only if-part: Suppose that $A \in l$ and $[l] = [\vec{a}]$. Since $\vec{a} \in [\vec{a}]$ it follows that $\vec{a} \in [l]$ and so, by Theorem 7-4, that $A + \vec{a} \in l$. Hence, if $A \in l$ and $[l] = [\vec{a}]$ then $A \in l$ and $A + \vec{a} \in l$.

Proof of (☆): Suppose that $[l_1] = [l_2]$. If there is a point — say, A — such that $A \in l_1 \cap l_2$ then $l_1 = \overleftrightarrow{A[l_1]}$ and $l_2 = \overleftrightarrow{A[l_2]}$ and, since $[l_1] = [l_2]$, $l_1 = l_2$. So, if $l_1 \neq l_2$ then there is no point common to l_1 and l_2. Hence,

$$(l_1 \neq l_2 \text{ and } [l_1] = [l_2]) \Longrightarrow \text{not } \exists_X\ X \in l_1 \cap l_2$$

— that is, two lines with the same direction have no common point.

The hint suggests that such a proof can be discovered by translating (☆) into the sentence displayed above, noting that this sentence is equivalent to:

$$\exists_X\ X \in l_1 \cap l_2 \Longrightarrow \text{not } (l_1 \neq l_2 \text{ and } [l_1] = [l_2])$$

and that, by the introduction rule for '∃', the latter is a theorem if:

$$A \in l_1 \cap l_2 \Longrightarrow \text{not } (l_1 \neq l_2 \text{ and } [l_1] = [l_2])$$

is a theorem. This last can, in many ways, be shown equivalent to:

$$(A \in l_1 \cap l_2 \text{ and } [l_1] = [l_2]) \Longrightarrow l_1 = l_2$$

[The simplest is to recall that 'not $(l_1 \neq l_2$ and $[l_1] = [l_2])$' is equivalent to '$[l_1] = [l_2] \Longrightarrow l_1 = l_2$' (see page 271).]

As has been repeatedly emphasized, the ability to give paragraph proofs should be rated much higher than that of giving formal logical analyses of reasoning. The only reason for developing the latter ability is to foster the former. A student who proves (☆) by saying "By Theorem 7-5, there is at most one line through a given point in a given direction. So, two lines with the same direction can't pass through the same point." is doing fine. Another student who doesn't see why the first can say 'So' with such confidence may, if he is acquainted with formal rules of logic, profit from an analysis.

A restatement of (☆) using 'parallel' is:

Two parallel lines have no common point.

Naturally, you will ask whether two lines with no common point need be parallel. As remarked earlier, the proof that two coplanar lines with no common point are parallel — that is, have the same direction — will be given after we define 'plane'.

Finally, let's recall the notion of parallel lines. Intuitively, parallel lines are lines which have the same direction. As pointed out in Chapter 1, it is convenient to agree that any line is parallel to itself. So, we shall adopt:

Definition 7-6 $l \parallel m \Longleftrightarrow [l] = [m]$

[Read '$\parallel$' as 'is parallel to'.] Restate (*) using 'parallel'. Restate Theorem 7-5(b) using 'parallel'.

Since Theorem 7-5(b) tells us that $\overleftrightarrow{A[l]}$ is *the* line through A which is parallel to l it follows from this theorem that

Theorem 7-6 There is one and only one line through a given point and parallel to a given line.

Exercises

Part A

1. Complete the details of the proof of Theorem 7-5(a) by proving:
   (a) $\overleftrightarrow{A[\vec{a}]} = \overleftrightarrow{A(A + \vec{a})}$
   (b) $[l] = [\vec{a}] \Longrightarrow A + \vec{a} \in l \quad [\vec{a} \neq \vec{0}, A \in l]$
2. (a) Prove: $A \in l \Longleftrightarrow l = \overleftrightarrow{A[l]}$
   (b) If, as suggested, you wrote a sentence having (*) as one of its "English translations", your sentence was probably much like this one:

   (1) $(l_1 \neq l_2 \text{ and } [l_1] = [l_2]) \Longleftrightarrow \text{not } \exists_X\, X \in l_1 \cap l_2$

   Show that (1) is a theorem if and only if the sentence:

   (2) $(l_1 \neq l_2 \text{ and } [l_1] = [l_2]) \Longrightarrow A \notin l_1 \cap l_2$

   is a theorem.
   (c) Prove sentence (2).

Part B

1. Draw an arrow to describe a proper translation $\vec{p}$ and mark a point P. Draw the line through P in the direction of $\vec{p}$. Locate points Q and R such that $Q \neq P$, $Q \in \overleftrightarrow{P[\vec{p}]}$ and $R \notin \overleftrightarrow{P[\vec{p}]}$.
2. Write a brace-notation name for $\overleftrightarrow{QR}$. Show that $\overleftrightarrow{QR} \neq \overleftrightarrow{PQ}$. That is, show that $\overleftrightarrow{QR}$ and $\overleftrightarrow{PQ}$ are different lines.
3. Do you think that $[Q - R] = [P - Q]$ or not? Explain your answer.
4. Add to your diagram a picture of $\overleftrightarrow{R[\vec{p}]}$. Are the lines $\overleftrightarrow{P[\vec{p}]}$ and $\overleftrightarrow{R[\vec{p}]}$ parallel or not? Do they intersect or not? Explain your answers.
5. Show that $\overleftrightarrow{QR}$ has the same direction as does the line through $Q + \vec{p}$ and $R + \vec{p}$. What, then, can be said of lines $\overleftrightarrow{QR}$ and $\overleftrightarrow{(Q + \vec{p})(R + \vec{p})}$?

Parts A - F are more than can be reasonably covered in one homework assignment. Here is one means for handling these exercises:

First day

(a) Part A as in-class exercises.

(b) Parts B and C as homework.

Second day

(a) Part D as in-class exercises using a stick model to illustrate the various lines in Exercise 1. Also discuss Definition 7 - 7.

(b) Parts E and F as homework.

Answers for Part A

1. (a) $\overleftrightarrow{A[\vec{a}]} = \{X\colon X - A \in [\vec{a}]\}$
   $= \{X\colon X - A \in [(A + \vec{a}) - A]\} = \overleftrightarrow{A(A + \vec{a})}$

   (b) For $\vec{a} \neq \vec{0}$, $A \neq A + \vec{a}$. So, for $A \in \ell$, it follows by Theorem 7-3 that if $A + \vec{a} \in \ell$ then $[\ell] = [(A + \vec{a}) - A] = [\vec{a}]$. Hence, for $\vec{a} \neq \vec{0}$ and $A \in \ell$, if $A + \vec{a} \in \ell$ then $[\ell] = [\vec{a}]$. If $[\ell] = [\vec{a}]$ then $\vec{a} \in [\ell]$ and, for $A \in \ell$, it follows by Theorem 7-4 that $A + \vec{a} \in \ell$. Hence, for $A \in \ell$, if $[\ell] = [\vec{a}]$ then $A + \vec{a} \in \ell$.

2. (a) Suppose that $A \in \ell$. Since $[\ell] = [\ell]$ it follows by (2b) that $\ell = \overleftrightarrow{A[\ell]}$. Hence, if $A \in \ell$ then $\ell = \overleftrightarrow{A[\ell]}$.

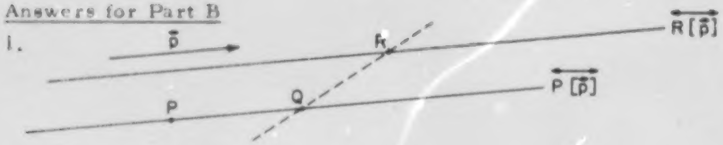
   On the other hand, suppose that $\ell = \overleftrightarrow{A[\ell]}$. Since, by (1b), $A \in \overleftrightarrow{A[\ell]}$ it follows that $A \in \ell$. Hence, if $\ell = \overleftrightarrow{A[\ell]}$ then $A \in \ell$.

   (b) Sentence (1) is of the form '$p \Longrightarrow \text{not } \exists_X\, FX$' and (2) is of the form '$p \Longrightarrow \text{not } FA$'. Since either (1) or (2) is a theorem if and only if its contrapositive is a theorem, we may consider the corresponding sentences of the forms '$\exists_X\, FX \Longrightarrow \text{not } p$' and '$FA \Longrightarrow \text{not } p$'. The second is a consequence of the first. So, if (1) is a theorem then (2) is a theorem. The first is a consequence of the second in case the latter is treated as an assertion. So, if (2) is a theorem then it implies (1), and (1) is also a theorem.

   (c) Suppose that $A \in \ell_1 \cap \ell_2$. It follows from part (a) that $\ell_1 = \overleftrightarrow{A[\ell_1]}$ and $\ell_2 = \overleftrightarrow{A[\ell_2]}$. So, if $[\ell_1] = [\ell_2]$ then $\ell_1 = \ell_2$. Hence, if $A \in \ell_1 \cap \ell_2$ then not both $\ell_1 \neq \ell_2$ and $[\ell_1] = [\ell_2]$. Taking the contrapositive, we obtain (2).

Answers for Part B

1.



2. $\overleftrightarrow{QR} = \{X: \exists_x\ X = Q + (R - Q)x\}$.

To show that $\overleftrightarrow{QR} \neq \overleftrightarrow{PQ}$, it is enough to show that $\overleftrightarrow{QR}$ contains a point not in $\overleftrightarrow{PQ}$. Since $Q \neq P$ and $\{Q, P\} \subseteq \overleftrightarrow{P[\vec{p}]}$, $\overleftrightarrow{P[\vec{p}]} = \overleftrightarrow{PQ}$. Since $R \notin \overleftrightarrow{P[\vec{p}]}$ it follows that $R \notin \overleftrightarrow{PQ}$. But, $R \in \overleftrightarrow{QR}$.

3. No. For if $[Q - R] = [P - Q]$ then $\overleftrightarrow{PQ}$ and $\overleftrightarrow{QR}$ have the same direction and, since both contain Q, $\overleftrightarrow{PQ} = \overleftrightarrow{QR}$. But, $\overleftrightarrow{PQ} \neq \overleftrightarrow{QR}$.

4. Yes, for each has the direction of $\vec{p}$.

No, for if $\overleftrightarrow{P[\vec{p}]}$ and $\overleftrightarrow{R[\vec{p}]}$ have a point in common then, by Theorem 7-6, they would contain exactly the same points (for they would be the same line). But, we know that $R \notin \overleftrightarrow{P[\vec{p}]}$.

5. $[\overleftrightarrow{QR}] = [R - Q] = [(R + \vec{p}) - (Q + \vec{p})] = [\overleftrightarrow{(Q + \vec{p})(R + \vec{p})}]$.

The lines are parallel.

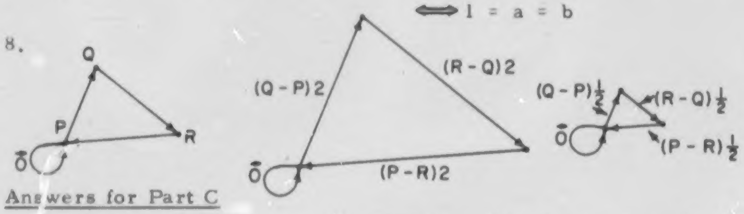
6. No, for $\overleftrightarrow{PQ}$ is the only line containing P and Q and $R \notin \overleftrightarrow{PQ}$.; Linearly independent.; Equals $\vec{0}$.

7. (a) See Theorem 6-12.

   (b) $Q - P = (Q - R)a + (R - P)b \iff (Q - P)1 + (R - Q)a + (P - R)b = \vec{0}$

   $\iff 1 = a = b$

8.



Answers for Part C

1. (a) By Definition 7-6, $\ell \parallel \ell \iff [\ell] = [\ell]$. Since $[\ell] = [\ell]$, it follows that $\ell \parallel \ell$.

   (b) Suppose that $\ell \parallel m$. Then $[\ell] = [m]$. So, $[m] = [\ell]$. By Definition 7-6, $m \parallel \ell$. Hence, if $\ell \parallel m$ then $m \parallel \ell$.

   (c) Suppose that $\ell \parallel m$ and $m \parallel n$. Then $[\ell] = [m]$ and $[m] = [n]$ so that $[\ell] = [n]$. Thus, $\ell \parallel n$. Hence, if ($\ell \parallel m$ and $m \parallel n$) then $\ell \parallel n$.

   (d) Suppose that $\ell \parallel m$ and $n \parallel m$. By (b) $m \parallel n$ so that, by (c), $\ell \parallel n$. Hence, if ($\ell \parallel m$ and $n \parallel m$) then $\ell \parallel n$.

   (e) Suppose that $\ell_1 \parallel \ell_2$. Either $\ell_1 \cap \ell_2 = \emptyset$ or $\ell_1 \cap \ell_2 \neq \emptyset$. Assume the former. It follows that $\ell_1 = \ell_2$ or $\ell_1 \cap \ell_2 = \emptyset$ [inferring an alternation]. Assume, then, that $\ell_1 \cap \ell_2 \neq \emptyset$. In this case, ℓ_1 and ℓ_2 have a common point and, since $\ell_1 \parallel \ell_2$, $\ell_1 = \ell_2$ by Theorem 7-6. It follows that, in this case, $\ell_1 = \ell_2$ or $\ell_1 \cap \ell_2 = \emptyset$. Since, in either case, $\ell_1 = \ell_2$ or $\ell_1 \cap \ell_2 = \emptyset$ it follows that if $\ell_1 \parallel \ell_2$ then $\ell_1 = \ell_2$ or $\ell_1 \cap \ell_2 = \emptyset$.

2. (a) No, for A and B may be the same point, in which case $C = D$.

   (b) Yes, for then $B - A \neq \vec{0}$ and since $B - A = C - D$, $C - D \neq \vec{0}$ and $C \neq D$. It follows that $[\ell] = [B - A] = [C - D] = [m]$. Hence, $\ell \parallel m$.

3. Since $(B - A, D - C)$ is linearly independent it follows that $B - A \neq \vec{0}$, $D - C \neq \vec{0}$, and $[B - A] \neq [D - C]$. [If $[\vec{a}] = [\vec{b}]$ then $\vec{a} \in [\vec{b}]$ and $(\vec{a}, \vec{b})$ is linearly dependent.] Since $B - A \neq \vec{0}$, $A \neq B$. So, since $\{A, B\} \subseteq \ell$, $[\ell] = [B - A]$. Similarly, $[m] = [D - C]$. Since $[B - A] \neq [D - C]$ it follows that $[\ell] \neq [m]$ and, so, that $\ell \nparallel m$.

4. For any two points A and B of ℓ, $[\ell] = [B - A]$. If $(B - A, \vec{a})$ is linearly dependent then, since $B - A \neq \vec{0}$, $\vec{a} \in [B - A] = [\ell]$. Hence, if $\vec{a} \notin [\ell]$ then $(B - A, \vec{a})$ is linearly independent.

[Exercises 3 and 4 have as a consequence that lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel if and only if $(B - A, D - C)$ is linearly dependent.]

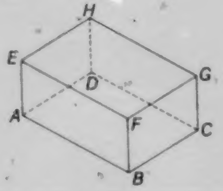
6. Is $\{P, Q, R\}$ collinear or not? How do you know? What can be said of $(Q - P, R - Q)$? Of $(Q - P) + (R - Q) + (P - R)$?
7. Given that $\{P, Q, R\}$ is noncollinear, prove each of the following. [*Hint*: There is a theorem in Chapter 6 which will help.]
   (a) $(Q - P)q + (R - Q)r + (P - R)p = \vec{0}$ if and only if $q = r = p$.
   (b) $Q - P = (Q - R)a + (R - P)b$ if and only if $a = b = 1$.
8. Draw figures to illustrate the theorem in Exercise 7(a) for $q = 2$ and for $q = \frac{1}{2}$.

Part C

1. Prove each of the following [(a), (b), and (c) are parts of *Theorem 7-7*.]
   (a) $l \parallel l$
   (b) $l \parallel m \longrightarrow m \parallel l$
   (c) $(l \parallel m$ and $m \parallel n) \longrightarrow l \parallel n$
   (d) $(l \parallel m$ and $n \parallel m) \longrightarrow l \parallel n$
   (e) $l_1 \parallel l_2 \longrightarrow (l_1 = l_2$ or $l_1 \cap l_2 = \emptyset)$ [*Hint*: Since $l_1 \cap l_2 \subseteq \mathscr{E}$, you can show that $l_1 \cap l_2 = \emptyset$ by showing that there does not exist any *point* in $l_1 \cap l_2$.]
2. Suppose that l and m are lines and $\{A, B\} \subseteq l$ and $\{C, D\} \subseteq m$. Also suppose that $B - A = C - D$.
   (a) From the above conditions does it follow that $l \parallel m$? If not, explain why. If so, give a proof.
   (b) Suppose we also assume that $A \neq B$. With this added condition does it follow that $l \parallel m$? If not, explain why. If so, give a proof.
3. Suppose that l and m are lines and $\{A, B\} \subseteq l$ and $\{C, D\} \subseteq m$ and $(B - A, D - C)$ is linearly independent. Prove that l is not parallel to m.
4. Suppose that l is a line and that $\vec{a} \notin [l]$. Prove that for any two points A and B of l, $(B - A, \vec{a})$ is linearly independent.

Part D

1. Consider the rectangular box $ABCD$-$EFGH$, shown at the right. Answer each of the following by referring to the points A through H which are the corners of the given box.



   (a) Give three lines each of which has the same direction as $\overleftrightarrow{AB}$. As $\overleftrightarrow{GC}$.
   (b) Give at least one line which has a direction different from $\overleftrightarrow{AB}$ and which intersects $\overleftrightarrow{AB}$. In how many points does this line intersect $\overleftrightarrow{AB}$? How do you know?
   (c) Give at least one line which has a direction different from $\overleftrightarrow{AB}$ and which *does not* intersect $\overleftrightarrow{AB}$. Is this line parallel to $\overleftrightarrow{AB}$? How do you know?
2. In Exercise 1 of Part C, you proved:

(*) $$l_1 \parallel l_2 \longrightarrow (l_1 = l_2 \text{ or } l_1 \cap l_2 = \emptyset)$$

Answers for Part D

1. (a) $\overleftrightarrow{CD}$, $\overleftrightarrow{EF}$, $\overleftrightarrow{GH}$; $\overleftrightarrow{AE}$, $\overleftrightarrow{BF}$, $\overleftrightarrow{DH}$
   (b) Any of the lines $\overleftrightarrow{AX}$ where $X \in \{C, D, E, F, G, H\}$ will do.; One.; If it had two points in common with $\overleftrightarrow{AB}$, it would be the same line and, so, would have the direction of $\overleftrightarrow{AB}$.
   (c) $\overleftrightarrow{EH}$ [There are lots of others.]; No, for it has a different direction from that of $\overleftrightarrow{AB}$.

(a) Write the converse of (*) and the contrapositive of (*). Which of these two statements is equivalent to (*)?

(b) Give a counter-example to show that one of the two statements written in (a) is not a theorem.

*

From the results in Exercise 2, above, it should be clear that the converse of (*) is not a theorem. This means that there are lines in space which do not intersect and which are not parallel. Such lines are called *skew lines*.

Part E

In Chapters 2 and 3 you proved various theorems about points and translations. Some of these are collected in the summary on pages 141 and 142. Before reading further, turn to this page and read over Theorems 2–1 through 2–13.

One of the main conclusions we reached in Chapter 2 was, roughly, that sentences about addition and subtraction of points and translations which look as though they should be true are actually theorems. Now, read the correct statement of this conclusion on page 136 and check your understanding of it by writing the real number sentence corresponding to:

(1) $(B + \vec{c}) - (A + \vec{c}) = B - A$

1. Draw a picture illustrating (1). Do you think that (1) is a theorem?
2. Prove (1) using only postulates and theorems on pages 141 and 142. [*Hint*: In Chapter 2 you learned that a promising attack on proving a theorem of the form '$\vec{a} = \vec{b}$' is to use Theorem 2–2, making a helpful choice for 'A'.]
3. Draw an illustrative picture, and prove:

   (2) $A + \vec{a} = B + \vec{b} \longleftrightarrow (A - B) + \vec{a} = \vec{b}$

   [*Hint*: Try Theorem 2–3.]

   (3) $(B - \vec{a}) - A = B - (A + \vec{a})$

4. Prove:
   (a) $\{A, B, C\}$ is collinear $\longleftrightarrow \{A + \vec{a}, B + \vec{a}, C + \vec{a}\}$ is collinear
   (b) $C \in \overleftrightarrow{A[l]} \longleftrightarrow C + \vec{a} \in \overleftrightarrow{(A + \vec{a})[l]}$
5. (a) Draw a line l and mark on it three points, A, B, and C. Draw an arrow to represent a translation $\vec{a}$ such that $\vec{a} \notin [l]$. Locate the images of A, B, and C under the mapping $\vec{a}$. Are the points $A + \vec{a}$, $B + \vec{a}$, and $C + \vec{a}$ collinear?
   (b) Mark a point D such that $D \notin l$. Are the images under $\vec{a}$ of A, B, and D collinear?
   (c) Would your answers for the questions in (a) and (b) be different if $\vec{a}$ were chosen in $[l]$?

---

2. (a) Converse of (*): $(\ell_1 = \ell_2 \text{ or } \ell_1 \cap \ell_2 = \emptyset) \Longrightarrow \ell_1 \parallel \ell_2$

   Contrapositive of (*): not $(\ell_1 = \ell_2 \text{ or } \ell_1 \cap \ell_2 = \emptyset) \Longrightarrow \ell_1 \not\parallel \ell_2$

   The contrapositive of (*) is equivalent to (*).

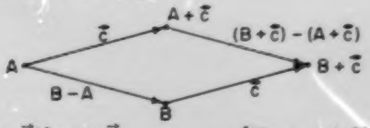
   (b) Let $\ell_1 = \overleftrightarrow{AB}$ and $\ell_2 = \overleftrightarrow{EH}$. Then $\ell_1 \cap \ell_2 = \emptyset$ but $\ell_1 \not\parallel \ell_2$. So, the converse of (*) is not a theorem.

Answers for Part E

The real number sentence corresponding to (1) is:

$$(b + c) - (a + c) = b - a$$

1. Here is a picture illustrating (1):



2. $(A + \vec{c}) + [(B + \vec{c}) - (A + \vec{c})] = B + \vec{c}$ [Post. 2(a)]

   $= [A + (B - A)] + \vec{c}$ [Post. 2(a)]

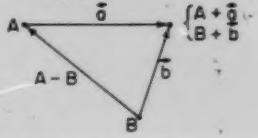
   $= (A + \vec{c}) + (B - A)$ [Th. 2-5(a), Post. 4_4, Th. 2-5(b)]

   So, by Theorem 2-2, $(B + \vec{c}) - (A + \vec{c}) = B - A$.

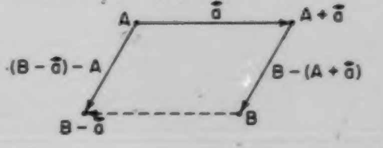
   [Motivation: A translation is determined by its effect on any one point. The "simplest point" to which to apply $(B + \vec{c}) - (A + \vec{c})$ is $A + \vec{c}$. Let's try to show that the image of this point under this translation is the same as its image under $B - A$.

   The choice of $A + \vec{c}$ as the point to work on is reasonable because Postulate 2(a) can be applied at once for a fast start. It is possible, however, to "reduce" '$A + [(B + \vec{c}) - (A + \vec{c})]$' to '$B$' by using Theorems 3-9, 3-5(b), 2-5(b), Postulate 4_3, and Theorem 3-1(a). One can then obtain the desired conclusion from Theorem 2-1.]

3. Picture for (2):



Picture for (3):



Proof of (2): $A + \vec{a} = B + \vec{b} \Longleftrightarrow (A + \vec{a}) - B = (B + \vec{b}) - B$ [Th. 2-3]

$\Longleftrightarrow (A - B) + \vec{a} = (B + \vec{b}) - B$ [Th. 3-8]

$\Longleftrightarrow (A - B) + \vec{a} = \vec{b}$ [Post. 2(b)]

Proof of (3): $A + [(B - \vec{a}) - A] = B - \vec{a}$ [Post. 2(a)]

$= B + -\vec{a}$ [Def. 3-1(a)]

$= B + [A - (A + \vec{a})]$ [Th. 3-5(b)]

$= A + [B - (A + \vec{a})]$ [Th. 3-9]

So, by Theorem 2-2, $(B - \vec{a}) - A = B - (A + \vec{a})$.

[Another proof proceeds by transforming '$(B - \vec{a}) - A$' by using Definition 3-1(a), Theorem 3-8, Theorem 3-10, Definition 3-1(a), and '$- - \vec{a} = \vec{a}$'. A third proof transforms '$(A + \vec{a}) + [(B - \vec{a}) - A]$' into '$(B - \vec{a}) + \vec{a}$' by use of Theorem 3-9 and Postulate 2(b). The latter term reduces to 'B', and Theorem 2-1 can be applied.]

*

Student response to Exercises 2 and 3 may suggest the desirability of a brief review of parts of Chapter 3. What is most important, however, is that students be convinced of the validity of the criterion on page 136 and be able to use it as a source for theorems of this kind.

*

4. (a) $\{A, B, C\}$ is collinear $\Longleftrightarrow$ $(B - A, C - A)$ is linearly dependent

$\Longleftrightarrow$ $((B + \vec{a}) - (A + \vec{a}), (C + \vec{a}) - (A + \vec{a}))$ is linearly dependent

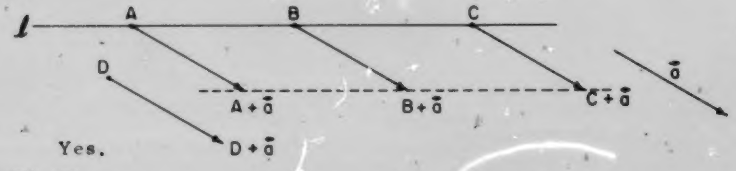
$\Longleftrightarrow$ $\{A + \vec{a}, B + \vec{a}, C + \vec{a}\}$ is collinear

(b) $C \in \overleftrightarrow{A[\ell]} \Longleftrightarrow C - A \in [\ell]$

$\Longleftrightarrow (C + \vec{a}) - (A + \vec{a}) \in [\ell]$

$\Longleftrightarrow C + \vec{a} \in \overleftrightarrow{(A + \vec{a})[\ell]}$

5. (a)



Yes.

(b) No.

(c) No.

Sample Quiz

1. Suppose that ℓ is a line and that $\vec{a}$ and $\vec{b}$ are linearly independent translations. Each of $\ell + \vec{a}$ and $\ell + \vec{b}$ is a line. Are these lines parallel or not? Explain your answer.

2. Suppose that $(\vec{p}, \vec{q})$ is linearly independent. Consider any point — say, A.

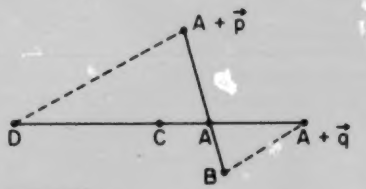
   (a) Draw an appropriate picture for these conditions and locate the points B and C, where $B = A + \vec{p} \cdot -\frac{1}{2}$ and $C = A + \vec{q} \cdot -\frac{1}{2}$.

   (b) What is the direction of the line through B and $A + \vec{q}$?

   (c) The line through $A + \vec{p}$ and parallel to $\overleftrightarrow{B(A + \vec{q})}$ intersects $\overleftrightarrow{AC}$ in a point — say, D. Find the number d such that $D = A + \vec{q}d$.

   (d) Is $\overleftrightarrow{BD}$ parallel to the line through $A + \vec{p}$ and $A + \vec{q}$? Explain your answer.

Key for Sample Quiz



1. $\ell + \vec{a} \parallel \ell + \vec{b}$, because each is parallel to ℓ.

2. (a) [Students should have a picture something like the one at the right.]

   (b) $[\vec{q} + \vec{p}\frac{1}{2}]$

   (c) $d = -2$ [Since $\overleftrightarrow{(A + \vec{p})D} \parallel \overleftrightarrow{(A + \vec{q})B}$, it follows that, for some a, $(\vec{q}d - \vec{p})a = \vec{q} + \vec{p}\frac{1}{2}$. From this, it follows that $a = \frac{1}{2}$ and $d = -2$.]

   (d) No. The direction of $\overleftrightarrow{BD}$ is $[\vec{p}\frac{1}{2} - \vec{q}2]$ and that of $\overleftrightarrow{(A + \vec{p})(A + \vec{q})}$ is $[\vec{p} - \vec{q}]$. Since there is no number a such that $(\vec{p}\frac{1}{2} - \vec{q}2)a = \vec{p} - \vec{q}$, the conclusion follows.

6. In Chapter 1 you discovered a number of important properties of translations by using tracing sheets and parallel rulers. Ten of these properties are described in (1)–(10) on page 47. Up to now we have been able to show that, on the basis of our postulates, (1)–(7) are theorems. Now you can show that (8) is also a theorem.
   (a) Given a line l and a translation $\vec{a}$, find a line m which is parallel to l and contains the image under $\vec{a}$ of each point of l.
   (b) Is each point of the line m of part (a) the image of some point of l? Explain.

7. In Exercise 6 you will have proved somewhat more than is stated in (8) on page 47.

Theorem 7–8 A translation maps any line *onto* a parallel line.

To deal with similar theorems it is convenient to introduce some notation. Recall that, by definition [Chapter 1], the image under a mapping f of a subset $\mathscr{K}$ of Df is the set of all images under f of members of $\mathscr{K}$. It is customary to denote the image of $\mathscr{K}$ under f by '$f(\mathscr{K})$'. So, since we are using '$A + \vec{a}$' instead of '$\vec{a}(A)$', it is natural to use '$\mathscr{K} + \vec{a}$' for the image of a subset $\mathscr{K}$ of $\mathscr{E}$ under the mapping $\vec{a}$. So, we adopt:

Definition 7–7 For $\mathscr{K} \subseteq \mathscr{E}$,
$\mathscr{K} + \vec{a} = \{X: \exists_Y\, (Y \in \mathscr{K} \text{ and } X = Y + \vec{a})\}$.

[Read '$\mathscr{K} + \vec{a}$' as 'the image of $\mathscr{K}$ under $\vec{a}$'.] Alternatively [Explain.]

$$C \in \mathscr{K} + \vec{a} \Longleftrightarrow \exists_Y\, (Y \in \mathscr{K} \text{ and } C = Y + \vec{a}).$$

(a) Use the new notation to reformulate Theorem 7–8. [Note that the theorem makes two claims; the image of l under $\vec{a}$ is a line, and this line is parallel to l.]
(b) Prove: $C \in \mathscr{K} + \vec{a} \Longleftrightarrow C - \vec{a} \in \mathscr{K}$ [$\mathscr{K} \subseteq \mathscr{E}$]. [*Hint*: Suppose that $B \in \mathscr{K}$ and $C = B + \vec{a}$. Does it follow that $C - \vec{a} \in \mathscr{K}$? If $C - \vec{a} \in \mathscr{K}$, does there exist a Y such that $Y \in \mathscr{K}$ and $C = Y + \vec{a}$?]

*

Theorem 7–9 For $\mathscr{K} \subseteq \mathscr{E}$, $\mathscr{K} + \vec{a} = \{X: X - \vec{a} \in \mathscr{K}\}$.

*

8. Prove, for $\mathscr{K} \subseteq \mathscr{E}$:
   (a) $C \in \mathscr{K} \Longleftrightarrow C + \vec{a} \in \mathscr{K} + \vec{a}$
   (b) $(\mathscr{K} + \vec{a}) + \vec{b} = \mathscr{K} + (\vec{a} + \vec{b})$

9. Prove:
   (a) $\overleftrightarrow{A[l]} + \vec{a} = \overleftrightarrow{(A + \vec{a})[l]}$ [*Theorem 7–10*]

Answers for Part E [cont.]

6. (a) Let A be a point of l. Then $l = \overleftrightarrow{A[l]}$ and if $m = \overleftrightarrow{(A + \vec{a})[l]}$ then it follows by Exercise 4(b) that $C \in l \Longleftrightarrow C + \vec{a} \in m$. So, m contains the image under $\vec{a}$ of each point of l.

   (b) Since $(D - \vec{a}) + \vec{a} = D$ it follows $D - \vec{a} \in l \Longleftrightarrow D \in m$. So, each point of m is the image under $\vec{a}$ of some point of l.

   [Since $[m] = [l]$ it follows from (a) and (b) that a translation maps any line onto a parallel line. [By (a), $\vec{a}$ maps l into m; by (b), the mapping is onto.] In other words, the image of a line under a translation is a parallel line. You might ask: What is the image of l under $\vec{a}$ in case $\vec{a} \in [l]$?]

7. (a) $l + \vec{a}$ is a line and $l + \vec{a} \parallel l$

   (b) Suppose that $B \in \mathscr{K}$ and $C = B + \vec{a}$. Since $C = B + \vec{a}$, $B = C - \vec{a}$. So, since $B \in \mathscr{K}$, $C - \vec{a} \in \mathscr{K}$. Hence, if $B \in \mathscr{K}$ and $C = B + \vec{a}$ then $C - \vec{a} \in \mathscr{K}$. Consequently, if there exists a Y such that $Y \in \mathscr{K}$ and $C = Y + \vec{a}$ then $C - \vec{a} \in \mathscr{K}$. In short, if $C \in \mathscr{K} + \vec{a}$ then $C - \vec{a} \in \mathscr{K}$.

   Suppose, on the other hand, that $C - \vec{a} \in \mathscr{K}$. Since $C = (C - \vec{a}) + \vec{a}$ it follows that there exists a Y such that $Y \in \mathscr{K}$ and $C = Y + \vec{a}$. Hence, if $C - \vec{a} \in \mathscr{K}$ it follows that $C \in \mathscr{K} + \vec{a}$.

   [The solution for Exercise 7(b) is, of course, a proof for Theorem 7-9.]

8. (a) By Exercise 7(b), $C + \vec{a} \in \mathscr{K} + \vec{a} \Longleftrightarrow (C + \vec{a}) - \vec{a} \in \mathscr{K}$. But, $(C + \vec{a}) - \vec{a} = C$. Hence, $C + \vec{a} \in \mathscr{K} + \vec{a} \Longleftrightarrow C \in \mathscr{K}$ — that is, $C \in \mathscr{K} \Longleftrightarrow C + \vec{a} \in \mathscr{K} + \vec{a}$.

   (b)
$$\begin{aligned} C \in (\mathscr{K} + \vec{a}) + \vec{b} &\Longleftrightarrow C - \vec{b} \in \mathscr{K} + \vec{a} \\ &\Longleftrightarrow (C - \vec{b}) - \vec{a} \in \mathscr{K}. \end{aligned}$$

   Since $(C - \vec{b}) - \vec{a} = C - (\vec{a} + \vec{b})$ it follows that
$$\begin{aligned} (C - \vec{b}) - \vec{a} \in \mathscr{K} &\Longleftrightarrow C - (\vec{a} + \vec{b}) \in \mathscr{K} \\ &\Longleftrightarrow C \in \mathscr{K} + (\vec{a} + \vec{b}). \end{aligned}$$

   Hence, $C \in (\mathscr{K} + \vec{a}) + \vec{b} \Longleftrightarrow C \in \mathscr{K} + (\vec{a} + \vec{b})$ — in short, $(\mathscr{K} + \vec{a}) + \vec{b} = \mathscr{K} + (\vec{a} + \vec{b})$.

9. (a)
$$\begin{aligned} C \in \overleftrightarrow{A[l]} + \vec{a} &\Longleftrightarrow C - \vec{a} \in \overleftrightarrow{A[l]} && \text{[Exercise 7(b)]} \\ &\Longleftrightarrow (C - \vec{a}) + \vec{a} \in \overleftrightarrow{(A + \vec{a})[l]} && \text{[Exercise 4(b)]} \\ &\Longleftrightarrow C \in \overleftrightarrow{(A + \vec{a})[l]} && \text{['}(C - \vec{a}) + \vec{a} = C\text{']} \end{aligned}$$

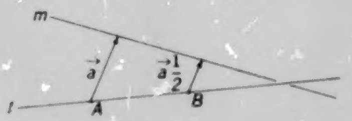
   Hence, $\overleftrightarrow{A[l]} + \vec{a} = \overleftrightarrow{(A + \vec{a})[l]}$. [This is, of course, a re-working of Exercise 6.]

(b) $l \parallel m \longrightarrow \exists_{\vec{x}}\, l + \vec{x} = m$ [*Hint:* How can you describe a lot of translations, each of which maps l onto m?]

(c) $l + \vec{a} = l \longleftrightarrow \vec{a} \in [l]$

Part F

1. Suppose that l is a line and that $\vec{a} \notin [l]$. Let A and B be two points of l.



(a) Show that $A + \vec{a} \neq B + \vec{a}\frac{1}{2}$

(b) Let $m = \overleftrightarrow{(A + \vec{a})(B + \vec{a}\frac{1}{2})}$. Find a point which belongs to $l \cap m$. [*Hint:* Since $l = \overleftrightarrow{AB}$, $C \in l \longleftrightarrow \exists_x\, C = A + (B - A)x$. A similar statement can be made about m. So, finding a point C which belongs to both l and m amounts to finding a pair (c_1, c_2) of numbers which satisfies:

$$A + (B - A)c_1 = (A + \vec{a}) + [\_\_\_\_ - \_\_\_\_]c_2$$

(Complete, and explain.) Transform this equation into one of the form '$(B - A) \cdot \_\_\_\_ + \vec{a} \cdot \_\_\_\_ = \vec{0}$' and, by inspection, find a solution.]

(c) Since l and m are two lines [Why?] and you have found one point in $l \cap m$, you know that $l \cap m$ consists of a single point. It is, however, important to realize that you can deduce this from your final equation in part (b) and a property of $B - A$ and $\vec{a}$. Explain.

2. Repeat Exercise 1 with '$\frac{1}{2}$' replaced by 'a'. [*Warning.* You may need to add a restriction.]

7.06 Some Theorems about Parallel Lines

The techniques you needed to solve the exercises of Part F will be useful throughout this course. In consequence, it will be worthwhile to illustrate their use now in a few more exercises. While we are about it, we shall prove some useful theorems about parallel lines.

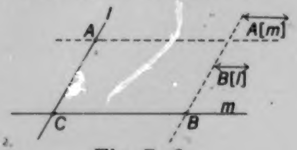


Fig. 7–6

Suppose that l and m are two lines through a point C and that $A \in l$ and $B \in m$. A natural question to ask is whether $\overleftrightarrow{A[m]}$ and $\overleftrightarrow{B[l]}$ have a common point. The interesting case is that in which $A \neq C$ and

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(b) Let $A \in \ell$, $B \in m$. Since $\ell = \overleftrightarrow{A[\ell]}$, $\ell + (B - A) = \overleftrightarrow{A[\ell]} + (B - A) = \overleftrightarrow{(A + (B - A))[\ell]} = \overleftrightarrow{B[\ell]}$. If $\ell \parallel m$ then $[\ell] = [m]$ and $\overleftrightarrow{B[\ell]} = \overleftrightarrow{B[m]} = m$. Hence, if $\ell \parallel m$ then $\ell + (B - A) = m$. Consequently, if $\ell \parallel m$ then $\exists_{\vec{x}}\, \ell + \vec{x} = m$.

(c) Suppose that $\ell + \vec{a} = \ell$. Let $P \in \ell$. It follows that $P + \vec{a} \in \ell + \vec{a}$ and, since $\ell + \vec{a} = \ell$, that $P + \vec{a} \in \ell$. Since $P \in \ell$ and $P + \vec{a} \in \ell$ it follows that $(P + \vec{a}) - P \in [\ell]$. But, $(P + \vec{a}) - P = \vec{a}$. Hence, if $\ell + \vec{a} = \ell$ then $\vec{a} \in [\ell]$.

Suppose that $\vec{a} \in [\ell]$. It follows [by Theorem 7-4] that if $A \in \ell$ then $A + \vec{a} \in \ell$. Suppose, then, that $A \in \ell$. Since $\ell + \vec{a} = \overleftrightarrow{(A + \vec{a})[\ell]}$ it follows that $A + \vec{a}$ belongs to both ℓ and $\ell + \vec{a}$. So, since $\ell + \vec{a} \parallel \ell$, $\ell + \vec{a} = \ell$. Hence, if $\vec{a} \in [\ell]$ then $\ell + \vec{a} = \ell$.

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The exercises of Part F may prove difficult and it may be well to treat Exercise 1 in class before assigning Exercise 2 and some of section 7.06 as homework. The technique introduced in these exercises and developed further in section 7.06 will be of frequent use in Chapter 8 and later.

Answers for Part F

1. (a) Note that $B + \vec{a}\frac{1}{2} = A + \vec{a} \Longleftrightarrow (B - A)2 = \vec{a}$

$$\Longleftrightarrow \vec{a} \in [B - A] = [\ell]$$

Since $\vec{a} \notin [\ell]$, it follows that $A + \vec{a} \neq B + \vec{a}\frac{1}{2}$.

(b) As indicated in the hint, $C \in \ell \cap m$ if and only if $C = A + (B - A)c_1$ [or, equivalently, $C = (A + \vec{a}) + [(B + \vec{a}\frac{1}{2}) - (A + \vec{a})]c_2$] where (c_1, c_2) is a solution of the equation:

$$A + (B - A)c_1 = (A + \vec{a}) + [(B + \vec{a}\tfrac{1}{2}) - (A + \vec{a})]c_2.$$

Using Theorem 2-5(b) and Theorem 2-2 this equation is seen to be equivalent to:

$$(B - A)c_1 = \vec{a} + [(B + \vec{a}\tfrac{1}{2}) - (A + \vec{a})]c_2$$

Using other theorems — or mere common sense — this reduces successively to:

$$(B - A)c_1 = \vec{a} + [(B - A) - \vec{a}\tfrac{1}{2}]c_2$$
$$(B - A)(c_1 - c_2) + \vec{a}(\tfrac{c_2}{2} - 1) = \vec{0}$$

This latter equation is satisfied if $c_2 = c_1$ and $c_2 = 2$. And, since $(B - A, \vec{a})$ is linearly independent it is satisfied only in this case. Hence,

$$C \in \ell \cap m \Longleftrightarrow C = A + (B - A)2.$$

So, the only point asked for is $A + (B - A)2$.

$B \neq C$. [Why?] So, we shall consider this case—that is, the case in which $\{A, B, C\}$ is noncollinear. In this case $[l] = [A - C]$ and $[m] = [B - C]$. So, our problem is that of determining,

> when $\{A, B, C\}$ is noncollinear, do $\overleftrightarrow{A[B - C]}$ and $\overleftrightarrow{B[A - C]}$ have a common point?

If it turns out that $\overleftrightarrow{A[B - C]} \cap \overleftrightarrow{B[A - C]} \neq \emptyset$, the next problem will be to describe the point [or points] which belong to this intersection.

Before reading further try, now, to find a point in $\overleftrightarrow{A[B - C]} \cap \overleftrightarrow{B[A - C]}$ [or to show that there is no such point]. If you fail, continue reading until you reach a place from which you can make another attempt.

Repeating the reasoning which led to the solution of Exercise 1 of Part F, we note that a point P belongs to $\overleftrightarrow{A[B - C]}$ if and only if $\exists_x\, P = A + (B - C)x$, and that it belongs to $\overleftrightarrow{B[A - C]}$ if and only if $\exists_x\, P = B + (A - C)x$. So, there will be a point in $\overleftrightarrow{A[B - C]} \cap \overleftrightarrow{B[A - C]}$ corresponding to any ordered pair (a, b) which satisfies the equation:

$$A + (B - C)a = B + (A - C)b \tag{1}$$

In fact, for any such solution, the point $A + (B - C)a$ [or, equivalently: the point $B + (A - C)b$] will belong to the intersection. Our problem, then, is to see whether (1) has a solution and, if it does, to find all solutions of (1). [You may be able to find a solution of (1) by inspection. If so, that's fine. You've solved the first part of the problem.]

Simplifying equations like (1) is not difficult if you remember that you can deal with expressions for points and translations in the same way you have learned to treat real number expressions *as long as the results of your dealings make sense.* [Recall the discussion preceding the exercises of Part E on page 294.] Following this lead it is easy to see that (1) is equivalent to:

$$[A + (B - C)a] - B = (A - C)b;$$

that this is equivalent to:

$$(A - B) + (B - C)a = (A - C)b;$$

and that this is equivalent to:

$$(A - B) + (B - C)a + (C - A)b = \vec{0} \tag{2}$$

In fact, with very little practice, you will be able to go immediately from (1) to (2). Equation (2) may be simplified still further by using

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2. (a) $A + \vec{a} = B + \vec{a}a \Longleftrightarrow B - A = \vec{a}(1 - a)$. So, if $a \neq 1$ then $A + \vec{a} \neq B + \vec{a}a$ because $\vec{a} \notin [B - A] = [\ell]$. And, if $a = 1$ then $A + \vec{a} \neq B + \vec{a}a$ because $A \neq B$.

   (b) As before, $C \in \ell \cap m$ if and only if $C = A + (B - A)c_1$ where, now,

   $$A + (B - A)c_1 = (A + \vec{a}) + [(B + \vec{a}a) - (A + \vec{a})]c_2$$

   — that is,

   $$(B - A)c_1 = \vec{a} + [(B - A) - \vec{a}(1 - a)]c_2$$
   $$(B - A)(c_1 - c_2) + \vec{a}[(1 - a)c_2 - 1] = \vec{0}.$$

   Since $(B - A, \vec{a})$ is linearly independent, this is the case if and only if

   $$c_1 = c_2 \text{ and } (1 - a)c_2 = 1.$$

   For $a = 1$ there is no solution [as one should have suspected since, in this case, ℓ and m are parallel]. For $a \neq 1$ the only solution is $c_1 = c_2 = 1/(1 - a)$. Hence, in this case, the sole point common to ℓ and m is

   $$A + (B - A)\cdot\frac{1}{1 - a}.$$

   [Note that this checks with the answer for Exercise 1, where $a = 1/2$.]

Answer to question. If either $A = C$ or $B = C$ then the lines in question have C in common.

TC 297 (1)

The proof that

$$(P = A + (B - C)a \text{ and } P = B + (A - C)b)$$
$$\Longleftrightarrow$$
$$(P = A + (B - C)a \text{ and } A + (B - C)a = B + (A - C)b)$$

involves mainly the rules for 'and' and the replacement rule for equations. Using these [as shown below] we can derive each of the displayed conjunctions from the other. The biconditional follows, once the deduction rule has been applied to each of the given derivations. To simplify [and generalize] the derivations we use 'sa' and 'tb' to stand for point-terms such as 'A + (B - C)a' and 'B + (A - C)b' in which the real number variables 'a' and 'b' occur. [This is an extension of the notation used in Chapter 6, where 'Fa', for example, stood for a sentence containing 'a'.]

$$\dfrac{\dfrac{P = sa \text{ and } P = tb}{P = sa} \qquad \dfrac{\dfrac{P = sa \text{ and } P = tb}{P = sa} \qquad \dfrac{P = sa \text{ and } P = tb}{P = tb}}{sa = tb}}{P = sa \text{ and } sa = tb}$$

TC 297 (2)

$$\dfrac{\dfrac{P = sa \text{ and } sa = tb}{P = sa} \qquad \dfrac{\dfrac{P = sa \text{ and } sa = tb}{sa = tb} \quad \dfrac{P = sa \text{ and } sa = tb}{P = sa}}{P = tb}}{P = sa \text{ and } P = tb}$$

So [for any replacement of 'sa' and 'tb' by point-terms], we have the theorem:

$$(P = sa \text{ and } P = tb) \Longleftrightarrow (P = sa \text{ and } sa = tb)$$

From the only if-part of this theorem we can infer:

$$(P = sa \text{ and } P = tb) \Longrightarrow sa = tb$$

from which follows, successively:

$$(P = sa \text{ and } P = tb) \Longrightarrow \exists_x \exists_y\, sx = ty$$

The converse of this last follows from the if-part of the biconditional:

$$\begin{aligned} (P = sa \text{ and } sa = tb) &\Longrightarrow (P = sa \text{ and } P = tb) \\ \left.\begin{aligned} (P = sa \text{ and } sa = tb) &\Longrightarrow \exists_X (\exists_x X = sx \text{ and } \exists_x X = tx) \\ (sa = sa \text{ and } sa = tb) &\Longrightarrow \exists_X (\exists_x X = sx \text{ and } \exists_x X = tx) \end{aligned}\right\} & \text{['}sa = sa\text{'} \\ sa = tb &\Longrightarrow \exists_X (\exists_x X = sx \text{ and } \exists_x X = tx) \quad \text{is a valid} \\ \exists_x \exists_y\, sx = ty &\Longrightarrow \exists_X (\exists_x X = sx \text{ and } \exists_x X = tx) \quad \text{sentence]} \end{aligned}$$

So, for example, equation (1) has a solution if and only if there exists a point common to $\overleftrightarrow{A[B - C]}$ and $\overleftrightarrow{B[A - C]}$.

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Students who recall that $A + (B - C) = B + (A - C)$ will see, ''by inspection'', that (1) is satisfied if $a = 1$ and $b = 1$ and be able to conclude that $A + (B - C)$ belongs to both $\overleftrightarrow{A[m]}$ and $\overleftrightarrow{B[\ell]}$. They can then argue that these lines contain no other common point. For, if they did, it would follow that $\overleftrightarrow{A[m]} = \overleftrightarrow{B[\ell]}$, $B \in \overleftrightarrow{A[m]}$, $\overleftrightarrow{A[m]} = \overleftrightarrow{B[m]} = m$, and, finally, $A \in m = \overleftrightarrow{BC}$ — contradicting the assumption that $\{A, B, C\}$ is noncollinear.

As in the case of students who are learning to solve ordinary algebraic equations, the ability to solve equations like (1) by inspection is a valuable one. For one thing, a student who displays this ability probably understands what is meant by ''solving an equation''. On the other hand, the more formal algebraic techniques of solving such equations must be mastered, for no student can solve all such equations by inspection.

$((A - B) + (B - C)) + (C - A) = (A - C) + (C - A) = C - C = \vec{0}$.

By Theorem 6-12, if $(A - B, B - C)$ is linearly independent and $(A - B) + (B - C) + (C - A) = \vec{0}$ then $(A - B)1 + (B - C)a + (C - A)b = \vec{0}$ if and only if $1 = a = b$.

Since $P = A + (B - C)a$, and $a = 1$, $P = A + (B - C)1$.

We have shown that $A + (B - C)1 = B + (A - C)1$. Suppose, then, that $P = A + (B - C)1$. It follows that ($P = A + (B - C)1$ and $P = B + (A - C)1$) and, so, that ($\exists_x\, P = A + (B - C)x$ and $\exists_x\, P = B + (A - C)x$) — that is, that $P \in \overleftrightarrow{A[B - C]} \cap \overleftrightarrow{B[A - C]}$. Hence, if $P = A + (B - C)1$ then $P \in \overleftrightarrow{A[B - C]} \cap \overleftrightarrow{B[A - C]}$.

TC 297 (3)

On the other hand, suppose that $P = A + (B - C)a$ and $P = B + (A - C)b$. It follows that $A + (B - C)a = B + (A - C)b$ and so, as we have shown, that $a = 1$. Hence, if ($P = A + (B - C)a$ and $P = B + (A - C)b$) then $P = A + (B - C)1$. Consequently, if ($\exists_x\, P = A + (B - C)x$ and $\exists_x\, P = B + (A - C)x$) then $P = A + (B - C)1$. In short, if $P \in \overleftrightarrow{A[B - C]} \cap \overleftrightarrow{B[A - C]}$ then $P = A + (B - C)1$.

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On TC 297(1) we showed that any sentence of the form:

$$(*) \qquad (P = sa \text{ and } P = tb) \Longleftrightarrow (P = sa \text{ and } sa = tb)$$

is valid. Equation (1) is of the form:

$$sa = tb$$

and we have proved a corresponding biconditional sentence of the form:

$$(**) \qquad sa = tb \Longleftrightarrow (a = c \text{ and } b = d)$$

So, for the particular choice of the terms 'sa' and 'tb' made here, we have the theorem:

$$(P = sa \text{ and } P = tb) \Longleftrightarrow (P = sa \text{ and } a = c \text{ and } b = d)$$

In particular, we have [since '$P = sa$ and $a = c$' implies '$P = sc$'] that

$$(P = sa \text{ and } P = tb) \Longrightarrow P = sc,$$

from which it follows that

$$(\exists_x\, P = sx \text{ and } \exists_x\, P = tx) \Longrightarrow P = sc.$$

On the other hand, suppose that $P = sc$. Since we have proved that $sc = td$, it follows that ($P = sc$ and $P = td$) and, so, that ($\exists_x\, P = sx$ and $\exists_x\, P = tx$). Hence,

$$P = sc \Longrightarrow (\exists_x\, P = sx \text{ and } \exists_x\, P = tx).$$

In summary, since any sentence of the form (*) is valid, any sentence of the form (**) has as a consequence:

$$(\exists_x\, P = sx \text{ and } \exists_x\, P = tx) \Longleftrightarrow P = sc$$

Postulate 3. [And this is a point at which you might, if you have not already done so, start off on your own.] Another way to continue is to note that

$$(A - B) + (B - C) + (C - A) = \overrightarrow{0} \qquad \text{[Explain.]}$$

and be reminded by this of an earlier theorem. ,Try to recall this theorem.] Since we are considering the case in which $\{A, B, C\}$ is noncollinear this theorem tells us that equation (2) is equivalent to:

$$1 = a = b \qquad \text{[Explain.]}$$

So [all at once] we have discovered that, for $\{A, B, C\}$ noncollinear,

$$P \in \overleftrightarrow{A[B - C]} \cap \overleftrightarrow{B[A - C]} \Longleftrightarrow P = A + (B - C)1.$$

[Explain.] In other words,

$$(*) \quad \overleftrightarrow{A[B - C]} \cap \overleftrightarrow{B[A - C]} = \{A + (B - C)\} \qquad [\{A, B, C\} \text{ noncollinear}].$$

Of course, we might replace the right side of this equation by '$\{B + (A - C)\}$'; but we don't need the work we have done here to tell us that. [Do you see why not? And, do you see how you *might* have shown that $A + (B - C)$ is *a* point of the intersection as soon as you had written down equation (1)? How could you have shown, then, that there is no other point in the intersection?] The sentence (*) is useful enough to rate a number. We shall call it 'Theorem 7–11'.

Exercises

Part A

1. Use equation (2) on page 297 and Postulate 3 to prove *Theorem 7–11.*
2. Suppose that $C \in l \cap m$, $[l] = [\overrightarrow{a}]$ and $[m] = [\overrightarrow{b}]$. Show that $\overleftrightarrow{D[l]} \cap m \neq \emptyset$ if and only if $D - C \in [\overrightarrow{a}, \overrightarrow{b}]$. [*Hint.* First find out what this theorem means by drawing a figure. Then, write an equation which is solvable if and only if $\overleftrightarrow{D[l]} \cap m \neq \emptyset$.]
3. Prove:

Theorem 7–12
$\overleftrightarrow{D[l]} \cap m \neq \emptyset \longleftrightarrow \overleftrightarrow{D[m]} \cap l \neq \emptyset \quad [l \cap m \neq \emptyset]$

[*Hint.* For $l \cap m \neq \emptyset$, there is a point – say C – such that $C \in l \cap m$. And, for any lines l and m there are non-$\overrightarrow{0}$ translations – say $\overrightarrow{a}$ and $\overrightarrow{b}$ – such that $[l] = [\overrightarrow{a}]$ and $[m] = [\overrightarrow{b}]$.]

Answers for Part A

1. Note that $B - C = (A - C) + (B - A)$. Using this and equation (2), we obtain, in turn:

$$(A - B) + [(A - C) + (B - A)]a + (C - A)b = \overrightarrow{0}$$
$$(B - A)[-1 + a] + (C - A)[-a + b] = \overrightarrow{0}$$

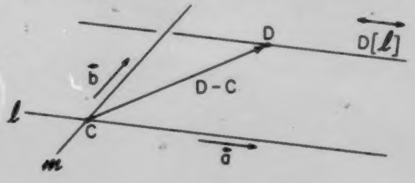
Since $(B - A, C - A)$ is linearly independent it follows from the latter that

$$-1 + a = 0 \text{ and } -a + b = 0.$$

That is, $a = 1$ and $b = 1$.

Hence, the only point in $\overleftrightarrow{A[B - C]} \cap \overleftrightarrow{B[A - C]}$ is $A + (B - C)$. This proves Theorem 7-11.

2. Here is a figure that illustrates the meaning of the theorem. [However, it is not assumed in the theorem that $(\overrightarrow{a}, \overrightarrow{b})$ is linearly independent.]



The theorem says that the lines $\overleftrightarrow{D[l]}$ and m have a point in common if and only if D - C is a linear combination of $\overrightarrow{a}$ and $\overrightarrow{b}$.

Since $P \in \overleftrightarrow{D[l]}$ if and only if $\exists_x\ P = D + \overrightarrow{a}x$, and $P \in m$ if and only if $\exists_x\ P = C + \overrightarrow{b}x$, it follows that

$$\overleftrightarrow{D[l]} \cap m \neq \emptyset \Longleftrightarrow \exists_x \exists_y\ D + \overrightarrow{a}x = C + \overrightarrow{b}y.$$

Since

$$D + \overrightarrow{a}a = C + \overrightarrow{b}b \Longleftrightarrow D - C = -\overrightarrow{a}a + \overrightarrow{b}b$$

it follows that

$$\begin{aligned}\overleftrightarrow{D[l]} \cap m \neq \emptyset &\Longleftrightarrow \exists_x \exists_y\ D - C = -\overrightarrow{a}x + \overrightarrow{b}y,\\ &\Longleftrightarrow D - C \in [-\overrightarrow{a}, \overrightarrow{b}],\\ &\Longleftrightarrow D - C \in [\overrightarrow{a}, \overrightarrow{b}].\end{aligned}$$

3. As suggested in the hint, we can — for a properly chosen point C — use the result of Exercise 2:

$$\overleftrightarrow{D[l]} \cap m \neq \emptyset \Longleftrightarrow D - C \in [\overrightarrow{a}, \overrightarrow{b}]$$

Since $m \cap l = l \cap m$ we can also — for the same point C — infer that

$$\overleftrightarrow{D[m]} \cap l \neq \emptyset \Longleftrightarrow D - C \in [\overrightarrow{b}, \overrightarrow{a}].$$

Since $[\overrightarrow{b}, \overrightarrow{a}] = [\overrightarrow{a}, \overrightarrow{b}]$ it follows that

$$\overleftrightarrow{D[l]} \cap m \neq \emptyset \Longleftrightarrow \overleftrightarrow{D[m]} \cap l \neq \emptyset.$$

[Of course, the existence of a point such as C depends on the assumption that $l \cap m \neq \emptyset$.]

4. Show that $\overleftrightarrow{A[\vec{a}]} \cap \overleftrightarrow{B[\vec{b}]} \neq \emptyset$ if and only if $B - A \in [\vec{a}, \vec{b}]$.

5. Given that $\{A, B, C\}$ is noncollinear, show that

   (a) $\overleftrightarrow{A[\vec{c}]} \cap \overleftrightarrow{BC} \neq \emptyset \longrightarrow (\vec{c} \in [B - A, C - B]$ and $\vec{c} \notin [C - B])$

   (b) $(\vec{c} \in [B - A, C - B]$ and $\vec{c} \notin [C - B]) \longrightarrow \overleftrightarrow{A[\vec{c}]} \cap \overleftrightarrow{BC} \neq \emptyset$

   [*Hint for (b)*: On assuming the antecedent of (b) we may suppose that there are real numbers – say c_1 and c_2 – such that $\vec{c}$ = ________ and ____ $\neq 0$.]

Part B

1. Suppose that l is a line, that A and B belong to l, and that $[l] = [\vec{a}]$. Suppose, also that $\vec{b}$ and $\vec{c}$ are translations such that $\vec{c} \notin [\vec{a}, \vec{b}]$.

   (a) Show that

   $$(B + \vec{b}) - (A + \vec{c}) \notin [\vec{a}, \vec{b}].$$

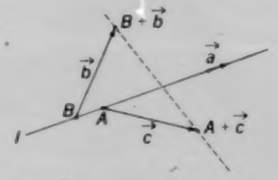
   [*Hint*: Why does $B - A \in [\vec{a}, \vec{b}]$?]

   (b) Use the result of part (a) to show that

   (i) $\overleftrightarrow{(A + \vec{c})(B + \vec{b})}$ is a line;

   (ii) $\overleftrightarrow{(A + \vec{c})(B + \vec{b})}$ is not parallel to l;

   (iii) $\vec{b} \notin [l] \longrightarrow \overleftrightarrow{(A + \vec{c})(B + \vec{b})} \cap l = \emptyset$.



2. Suppose that l and m are two lines through C, that $P \in l$ and $P \neq C$, and that $[l] = [\vec{a}]$, $[m] = [\vec{b}]$, and $\vec{c} \neq \vec{0}$. [Draw a picture!]

   (a) Under what conditions would you expect $\overleftrightarrow{P[\vec{c}]}$ to intersect m?

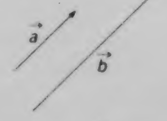
   (b) Show that

   $$\overleftrightarrow{P[\vec{c}]} \cap m \neq \emptyset \longleftrightarrow (\vec{c} \in [\vec{a}, \vec{b}] \text{ and } \vec{c} \notin [\vec{b}]).$$

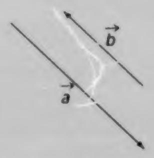
Part C

1. In each of the following exercises you are given two non-$\vec{0}$ vectors $\vec{a}$ and $\vec{b}$ such that $[\vec{b}] = [\vec{a}]$. In each exercise estimate the value of 'b' such that $\vec{b} = \vec{a}b$.

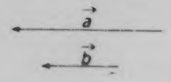
(a)



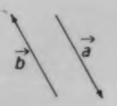
(b)



(c)



(d)



2. In Exercise 1, list the parts for which the value of 'b' such that $\vec{b} = \vec{a}b$ is greater than 0. Also list the parts for which $b < 0$.

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4. [The proof is that of Exercise 2:]

$$\begin{aligned} \overleftrightarrow{A[\vec{a}]} \cap \overleftrightarrow{B[\vec{b}]} \neq \emptyset &\Longleftrightarrow \exists_x \exists_y\ B + \vec{b}x = A + \vec{a}y \\ &\Longleftrightarrow \exists_x \exists_y\ B - A = \vec{a}y - \vec{b}x \\ &\Longleftrightarrow B - A \in [\vec{a}, \vec{b}] \end{aligned}$$

[It is perhaps worth noting that it has not been assumed that either $\vec{a}$ or $\vec{b}$ is non-$\vec{0}$. So, since $\overleftrightarrow{A[\vec{0}]} = \{A\}$ and $[\vec{0}, \vec{b}] = [\vec{b}]$, the theorem tells us that $A \in \overleftrightarrow{B[\vec{b}]} \Longleftrightarrow B - A \in [\vec{b}]$. Since $\overleftrightarrow{B[\vec{0}]} = \{B\}$ and $[\vec{0}] = \{\vec{0}\}$ the theorem also tells us that $A = B \Longleftrightarrow B - A = \vec{0}$. Neither of these results is new, but it is interesting to see how apparently unrelated results are, in fact, connected with one another. It will be seen that one of the advantages of the present approach to geometry is that it shows unexpected connections among seemingly unrelated theorems. (The preceding is, admittedly, a very trivial example.)]

5. (a) Suppose that $\overleftrightarrow{A[\vec{c}]} \cap \overleftrightarrow{BC} \neq \emptyset$, where $\{A, B, C\}$ is noncollinear. It follows that there are numbers — say, p and q — such that $A + \vec{c}p = B + (C - B)q$ and, since $A \notin \overleftrightarrow{BC}$, $p \neq 0$. So, $\vec{c}p = (B - A) + (C - B)q$ and, since $p \neq 0$, $\vec{c} \in [B - A, C - B]$. If $\vec{c} \in [C - B]$ then so does $\vec{c}p$ and it follows that for some number — say, r — $(C - B)r = \vec{c}p = (B - A) + (C - B)q$, contrary to the linear independence of $(B - A, C - B)$. So, $\vec{c} \notin [C - B]$. Hence, for $\{A, B, C\}$ noncollinear, if $\overleftrightarrow{A[\vec{c}]} \cap \overleftrightarrow{BC} \neq \emptyset$ then $\vec{c} \in [B - A, C - B]$ and $\vec{c} \notin [C - B]$.

   (b) Suppose that $\vec{c} \in [B - A, C - B]$ and $\vec{c} \notin [C - B]$. It follows that there are numbers — say, c_1 and c_2 — such that $\vec{c} = (B - A)c_1 + (C - B)c_2$ and $c_1 \neq 0$. So $B - A = \vec{c} \cdot / c_1 + (C - B) \cdot -(c_2/c_1)$, from which it follows that $B + (C - B)(c_2/c_1) = A + \vec{c} \cdot / c_1$. So, there is a point [the point $A + \vec{c} \cdot / c_1$] which belongs both to $\overleftrightarrow{A[\vec{c}]}$ and to $\overleftrightarrow{BC}$. Hence, if $\vec{c} \in [B - A, C - B]$ and $\vec{c} \notin [C - B]$ then $\overleftrightarrow{A[\vec{c}]} \cap \overleftrightarrow{BC} \neq \emptyset$.

* * *

Exercise 5 can also be solved by relating it to Exercise 4. Doing so will bring to light a result which we shall come upon later when dealing with planes. Since $\overleftrightarrow{BC} = \overleftrightarrow{B[C - B]}$ it follows from Exercise 4 that

$$\overleftrightarrow{A[\vec{c}]} \cap \overleftrightarrow{BC} \neq \emptyset \Longleftrightarrow B - A \in [\vec{c}, C - B].$$

So, for (a) and (b), respectively, it is sufficient to establish the if-part and the only if-part of:

$$(*) \quad (\vec{c} \in [B - A, C - B] \text{ and } \vec{c} \notin [C - B]) \Longleftrightarrow B - A \in [\vec{c}, C - B],$$

for $\{A, B, C\}$ noncollinear. These are easily established by arguments similar to those given above for (a) and (b). Inspection of (*) suggests, however, that it is merely an instance of a theorem which has nothing, explicitly, to do with points:

$$(**) \quad (\vec{c} \in [\vec{a}, \vec{b}] \text{ and } \vec{c} \notin [\vec{b}]) \Longleftrightarrow \vec{a} \in [\vec{c}, \vec{b}] \quad [(\vec{a}, \vec{b}) \text{ linearly independent}]$$

This is the case, and a merely notational modification of the arguments which would establish (*) will serve to prove (**). As it will turn out,

TC 299 (2)

the only if-part of (**) — which does not require the linear independence of $(\vec{a}, \vec{b})$ — is the more interesting. Its proof is essentially that given for part (b):

Suppose that $\vec{c} \in [\vec{a}, \vec{b}]$ and $\vec{c} \notin [\vec{b}]$. It follows that there are numbers — say, c_1 and c_2 — such that $\vec{c} = \vec{a}c_1 + \vec{b}c_2$ and $c_1 \neq 0$. So $\vec{a} = \vec{c}\cdot 1/c_1 + \vec{b}\cdot -(c_2/c_1)$, from which it follows that $\vec{a} \in [\vec{c}, \vec{b}]$. Hence,

$$(\vec{c} \in [\vec{a}, \vec{b}] \text{ and } \vec{c} \notin [\vec{b}]) \Longrightarrow \vec{a} \in [\vec{c}, \vec{b}].$$

From this result it follows easily that

$$(***) \qquad (\vec{c} \in [\vec{a}, \vec{b}] \text{ and } \vec{c} \notin [\vec{b}]) \Longrightarrow [\vec{c}, \vec{b}] = [\vec{a}, \vec{b}].$$

For, if $\vec{c} \in [\vec{a}, \vec{b}]$ then, since $\vec{b} \in [\vec{a}, \vec{b}]$, $[\vec{c}, \vec{b}] \subseteq [\vec{a}, \vec{b}]$. And, similarly, if $\vec{a} \in [\vec{c}, \vec{b}]$ then $[\vec{a}, \vec{b}] \subseteq [\vec{c}, \vec{b}]$.

The importance of (***) is that it shows one way of choosing a new "basis" in a given bidirection $[\vec{a}, \vec{b}]$ — equivalently, of making a "change of coordinates" in a given plane. Moreover, it leads, as we shall see, to more general theorems of this nature. As a first step, note that, by two applications of (***) we obtain:

$$(\{\vec{c}, \vec{d}\} \subseteq [\vec{a}, \vec{b}] \text{ and } \vec{c} \notin [\vec{b}] \text{ and } \vec{d} \notin [\vec{c}]) \Longrightarrow [\vec{c}, \vec{d}] = [\vec{a}, \vec{b}]$$

For, assuming the antecedent, we have — directly from (***) — that $[\vec{c}, \vec{b}] = [\vec{a}, \vec{b}]$. Since, by assumption, $\vec{d} \in [\vec{a}, \vec{b}]$ it now follows that $\vec{d} \in [\vec{c}, \vec{b}] = [\vec{b}, \vec{c}]$. So, since $\vec{d} \notin [\vec{c}]$ it follows — from an instance of (***) — that $[\vec{d}, \vec{c}] = [\vec{b}, \vec{c}]$. Since $[\vec{c}, \vec{d}] = [\vec{d}, \vec{c}]$ it follows that $[\vec{c}, \vec{d}] = [\vec{a}, \vec{b}]$.

Finally, we obtain the very important result:

$$(\{\vec{c}, \vec{d}\} \subseteq [\vec{a}, \vec{b}] \text{ and } (\vec{c}, \vec{d}) \text{ is linearly independent})$$
$$\Longrightarrow$$
$$[\vec{c}, \vec{d}] = [\vec{a}, \vec{b}].$$

For, assuming the antecedent, since $(\vec{c}, \vec{d})$ is linearly independent it follows that $\vec{d} \notin [\vec{c}]$. So the desired conclusion will follow from the preceding result unless $\vec{c}$ belongs both to $[\vec{b}]$ and to $[\vec{a}]$. [If $\vec{c} \in [\vec{a}]$ then the conclusion follows from the instance obtained by interchanging '$\vec{a}$' and '$\vec{b}$', together with the fact that $[\vec{b}, \vec{a}] = [\vec{a}, \vec{b}]$.] Now, since $(\vec{c}, \vec{d})$ is linearly independent, $\vec{c} \neq \vec{0}$. So, if $\vec{c}$ belongs to both $[\vec{a}]$ and $[\vec{b}]$ it follows that $[\vec{a}] = [\vec{c}] = [\vec{b}]$ and it follows from this that $[\vec{a}, \vec{b}] = [\vec{c}]$. But, since $\vec{d} \in [\vec{a}, \vec{b}]$, this implies that $\vec{d} \in [\vec{c}]$, contradicting the assumed linear independence of $(\vec{c}, \vec{d})$. So, $\vec{c} \notin [\vec{a}] \cap [\vec{b}]$ and, as remarked above, the desired conclusion follows.

Our final result can be stated as "Any two linearly independent members of a bidirection form a basis for the bidirection". It will be of use in proving a theorem for planes analogous to Theorem 7-1 for lines. In particular, it has as one consequence the fact that three noncollinear points [or: two intersecting lines] determine a unique plane.

*

TC 299 (3)

Answers for Part B

1. (a) Since $(B + \vec{b}) - (A + \vec{c}) = (B - A) + \vec{b} - \vec{c}$ and since $B - A \in [\ell] = [\vec{a}]$ it follows that if $(B + \vec{b}) - (A + \vec{c}) \in [\vec{a}, \vec{b}]$ then $\vec{c} \in [\vec{a}, \vec{b}]$. Hence, since $\vec{c} \notin [\vec{a}, \vec{b}]$, $(B + \vec{b}) - (A + \vec{c}) \notin [\vec{a}, \vec{b}]$.

   (b) (i) Since $\vec{0} \in [\vec{a}, \vec{b}]$ it follows from part (a) that $(B + \vec{b}) - (A + \vec{c}) \neq \vec{0}$. Hence, $A + \vec{c}$ and $B + \vec{b}$ are two points and, so, $\overleftrightarrow{(A + \vec{c})(B + \vec{b})}$ is a line.

   (ii) $[\overleftrightarrow{(A + \vec{c})(B + \vec{b})}] = [(B + \vec{b}) - (A + \vec{c})]$ and, by part (a), this direction is not a subset of $[\vec{a}, \vec{b}]$. But, $[\ell] = [\vec{a}] \subseteq [\vec{a}, \vec{b}]$. Hence, the two lines have different directions, and, so, are not parallel.

   (iii) Suppose that $\overleftrightarrow{(A + \vec{c})(B + \vec{b})} \cap \ell \neq \emptyset$. It follows that there are numbers — say, p and q — such that

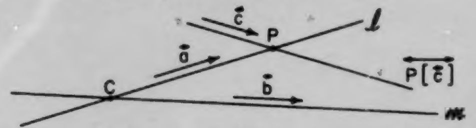
$$(A + \vec{c}) + ((B + \vec{b}) - (A + \vec{c}))p = A + \vec{a}q,$$
$$\vec{c} + ((B - A) + (\vec{b} - \vec{c}))p = \vec{a}q,$$
$$\vec{c}(1 - p) = \vec{a}q - \vec{b}p - (B - A)p.$$

   Since $B - A \in [\vec{a}]$ it follows that $\vec{c} \in [\vec{a}, \vec{b}]$ unless $p = 1$. So, since $\vec{c} \notin [\vec{a}, \vec{b}]$, $p = 1$ and [from the last equation]

$$\vec{b} = \vec{a}q - (B - A).$$

   So, since $B - A \in [\vec{a}]$, $\vec{b} \in [\vec{a}] = [\ell]$. Hence, if $\overleftrightarrow{(A + \vec{c})(B + \vec{b})} \cap \ell \neq \emptyset$ then $\vec{b} \in [\ell]$ — that is, if $\vec{b} \notin [\ell]$ then $\overleftrightarrow{(A + \vec{c})(B + \vec{b})} \cap \ell = \emptyset$. [Note that the converse also holds. For, if $\vec{b} \in [\ell]$ then, since $B \in \ell$, $B + \vec{b} \in \ell$.]

2.



   (a) When $\vec{c} \in [\vec{a}, \vec{b}]$ and $\vec{c} \notin [\vec{b}]$.

   (b) Since P and any two points of m are noncollinear, this follows directly from parts (a) and (b) of Exercise 5, Part A, page 299.

Answers for Part C

1. (a) 2 (b) (about) $-\frac{2}{3}$
   (c) $\frac{1}{2}$ (d) -1
2. In (a) and (c), $b > 0$; in (b) and (d), $b < 0$.

TC 300 (1)

In section 7.04 and its commentary we gave reasons for interpreting 'the direction of $\vec{a}$' as referring to $[\vec{a}]$. The somewhat arbitrary choice of including $\vec{0}$ in the direction of every vector [and of every line] has paid off in added simplicity. For example, with the chosen definition we can say that, for $\vec{a} \neq \vec{0}$, the line through A in the direction of $\vec{a}$ is the set of all points X such that X - A is in the direction of $\vec{a}$

7.07 The Sense of a Vector

We have already agreed that the direction of a vector $\vec{a}$ is the set $[\vec{a}]$ of all multiples of $\vec{a}$. As Part C on page 299 reminds us, proper translations which have the same direction may have either the same sense or opposite senses. For example, the vectors $\vec{a}$ and $\vec{b}$ in all four parts of of Exercise 1 have the same direction, but those in part (a) have the same sense while those in part (b) have opposite senses. Intuitively, non-$\vec{0}$ vectors have the same sense if and only if each is a multiple of the other by some positive number. So, as in the case of 'direction', it seems reasonable to define the sense of a non-$\vec{0}$ vector to be the set of all multiples of this vector by positive numbers. To avoid restrictions, we need also to define the sense of $\vec{0}$ and, again as in the case of 'direction', the choice of this definition is dictated by convenience. As it turns out, it is most convenient to define 'sense' so that the sense of $\vec{0}$ is the empty set. These considerations lead to:

Definition 7-8 $[\vec{a}]^+ = \{\vec{x}: \vec{x} \neq \vec{0} \text{ and } \exists_{x>0}\, \vec{x} = \vec{a}x\}$

and to reading '$[\vec{a}]^+$' as 'the sense of $\vec{a}$'. [Read '$\exists_{x>0}$' as 'there exists a real number x greater than 0 such that'. More formally, a sentence of the form '$\exists_{x>0}$ ______' is equivalent to one of the form '$\exists_x$ ($x > 0$ and ______)'.] In words, Definition 7-8 says that $[\vec{a}]^+$ is the set of all non-$\vec{0}$ translations that are positive multiples of $\vec{a}$.

The somewhat complicated form of Definition 7-8 yields the desired results on senses of non-$\vec{0}$ vectors and of $\vec{0}$:

Theorem 7-13 (a) $[\vec{a}]^+ = \{\vec{x}: \exists_{x>0}\, \vec{x} = \vec{a}x\}$ $[\vec{a} \neq \vec{0}]$
(b) $[\vec{0}]^+ = \varnothing$

Translations $\vec{a}$ and $\vec{b}$ have the same sense if and only if $[\vec{a}]^+ = [\vec{b}]^+$ – otherwise, $\vec{a}$ and $\vec{b}$ have different senses. For $\vec{a}$ and $\vec{b}$ to have *opposite* senses, each – or, either – must have the same sense as the opposite of the other.

As an example, let's consider a proof of Theorem 7-13(a). Since it is clearly the case that, for any $\vec{a}$, $[\vec{a}]^+ \subseteq \{\vec{x}: \exists_{x>0}\, \vec{x} = \vec{a}x\}$ [Why?], what we need to show is that, for $\vec{a} \neq \vec{0}$, if $\exists_{x>0}\, \vec{b} = \vec{a}x$ then $\vec{b} \neq \vec{0}$. [Explain.] For this we need to show that, for $\vec{a} \neq \vec{0}$, if $b > 0$ and $\vec{b} = \vec{a}b$ then $\vec{b} \neq \vec{0}$. More simply, we must show that if $\vec{a} \neq \vec{0}$ and $b > 0$ then $\vec{a}b \neq \vec{0}$. This last follows from an earlier theorem and the fact that if $b > 0$ then $b \neq 0$.

An analysis like the preceding is as good as a proof – assuming that you can give the required explanations – and is often easier to under-

[Theorem 7-5(a)]. If the direction of $\vec{a}$ had been specified as consisting of the proper translations in $[\vec{a}]$ then the same line would have to be described [in terms of 'direction'] as the set of all points X such that either X = A or X - A is in the direction of $\vec{a}$. Complications of this nature would have occurred throughout and, in addition, we would have needed a new notation — say, '$d(\vec{a})$' or '$[\vec{a}]_\circ$' — to use in referring to directions of vectors. Were it not for these two objections, a good case could have been made — on the grounds of the intuitive connotation of 'direction' — for excluding $\vec{0}$ from all directions.

Objections of the kind just brought up do not apply to limit our choice of a definition of 'sense'. We shall, in any case, need to adopt a new notation. Furthermore, the sense of a vector will be referred to in describing both rays [which contain their vertices] and half-lines [which do not]. So we can expect that, whether we exclude $\vec{0}$ from senses or include it in them, we shall encounter the same complications, in describing either rays or half-lines, which we avoided in the case of lines by including $\vec{0}$ in all directions. These considerations set us free to follow intuition more closely and to exclude $\vec{0}$ from the senses of, at best, non-$\vec{0}$ vectors. Intuition would probably urge that "$\vec{0}$ has no sense". But to leave the phrase 'the sense of $\vec{0}$' undefined would lead to further complications. Applying to $\vec{0}$ the same definition of 'sense' which seems satisfactory for non-$\vec{0}$ vectors leads us to take the sense of $\vec{0}$ to be $\varnothing$.

The notation '$[\vec{a}]^+$' seems a natural one, but it has one formal disadvantage. It suggests that the sense of a vector results from carrying out some operation on its direction. This is, of course, not the case. If $\vec{a} \neq \vec{0}$ then, even though $[\vec{a}] = [-\vec{a}]$, it is not the case that $[\vec{a}]^+ = [-\vec{a}]^+$. This apparent exception to the usual rules of logic for equations can be explained by noting that the '$^+$' in '$[\vec{a}]^+$' is not an operator. Rather, one should think of the complete symbol '$[\ \ ]^+$' as an operator.

The captions 'Definition 7-8' and 'Theorem 7-13' could perfectly well be interchanged. We shall, in fact, make most use of Theorem 7-13. The only reason for not adopting it as a definition is that we have a prejudice against what might be called "definition by cases" in contrast to "monolithic definitions".

See TC 289(2) for remarks on restricted quantifiers like '$\exists_{x>0}$'.

Answer for 'Why?': If $\vec{c} \neq \vec{0}$ and $\exists_{x>0}\, \vec{c} = \vec{a}x$ then $\exists_{x>0}\, \vec{c} = \vec{a}x$. In other words, if $\vec{c} \in [\vec{a}]^+$ then $\vec{c} \in \{\vec{x}: \exists_{x>0}\, \vec{x} = \vec{a}x\}$.

Analyses like that referred to here have been discussed in the commentary in connection with the proof of Theorem 7-1. As an indication of an understanding of what is going on, such analyses are often better than more formal proofs. Unless you have good reasons to the contrary, allow students to present such analyses when proofs are asked for.

Proof of Theorem 7-13(b): By definition, if $\vec{c} \in [\vec{0}]^+$ then $\vec{c} \neq \vec{0}$ and there is a number — say, a — such that a > 0 and $\vec{c} = \vec{0}a$. Since $\vec{0}a = \vec{0}$ it follows that if $\vec{c} \in [\vec{0}]^+$ then $\vec{c} \neq \vec{0}$ and $\vec{c} = \vec{0}$. Since it is not the case that $\vec{c} \neq \vec{0}$ and $\vec{c} = \vec{0}$ it follows that [for any $\vec{c}$] $\vec{c} \notin [\vec{0}]^+$. Since $[\vec{0}]^+ \subseteq \mathcal{T}$, $[\vec{0}]^+ = \varnothing$.

stand. It is easily turned into a more conventional proof by merely turning it around:

Suppose that $\vec{a} \neq \vec{0}$ and that $b > 0$. Since $0 \not> 0$ it follows that $b \neq 0$ and so, by Theorem 5-5, that $\vec{a}b \neq \vec{0}$. So, for $\vec{a} \neq \vec{0}$, if $b > 0$ and $\vec{b} = \vec{a}b$ then $\vec{b} \neq \vec{0}$—that is, if $\exists_{x>0}\, \vec{b} = \vec{a}x$ then $\vec{b} \neq \vec{0}$. Hence, for $\vec{a} \neq \vec{0}$,

$$(\vec{b} \neq \vec{0} \text{ and } \exists_{x>0}\, \vec{b} = \vec{a}x) \Longleftrightarrow \exists_{x>0}\, \vec{b} = \vec{a}x.$$

Consequently, Theorem 7-13(a).

Exercises

Part A

Draw an arrow describing a proper translation $\vec{b}$ and mark four points A, B, C, and D.

1. Draw the set of all images of A under translations which belong to $[\vec{b}]^+$ and the set of all images of B under translations which belong to $[-\vec{b}]^+$. Does A belong to the first set? Does B belong to the second set?
2. (a) Choose a translation $\vec{a}_1 \in [\vec{b}]^+$ and draw an arrow to locate the image of C under $\vec{a}_1$.
   (b) Similarly, locate the image of D under a chosen translation $\vec{a}_2 \in [-\vec{b}]^+$.
   (c) In the same way, locate the images of B under a translation $\vec{a}_3 \in [-\vec{b}]^+$ and under a translation $\vec{a}_4 \in [\vec{b}]^+$.
3. Referring to the translations $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, and $\vec{a}_4$ of Exercise 2, tell which of the following have the same sense as $\vec{b}$ and which have the sense opposite to that of $\vec{b}$.
   (a) $\vec{a}_1$ (b) $\vec{a}_2$ (c) $\vec{a}_3$ (d) $\vec{a}_4$
   (e) $\vec{a}_1 + \vec{b}$ (f) $\vec{a}_4 \cdot -3$ (g) $\vec{a}_2 \cdot 5$ (h) $\vec{a}_2 \cdot -5$
   (i) $\vec{a}_3 + \vec{a}_2$ (j) $(\vec{a}_3 + \vec{a}_2) \cdot -2$ (k) $\vec{a}_1 + \vec{a}_4$ (l) $(\vec{a}_1 + \vec{a}_4)0$
4. Which of the vectors of Exercise 3 have the same direction as $\vec{b}$?

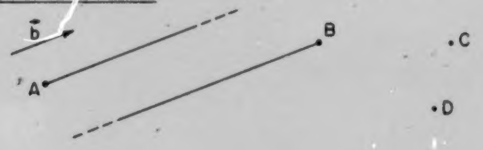
Part B

1. Show that
   (a) $\vec{0} \notin [\vec{a}]^+$, (b) $\vec{a} \neq \vec{0} \Longrightarrow \vec{a} \in [\vec{a}]^+$,
   (c) $[\vec{a}]^+ = \emptyset \Longrightarrow \vec{a} = \vec{0}$ [*Hint*: Use Part (b).],
   (d) $\vec{b} \in [\vec{a}]^+ \Longrightarrow \vec{a} \neq \vec{0}$
   [*Hint*: Complete: $(\vec{a} = \vec{0}$ and $\vec{b} = \vec{a}b) \Longrightarrow$ ________],
   (e) $[\vec{0}]^+ = \emptyset$ [*Hint*: Use part (d).],
   (f) $[\vec{a}]^+ \cap [-\vec{a}]^+ = \emptyset$.
2. Prove:
   (a) $\vec{b} \in [\vec{a}]^+ \Longrightarrow \vec{a} \in [\vec{b}]^+$ [*Hint*: Use the fact that if $b > 0$ then $b \neq 0$ and $/b > 0$.]
   (b) $\vec{b} \in [\vec{a}]^+ \Longrightarrow [\vec{a}]^+ = [\vec{b}]^+$

Parts A and B represent a reasonable homework assignment. Part C is recommended as an in-class activity. Parts D and E make another reasonable homework assignment.

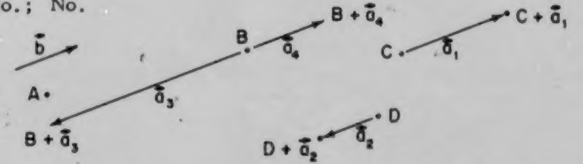
Answers for Part A

1.



No.; No.

2.



3. $\vec{a}_1$, $\vec{a}_4$, $\vec{a}_1 + \vec{b}$, $\vec{a}_2 \cdot -5$, $(\vec{a}_3 + \vec{a}_2) \cdot -2$, $\vec{a}_1 + \vec{a}_4$ have same sense as $\vec{b}$, $\vec{a}_2$, $\vec{a}_3$, $\vec{a}_4 \cdot -3$, $\vec{a}_2 \cdot 5$, $\vec{a}_3 + \vec{a}_2$ have the sense opposite to that of $\vec{b}$.
4. All but $(\vec{a}_1 + \vec{a}_4)0$ have the same direction as $\vec{b}$.

Answers for Part B

1. (a) Suppose that $\vec{p} \in [\vec{a}]^+$. Then, $\vec{p} \neq \vec{0}$ and $\exists_{x>0}\, \vec{p} = \vec{a}x$. So, $\vec{p} \neq \vec{0}$. Thus, if $\vec{p} = \vec{0}$ then $\vec{p} \notin [\vec{a}]^+$. That is, $\vec{0} \notin [\vec{a}]^+$.
   (b) Suppose that $\vec{a} \neq \vec{0}$. Since $\vec{a} = \vec{a}1$ and $1 \geq 0$, it follows that $\exists_{x>0}\, \vec{a} = \vec{a}x$. So, $\vec{a} \neq \vec{0}$ and $\exists_{x>0}\, \vec{a} = \vec{a}x$. By definition, then, $\vec{a} \in [\vec{a}]^+$. Hence, if $\vec{a} \neq \vec{0}$ then $\vec{a} \in [\vec{a}]^+$.
   (c) Suppose that $\vec{a} \neq \vec{0}$. Then, by (b), $\vec{a} \in [\vec{a}]^+$ so that $[\vec{a}] \neq \emptyset$. Hence, if $\vec{a} \neq \vec{0}$ then $[\vec{a}]^+ \neq \emptyset$. By contraposition, if $[\vec{a}]^+ = \emptyset$ then $\vec{a} = \vec{0}$.
   (d) If $\vec{a} = \vec{0}$ and $\vec{b} = \vec{a}b$ then $\vec{b} = \vec{0}$. So, if $\vec{a} = \vec{0}$ and $\vec{b} \in [\vec{a}]^+$ then $\vec{b} = \vec{0}$. But, if $\vec{b} \in [\vec{a}]^+$ then $\vec{b} \neq \vec{0}$. Hence, it is not the case that $\vec{a} = \vec{0}$ and $\vec{b} \in [\vec{a}]^+$. In other words, if $\vec{b} \in [\vec{a}]^+$ then $\vec{a} \neq \vec{0}$.
   (e) By part (d), if $\vec{a} = \vec{0}$ then $\vec{b} \notin [\vec{a}]^+$ — that is, $\vec{b} \notin [\vec{0}]^+$. Since $[\vec{0}]^+ \subseteq T$ it follows that $[\vec{0}]^+ = \emptyset$.
   (f) Suppose that $\vec{b} \in [\vec{a}]^+ \cap [-\vec{a}]^+$. By definition, $\vec{b} \neq \vec{0}$ and there are numbers — say, p and q — such that $p \geq 0$, $q > 0$, $\vec{b} = \vec{a}p$, and $\vec{b} = -\vec{a}q$. It follows that $\vec{a}p = -\vec{a}q$ and, so, that $\vec{a}(p + q) = \vec{0}$. Since $p > 0$ and $q > 0$, $p + q \neq 0$ and, so, $\vec{a} = \vec{0}$. Since $\vec{b} \in [\vec{a}]^+ \cap [-\vec{a}]^+$ it follows that $\vec{b} \in [\vec{0}]^+ \cap [-\vec{0}]^+ = [\vec{0}]^+$. Since, by part (e), $\vec{b} \notin [\vec{0}]^+$ it follows that $\vec{b} \notin [\vec{a}]^+ \cap [-\vec{a}]^+$ — that is, $[\vec{a}]^+ \cap [-\vec{a}]^+ = \emptyset$.

2. (a) Suppose that $\vec{b} \in [\vec{a}]^*$. It follows, by definition, that $\vec{b} \neq \vec{0}$ and that there is a number — say, b — such that $b > 0$ and $\vec{b} = \vec{a}b$. Since $b > 0$, $b \neq 0$ and it follows that $\vec{a} = \vec{b} \cdot /b$. Since $b > 0$, $/b > 0$ and it follows that $\exists_{x > 0}\ \vec{a} = \vec{b}x$.

   Moreover, $\vec{a} \neq \vec{0}$ [either by Exercise 1(d) or because $\vec{a} = \vec{b} \cdot /b$, $\vec{b} \neq \vec{0}$, and $/b \neq 0$]. So, by definition, $\vec{a} \in [\vec{b}]^*$. Hence, if $\vec{b} \in [\vec{a}]^*$ then $\vec{a} \in [\vec{b}]^*$.

   (b) Suppose that $\vec{b} \in [\vec{a}]^*$. Suppose, also, that $\vec{c} \in [\vec{b}]^*$. It follows that $\vec{c} \neq \vec{0}$ and that there is a number — say, c — such that $c > 0$ and $\vec{c} = \vec{b}c$. Since $\vec{b} \in [\vec{a}]^*$ there is a number — say, b — such that $b > 0$ and $\vec{b} = \vec{a}b$. It follows that $\vec{c} = (\vec{a}b)c = \vec{a}(bc)$ and, since $b > 0$ and $c > 0$, $bc > 0$. Since $\vec{c} \neq \vec{0}$ it follows that $\vec{c} \in [\vec{a}]^*$. Hence, if $\vec{c} \in [\vec{b}]^*$ then $\vec{c} \in [\vec{a}]^*$ — that is, $[\vec{b}]^* \subseteq [\vec{a}]^*$.

   It follows that if $\vec{b} \in [\vec{a}]^*$ then $[\vec{b}]^* \subseteq [\vec{a}]^*$. So, if $\vec{a} \in [\vec{b}]^*$ then $[\vec{a}]^* \subseteq [\vec{b}]^*$. But, by part (a), if $\vec{b} \in [\vec{a}]^*$ then $\vec{a} \in [\vec{b}]^*$. Hence, if $\vec{b} \in [\vec{a}]^*$ then $[\vec{a}]^* = [\vec{b}]^*$.

Answers for Part B [cont.]

3. Yes [it is even an instance of Exercise 2(a)].; No [$[\vec{0}]^* = [\vec{0}]^*$, but $\vec{0} \notin [\vec{0}]^*$].

4. (a) Suppose that $\vec{c} \in [-\vec{a}]^*$. It follows that $\vec{c} \neq \vec{0}$ and that, for some number — say, c — $c > 0$ and $\vec{c} = -\vec{a}c$. Since $-\vec{a}c = -(\vec{a}c)$ it follows that $-\vec{c} = \vec{a}c$, where $c > 0$. Since $\vec{c} \neq \vec{0}$, $-\vec{c} \neq \vec{0}$. So, $-\vec{c} \in [\vec{a}]^*$. Hence, if $\vec{c} \in [-\vec{a}]^*$ then $-\vec{c} \in [\vec{a}]^*$.

   (b) After (a), what remains to be shown is that if $-\vec{c} \in [\vec{a}]^*$ then $\vec{c} \in [-\vec{a}]^*$. But, by (a), if $-\vec{c} \in [--\vec{a}]^*$ then $--\vec{c} \in [-\vec{a}]^*$. Hence, since $--\vec{a} = \vec{a}$ and $--\vec{c} = \vec{c}$, if $-\vec{c} \in [\vec{a}]^*$ then $\vec{c} \in [-\vec{a}]^*$.

   (c) By (b), $\vec{c} \in [-\vec{a}]^*$ if and only if $-\vec{c} \in [\vec{a}]^*$ and $-\vec{c} \in [\vec{b}]^*$ if and only if $\vec{c} \in [-\vec{b}]^*$. Hence, if $[\vec{a}]^* = [\vec{b}]^*$ then $\vec{c} \in [-\vec{a}]^*$ if and only if $\vec{c} \in [-\vec{b}]^*$ — that is, if $[\vec{a}]^* = [\vec{b}]^*$ then $[-\vec{a}]^* = [-\vec{b}]^*$.

   (d) By (c), if $[\vec{a}]^* = [-\vec{b}]^*$ then $[-\vec{a}]^* = [--\vec{b}]^*$. Since $--\vec{b} = \vec{b}$ it follows that if $[\vec{a}]^* = [-\vec{b}]^*$ then $[\vec{b}]^* = [-\vec{a}]^*$. From this it follows [by instantiation] that if $[\vec{b}]^* = [-\vec{a}]^*$ then $[\vec{a}]^* = [-\vec{b}]^*$. Hence, $[\vec{a}]^* = [-\vec{b}]^*$ if and only if $[\vec{b}]^* = [-\vec{a}]^*$.

5. No. If $\vec{a}$ and $\vec{b}$ have different directions then they will have different senses but will not have opposite senses. [If, however, $\vec{a}$ and $\vec{b}$ have the same direction then either they have the same sense or they have opposite senses.]

6. No. If $\vec{a} = \vec{0} = \vec{b}$ then $\vec{a}$ and $\vec{b}$ have the same sense and, since $\vec{b} = -\vec{a}$, they also have opposite senses. [But, proper translations which have the same sense do not have opposite senses.]

Answers for Part C

$$\dfrac{\dfrac{\dfrac{\overset{\dagger}{p \text{ and } q}}{p}}{(p \text{ and } q) \Longrightarrow p}{\,\dagger} \qquad \dfrac{\dfrac{\overset{*}{p} \qquad \dfrac{\overset{*}{p} \quad p \Longrightarrow q}{q}}{p \text{ and } q}}{p \Longrightarrow (p \text{ and } q)}{\,*}}{p \Longleftrightarrow (p \text{ and } q)}$$

$$\dfrac{\dfrac{\dfrac{\overset{\dagger}{q}}{p \text{ or } q}}{q \Longrightarrow (p \text{ or } q)}{\,\dagger} \qquad \dfrac{\dfrac{\overset{*}{p \text{ or } q} \quad p \Longrightarrow q \quad q \overset{\#}{\Longrightarrow} q}{q}}{(p \text{ or } q) \Longrightarrow q}{\,*}}{(p \text{ or } q) \Longleftrightarrow q}$$

$$\dfrac{\dfrac{\dfrac{p \Longleftrightarrow q}{p \Longrightarrow q}}{(p \text{ or } q) \Longleftrightarrow q} \qquad \dfrac{\dfrac{p \Longleftrightarrow q}{q \Longrightarrow p}}{q \Longleftrightarrow (q \text{ and } p)}}{(p \text{ or } q) \Longleftrightarrow (q \text{ and } p)}$$

3. Is the converse of Exercise 2(a) a theorem? How about the converse of Exercise 2(b)?
4. Prove:
   (a) $\vec{c} \in [-\vec{a}]^+ \longrightarrow -\vec{c} \in [\vec{a}]^+$
   (b) $\vec{c} \in [-\vec{a}]^+ \longleftrightarrow -\vec{c} \in [\vec{a}]^+$ [*Hint*: Recall that oppositing is its own inverse.]
   (c) $[\vec{a}]^+ = [\vec{b}]^+ \longrightarrow [-\vec{a}]^+ = [-\vec{b}]^+$
   (d) $[\vec{a}]^+ = [-\vec{b}]^+ \longleftrightarrow [\vec{b}]^+ = [-\vec{a}]^+$
5. Suppose that $\vec{a}$ and $\vec{b}$ do not have the same sense. Does it follow that $\vec{a}$ and $\vec{b}$ have opposite senses? Explain.
6. Suppose that $\vec{a}$ and $\vec{b}$ have the same sense. Does it follow that they do not have opposite senses? Explain.

Part C

Show that inferences of any of the following forms are valid:

$$\frac{p \longrightarrow q}{p \longleftrightarrow (p \text{ and } q)} \qquad \frac{p \longrightarrow q}{(p \text{ or } q) \longleftrightarrow q}$$

$$\frac{p \longleftrightarrow q}{(p \text{ or } q) \longleftrightarrow (q \text{ and } p)}$$

[An inference of the first kind was used in proving Theorem 7-13; an inference of the third kind—with Exercise 4(d) of Part B as a premiss—justifies the phrase 'each—or either' in the paragraph following Theorem 7-13 on page 300.]

Part D

1. Consider a proper translation $\vec{a}$ and suppose that $\vec{b}$ is in the direction of $\vec{a}$.
   (a) Can $\vec{b} = \vec{0}$?
   (b) If $\vec{b} \neq \vec{0}$ does it follow that $\vec{b}$ has the same sense as $\vec{a}$?
   (c) If your answer to (b) is 'No.', what can you say about the sense of any non-$\vec{0}$ vector in the direction of $\vec{a}$?
2. Prove:

   Theorem 7-14 $[\vec{a}] = [\vec{a}]^+ \cup \{\vec{0}\} \cup [-\vec{a}]^+$

   [*Hint*: In proving "$\subseteq$", consider two cases, $\vec{a} = \vec{0}$ and $\vec{a} \neq \vec{0}$.]
3. What is true of $(\vec{a}, \vec{b})$ if $\vec{a}$ and $\vec{b}$ have the same sense? If $\vec{a}$ and $\vec{b}$ have opposite senses?
4. Show that $(\vec{a}, \vec{b})$ is linearly dependent if and only if $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ have the same sense or opposite senses. [*Hint*. If $(\vec{a}, \vec{b})$ is linearly dependent then $\vec{a} = \vec{0}$ or . . . .]
5. Show that $\vec{a}$ and $\vec{b}$ have the same direction if and only if they have the same sense or opposite senses.

Answers for Part D

1. (a) Yes. (b) No.

   (c) Any non-$\vec{0}$ vector in the direction of $\vec{a}$ has the same sense as $\vec{a}$ or has the sense opposite to that of $\vec{a}$. [For, if $\vec{0} \neq \vec{b} \in [\vec{a}]$ then there is a number — say, b — such that $\vec{b} = \vec{a}b$ and $b \neq 0$. Since $b \neq 0$, either $b > 0$ or $-b > 0$. In the first case, since $\vec{b} \neq \vec{0}$, $\vec{b} \in [\vec{a}]^+$. In the second case, since $\vec{b} \neq \vec{0}$ and since $\vec{a}b = -\vec{a} \cdot -b$, $\vec{b} \in [-\vec{a}]^+$.]

2. Suppose that $\vec{a} = \vec{0}$. Since $[\vec{0}] = \{\vec{0}\}$, $[\vec{0}]^+ = \emptyset$, and $-\vec{0} = \vec{0}$, it follows that $[\vec{a}] = [\vec{a}]^+ \cup \{\vec{0}\} \cup [-\vec{a}]^+$.

   Suppose that $\vec{a} \neq \vec{0}$. Suppose that $\vec{b} \in [\vec{a}]$. By definition, there is a number — say, b — such that $\vec{b} = \vec{a}b$. If $\vec{b} \neq \vec{0}$ then [as in the explanation, above, for Exercise 1(c)], $\vec{b} \in [\vec{a}]^+$ or $\vec{b} \in [-\vec{a}]^+$. If $\vec{b} = \vec{0}$ then $\vec{b} \in \{\vec{0}\}$. Hence, if $\vec{b} \in [\vec{a}]$ then $\vec{b} \in [\vec{a}]^+ \cup \{\vec{0}\} \cup [-\vec{a}]^+$. On the other hand, if $\vec{b} \in [\vec{a}]^+$ then $\exists_{x>0}\ \vec{b} = \vec{a}x$ and, so, $\exists_x\ \vec{b} = \vec{a}x$ — that is, $\vec{b} \in [\vec{a}]$. Also, if $\vec{b} \in [-\vec{a}]^+$ then $\vec{b} \in [-\vec{a}] = [\vec{a}]$. Since, finally, $\vec{0} \in [\vec{a}]$ it follows that $[\vec{a}]^+ \cup \{\vec{0}\} \cup [-\vec{a}]^+ \subseteq [\vec{a}]$. Hence, the theorem. [For $\vec{a} \neq \vec{0}$, this theorem will tell us that if $A \in \ell$ then ℓ is the union of two opposite rays whose common vertex is A. See page 28 and Exercise 2 of Part C on page 307. Since $[\vec{a}]^+ \cap [-\vec{a}]^+ = \emptyset$, it also will tell us that a point divides a line into two disjoint half-lines.]

3. If $\vec{a}$ and $\vec{b}$ have the same sense [or have opposite senses] then $(\vec{a}, \vec{b})$ is linearly dependent.

4. If $(\vec{a}, \vec{b})$ is linearly dependent then $\vec{a} = \vec{0}$ or $\vec{b} \in [\vec{a}]$. [Theorem 6-14] So, by Theorem 7-14, if $(\vec{a}, \vec{b})$ is linearly dependent then $\vec{a} = \vec{0}$ or $\vec{b} \in [\vec{a}]^+$ or $\vec{b} = \vec{0}$ or $\vec{b} \in [-\vec{a}]^+$ — that is, $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ have the same sense or opposite senses.

5. Suppose that $\vec{a}$ and $\vec{b}$ have the same sense. It follows that $[\vec{a}]^+ = [\vec{b}]^+$ and, by Exercise 4(c) of Part B, that $[-\vec{a}]^+ = [-\vec{b}]^+$. So, by Theorem 7-14, $[\vec{a}] = [\vec{b}]$. Hence, if $\vec{a}$ and $\vec{b}$ have the same sense then they have the same direction. A similar argument using Exercise 4(d) shows that if $\vec{a}$ and $\vec{b}$ have opposite senses then they have the same direction.

   Suppose, now, that $\vec{a}$ and $\vec{b}$ have the same direction. It follows that either $\vec{a} = \vec{0} = \vec{b}$ or $\vec{a} \neq \vec{0} \neq \vec{b}$. In the first case both $\vec{a}$ and $\vec{b}$ have the sense $\emptyset$. [The second case has been treated previously. See explanation for Exercise 1(c).]

Part E

Given three noncollinear points A, B, and C, let $\vec{a} = B - A$, $\vec{b} = C - B$, $\vec{c} = C - A$, $M = A + \vec{a}\frac{3}{2}$, $N = B + \vec{b} \cdot -\frac{1}{2}$, $R = A + \vec{a}\frac{1}{3}$, and $S = B + \vec{b}\frac{2}{3}$.

1. Draw an appropriate picture for these conditions.
2. (a) Find numbers m and n such that $N - M = \vec{a}m + \vec{b}n$.
   (b) Find numbers r and s such that $S - R = \vec{a}r + \vec{b}s$.
   (c) Use the results of (a) and (b) to express $S - R$ as a linear combination of $N - M$.
3. Show the following.
   (a) $\overleftrightarrow{MN} \parallel \overleftrightarrow{RS}$ (b) $\overleftrightarrow{RS} \parallel \overleftrightarrow{AC}$
   (c) $N - M$ and $\vec{c}$ are oppositely sensed.
   (d) $S - R$ and $\vec{c}$ have the same sense.
4. Suppose that P and Q are points such that, for some p and q, $P = A + \vec{a}p$ and $Q = B + \vec{b}q$.
   (a) For what values of 'p' and 'q' is it the case that $P = Q$?
   (b) Express $Q - P$ as a linear combination of $\vec{a}$ and $\vec{b}$.
   *(c) Show that $Q - P$ is in the direction of $C - A$ if and only if $p + q = 1$. [*Hint*: Note that $C - A = \vec{a} + \vec{b}$ and that $Q - P$ is in the direction of $\vec{a} + \vec{b}$ if and only if $Q - P = (\vec{a} + \vec{b})t$, for for some t.]

7.08 Subsets of Lines

In Chapter 1 we developed intuitive notions of rays, half-lines, intervals, and segments. In this section we shall arrive at definitions for these kinds of subsets of lines. Perhaps you can guess what will be our definition of, say, '$\overrightarrow{AB}$'. Try to do so.

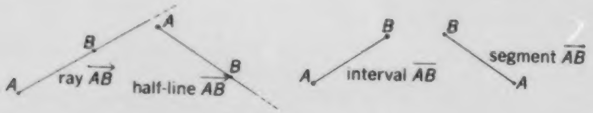


Fig. 7–7

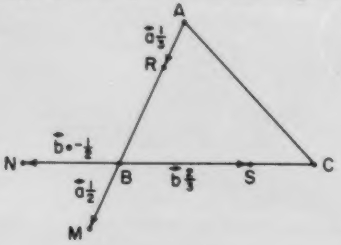
Exercises

Suppose that $P \neq Q$. As you know, there is a unique line, $\overleftrightarrow{PQ}$ which contains P and Q, and

$$\begin{aligned}\overleftrightarrow{PQ} &= \{X: \exists_x\, X = P + (Q - P)x\}\\ &= \{X: X - P \in [Q - P]\}.\end{aligned}$$

Answers for Part E

1. Here is an appropriate picture for the given conditions.



2. (a) From the diagram [or the given information] it follows that $N - M = \vec{a} \cdot -\frac{1}{2} + \vec{b} \cdot -\frac{1}{2}$. So, $m = -\frac{1}{2}$ and $n = -\frac{1}{2}$.
   (b) From the diagram, $S - R = \vec{a}\frac{2}{3} + \vec{b}\frac{2}{3}$. So, $r = \frac{2}{3} = s$.
   (c) $S - R = (\vec{a} + \vec{b})\frac{2}{3} = (\vec{a} + \vec{b}) \cdot -\frac{1}{2} \cdot -\frac{4}{3} = (\vec{a} \cdot -\frac{1}{2} + \vec{b} \cdot -\frac{1}{2}) \cdot -\frac{4}{3}$ $= (N - M) \cdot -\frac{4}{3}$.
3. (a) $[\overleftrightarrow{MN}] = [N - M] = [\vec{a} + \vec{b}] = [S - R] = [\overleftrightarrow{RS}]$. So, $\overleftrightarrow{MN} \parallel \overleftrightarrow{RS}$.
   (b) $[\overleftrightarrow{RS}] = [S - R] = [\vec{a} + \vec{b}] = [C - A] = [\overleftrightarrow{AC}]$. So, $\overleftrightarrow{RS} \parallel \overleftrightarrow{AC}$.
   (c) From the given information and Postulate 3, $\vec{c} = \vec{a} + \vec{b}$ and $N - M = (\vec{a} + \vec{b}) \cdot -\frac{1}{2}$. So, $\exists_{x<0}\, N - M = \vec{c}x$. Hence $N - M$ and $\vec{c}$ are oppositely sensed.
   (d) From the given information, $S - R = (\vec{a} + \vec{b})\frac{2}{3} = \vec{c}\frac{2}{3}$. So, $\exists_{x>0}\, S - R = \vec{c}x$. Hence $S - R$ and $\vec{c}$ have the same sense.
4. (a) 1 [for 'p'] and 0 [for 'q']. [$A + \vec{a}p = B + \vec{b}q$ if and only if $A + (B - A)p = B + (C - B)q$ — that is, if and only if $(A - B)(1 - p) + (C - B)q = \vec{0}$. Since $\{A, B, C\}$ is noncollinear, $(A - B, C - B)$ is linearly independent and, so, the preceding equation is satisfied if and only if $1 - p = 0 = q$.]
   (b) $Q - P = (B + \vec{b}q) - (A + \vec{a}p) = (B - A) + (\vec{b}q - \vec{a}p) = \vec{a}(1 - p) + \vec{b}q$
   (c) $Q - P \in [C - A]$ if and only if $\vec{a}(1 - p) + \vec{b}q \in [\vec{c}]$. Since $\vec{c} = \vec{a} + \vec{b}$ it follows that $Q - P \in [C - A]$ if and only if $\vec{a}[(1 - p) - t] + \vec{b}(q - t) = \vec{0}$, for some t. Since $(\vec{a}, \vec{b})$ is linearly independent, this is the case if and only if $1 - p = q$.

We shall define '$\overrightarrow{AB}$' so that

$$\overrightarrow{AB} = \{X: X - A \in [B - A]^+\}.$$

[See Definition 7-9 on page 305.]

In these exercises we shall consider sets p_1, p_2, p_3, p_4, p_5, and p_6, where

$$\begin{aligned} p_1 &= \{X: \exists_{x > 0}\, X = P + (Q - P)x\},\\ p_2 &= \{X: \exists_{x \geq 0}\, X = P + (Q - P)x\},\\ p_3 &= \{X: \exists_{x \leq 0}\, X = P + (Q - P)x\},\\ p_4 &= \{X: \exists_{x < 1}\, X = P + (Q - P)x\},\\ p_5 &= \{X: \exists_{x \leq 1}\, X = P + (Q - P)x\},\\ \text{and}\quad p_6 &= \{X: \exists_{x \leq 1}\, X = Q + (P - Q)x\}. \end{aligned}$$

1. Are any of these sets *not* subsets of $\overleftrightarrow{PQ}$?
2. Graph each of the sets. [This calls for six pictures. It will help if you line up the six marks you make for P vertically and do the same with your marks for Q.]
3. How many sets have you pictured? Explain.
4. Each of the sets we are considering is one of the kinds illustrated preceding these exercises. Which are rays? Which are half-lines? Which are intervals? Segments?
5. Intuitively, $p_1 = \overrightarrow{PQ}$; and this will turn out to be so, formally, once we have adopted the appropriate definition for 'half-line'. Write an equation like that in the preceding sentence for each of the other sets we are considering. [In one case use an oppositing sign.]
6. Use your answers for Exercise 2 to aid you in graphing each of the following sets.

   (a) $p_1 \cap p_3$ (b) $p_1 \cup p_3$ (c) $p_1 \cap p_4$ (d) $p_1 \cup p_4$

   (e) $p_2 \cap p_3$ (f) $p_2 \cup p_3$ (g) $p_2 \cap p_5$ (h) $p_2 \cup p_5$
7. For each of the sets listed in Exercise 6, tell what kind of set it is and give a simple name for it. [When possible use the kind of notation illustrated preceding the exercises.]
8. As you discovered from your graphs, it appears that $p_2 = p_6$. Show that this is the case by completing the following argument:

$$\begin{aligned} P + (Q - P)a &= P + (P - Q)\cdot \underline{\quad\quad} \\ &= [Q + \underline{\quad\quad}] + (P - Q)\cdot -a \\ &= Q + (P - Q)\cdot \underline{\quad\quad}, \end{aligned}$$

   and

$$a \geq 0 \longleftrightarrow 1 - a \underline{\quad\quad}.$$

9. (a) Draw a horizontal picture of $\overleftrightarrow{PQ}$. Below it draw a picture of the number line with the marks for 0 and 1 vertically below those for P and Q, respectively. On your picture of $\overleftrightarrow{PQ}$, graph p_2; on your picture of the number line, graph q_2, where

$$q_2 = \{x: x \geq 0\}.$$

   The similarity of appearance should be striking.

   (b) Repeat part (a) for p_5 and an appropriately defined subset q_5 of ℛ.

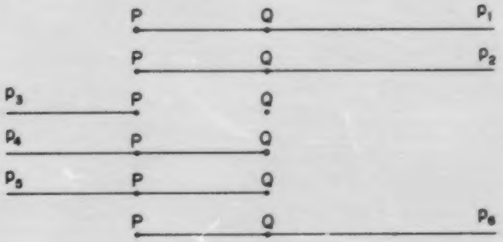
   Write a brace description (c) Of $q_2 \cap q_5$. *(d) Of $p_2 \cap p_5$.
10. Prove:

Theorem 7-15 $A + (B - A)a = A + (B - A)b \longleftrightarrow a = b\ [A \neq B]$

Answers for Exercises

1. No.

2.

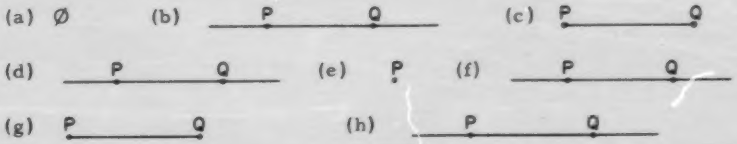


3. Five. $p_2 = p_6$.

4. p_2, p_3, p_5, and p_6 are rays; p_1 and p_4 are half-lines; there are no intervals or segments pictured.

5. $p_2 = \overrightarrow{PQ}$; $p_3 = -\overrightarrow{PQ}$; $p_4 = \overrightarrow{QP}$; $p_5 = \overrightarrow{QP}$; $p_6 = \overrightarrow{PQ}$.

6.



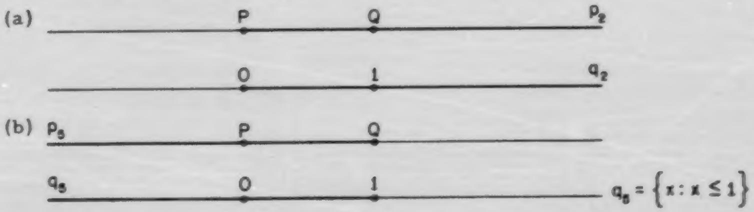
7. (a) empty set; Ø (b) line; $\overleftrightarrow{PQ}$ (c) interval; $\overline{PQ}$

   (d) line; $\overleftrightarrow{PQ}$ (e) singleton; $\{P\}$ (f) line; $\overleftrightarrow{PQ}$

   (g) segment; $\overline{PQ}$ (h) line; $\overleftrightarrow{PQ}$

8. $-a$; $(P - Q)$; $(1 - a)$; ≤ 1

9.



   (c) $q_2 \cap q_5 = \{x: 0 \leq x \leq 1\}$

   ☆(d) $p_2 \cap p_5 = \{X: \exists_x\ (0 \leq x \leq 1 \text{ and } X = P + (Q - P)x)\}$

10. Suppose that $A + (B - A)a = A + (B - A)b$ and $A \neq B$. It follows that $(B - A)(a - b) = \vec{0}$ and $B - A \neq \vec{0}$. So, $a - b = 0$. Thus, $a = b$. Hence, if $A + (B - A)a = A + (B - A)b$ then $a = b\ [A \neq B]$.

    Suppose that $a = b$. Then $(B - A)a = (B - A)b$ so that $A + (B - A)a = A + (B - A)b$. Hence, if $a = b$ then $A + (B - A)a = A + (B - A)b$.

Half-Lines and Rays

Theorem 7–15 shows how, given an ordered pair (P, Q) of *distinct* points, one can establish a one-to-one correspondence between $\mathscr{R}$ and $\{X: \exists_x\, X = P + (Q - P)x\}$ – that is, between $\mathscr{R}$ and the line $\overleftrightarrow{PQ}$ – in which 0 corresponds with P and 1 with Q. Intuitively, this correspondence has a much more special property than that of being merely one-to-one – it seems to preserve order. For example, points of $\overleftrightarrow{PQ}$ which we would think of as being on "the Q-side of P" are just those which correspond with positive numbers and the greater the corresponding number, the farther a point is from P. It would seem natural then, to formalize the first part of this intuition by defining the Q-side of P – that is, the half-line $\overrightarrow{PQ}$ – so that

$$(1) \qquad \overrightarrow{PQ} = \{X: \exists_{x > 0}\, X = P + (Q - P)x\} \qquad [P \neq Q]$$

[To formalize the second part of our intuition – that greater positive numbers correspond with points which are farther from P – we would need a notion of distance. Such a notion will be introduced later in the course.]

The only difficulty with taking (1), itself, as a definition is that it bears the restriction '$P \neq Q$'. Intuitively, in case $P = Q$ there should be no Q-side of P or, better, the Q-side of P should be $\varnothing$. This is a rather trivial case but would be troublesome if we didn't take care of it. Our discussion of sense in the preceding section shows how to do this; we should formulate our definition so that, for any P and Q,

$$\overrightarrow{PQ} = \{X: X \neq P \text{ and } \exists_{x > 0}\, X = P + (Q - P)x\}.$$

The easy way is to adopt:

Definition 7–9 (a) $\overrightarrow{A[\vec{a}]^+} = \{X: X - A \in [\vec{a}]^+\}$
(b) $\overrightarrow{AB} = \overrightarrow{A[B - A]^+}$

[Just as you read '$\overleftrightarrow{AB}$' as 'double arrow AB', read '$\overrightarrow{AB}$' as 'arrow AB'.] Note that (1) is a consequence of Definition 7–9 and an earlier theorem. [What theorem?]

When $A \neq B$ it is proper to read '$\overrightarrow{AB}$' as 'half-line AB' [and to read '$\overleftrightarrow{AB}$' as 'line AB'.].

Having introduced half-lines it is easy for us to introduce rays:

Definition 7–10 (a) $\overset{\bullet}{\overrightarrow{A[\vec{a}]^+}} = \{A\} \cup \overrightarrow{A[\vec{a}]^+}$
(b) $\overset{\bullet}{\overrightarrow{AB}} = \{A\} \cup \overrightarrow{AB}$

[Read '$\overset{\bullet}{\overrightarrow{AB}}$' as 'dot arrow AB'.] As in the case of lines and half-lines, when $A \neq B$ it is proper to read '$\overset{\bullet}{\overrightarrow{AB}}$' as 'ray AB'. What set is $\overset{\bullet}{\overrightarrow{AA}}$?

Sentence (1) on page 305 is a consequence of Definition 7-9 — which gives:

$$\overrightarrow{PQ} = \{X: X - P \in [Q - P]^+\}$$

and Theorem 7-13(a) which, for $Q - P \neq \vec{0}$, gives:

$$\vec{a} \in [Q - P]^+ \Longleftrightarrow \exists_{x > 0}\, \vec{a} = (Q - P)x$$

[Of course, we also need '$P \neq Q \Longleftrightarrow Q - P \neq \vec{0}$' and '$C - P = \vec{b} \Longleftrightarrow C = P + \vec{b}$' in addition to Definition 7-9 and Theorem 7-13(a).]

For $Q - P = \vec{0}$, Definition 7-9 and Theorem 7-13(a) yield '$\overrightarrow{PQ} = \varnothing$', as expected. In short, for any point A, $\overrightarrow{AA} = \varnothing$ and, by Definition 7-10(b), $\overset{\bullet}{\overrightarrow{AA}} = \{A\}$.

Theorem 7-16 (a) $\overrightarrow{AB} = \{X: \exists_{x>0}\, X = A + (B - A)x\}$ $[A \neq B]$

(b) $\overset{\bullet\!\!\longrightarrow}{AB} = \{X: \exists_{x\geq 0}\, X = A + (B - A)x\}$

Exercises

Part A

1. **(a)** Complete: $\overrightarrow{AB} = \{X: X - A \in$ ____$\}$
   (b) Prove Theorem 7-16(a). [*Hint*: See the note following Definition 7-9.]
   (c) Prove Theorem 7-16(b). [*Hint*: Consider two cases.]
2. **(a)** Prove:

   Theorem 7-17 $C \in \overrightarrow{AB} \Longrightarrow (\overrightarrow{AC} = \overrightarrow{AB}$ and $\overset{\bullet\!\!\longrightarrow}{AC} = \overset{\bullet\!\!\longrightarrow}{AB})$

   [*Hint*: By Definition 7-9(b), $\overrightarrow{AC} = \overrightarrow{AB}$ if $[C - A]^+ = [B - A]^+$.]
   (b) If $\overrightarrow{AC} = \overrightarrow{AB}$ does it follow that $C \in \overrightarrow{AB}$?
   (c) If $C \in \overrightarrow{AB}$ what follows about A and B?
3. Show that if $A \neq B$ then $\overset{\bullet\!\!\longrightarrow}{AB} \neq \overset{\bullet\!\!\longrightarrow}{BA}$. [*Hint*: It is not difficult to describe a point of $\overset{\bullet\!\!\longrightarrow}{AB}$ which, *if* $A \neq B$, does not belong to $\overset{\bullet\!\!\longrightarrow}{BA}$.]

*4. Show that if $A \neq C$ then $\overrightarrow{AB} \neq \overrightarrow{CB}$. [*Hint*: The case in which $A = B$ is very easy. Assuming that $A \neq B$, the case in which $C \notin \overleftrightarrow{AB}$ is easy; that in which $C \in \overleftrightarrow{AB}$ is like Exercise 3, but splits into two cases, one of which is more difficult.]

5. If $\overrightarrow{AB} = \overrightarrow{CD}$ does it follow that $A = C$? Explain your answer.

Part B

In Chapter 1 we dealt with vertices of rays and half-lines. As you recall, for $A \neq B$, A is the *vertex* of the ray $\overset{\bullet\!\!\longrightarrow}{AB}$ and of the half-line $\overrightarrow{AB}$. We also spoke of the senses of rays and half-lines and, as you probably guess, the *sense* of $\overset{\bullet\!\!\longrightarrow}{AB}$ [and of $\overrightarrow{AB}$] is $[B - A]^+$.

Before, however, we are entitled to call A *the* vertex of $\overset{\bullet\!\!\longrightarrow}{AB}$ we need to be sure that there is no other point which the same conventions will require us to call the vertex of $\overset{\bullet\!\!\longrightarrow}{AB}$. Explicitly, we need to make sure that if $\overset{\bullet\!\!\longrightarrow}{AB} = \overset{\bullet\!\!\longrightarrow}{CD}$ then $A = C$. That rays and half-lines do have unique senses follows from the next two theorems and their two corollaries.

Theorem 7-18 $\overset{\bullet\!\!\longrightarrow}{AB} = \overset{\bullet\!\!\longrightarrow}{CD} \Longrightarrow A = C$

Corollary $\overset{\bullet\!\!\longrightarrow}{AB} = \overset{\bullet\!\!\longrightarrow}{CD} \Longrightarrow [B - A]^+ = [D - C]^+$

Theorem 7-19 $\overrightarrow{AB} = \overrightarrow{CD} \Longrightarrow A = C$ $[A \neq B]$

Corollary $\overrightarrow{AB} = \overrightarrow{CD} \Longrightarrow [B - A]^+ = [D - C]^+$

We recommend that if Parts A and B are made part of a single homework assignment, that you assign a team of students to each derivation rather than require each student to do each derivation. Part C makes a nice class activity. Parts D and E constitute a homework assignment that is reasonable for all to do.

Answers for Part A

1. (a) $[B - A]^+$

   (b) By Definition 7-9, $C \in \overrightarrow{AB}$ if and only if $C - A \in [B - A]^+$. For $A \neq B$, $B - A \neq \vec{0}$ and it follows by Theorem 7-13(a) that $C - A \in [B - A]^+$ if and only if $\exists_{x>0}\, C - A = (B - A)x$. Since ($a > 0$ and $C - A = (B - A)a$) if and only if ($a > 0$ and $C = A + (B - A)a$) it follows that $\exists_{x>0}\, C - A = (B - A)x$ if and only if $\exists_{x>0}\, C = A + (B - A)x$. Hence,

   $$C \in \overrightarrow{AB} \Longleftrightarrow \exists_{x>0}\, C = A + (B - A)x$$

   and, consequently, $\overrightarrow{AB} = \{X: \exists_{x>0}\, X = A + (B - A)x\}$ [for $A \neq B$].

   (c) For $A = B$, $\overset{\bullet\!\!\longrightarrow}{AB} = \{A\} = \{X: X = A\} = \{X: \exists_{x\geq 0}\, X = A + (B - A)x\}$. For, in this case, $A = A + \vec{0}a = A + (B - A)a$.

   For $A \neq B$, $C \in \overset{\bullet\!\!\longrightarrow}{AB} \Longleftrightarrow (C = A$ or $\exists_{x>0}\, C = A + (B - A)x)$, by Definition 7-10(b) and part (b), above. Suppose, now, that $a \geq 0$ and $C = A + (B - A)a$. From this it follows that

   $$(a = 0 \text{ and } C = A + (B - A)a) \text{ or } (a > 0 \text{ and } C = A + (B - A)a),$$

   from which it follows that $C = A$ or $\exists_{x>0}\, C = A + (B - A)x$. Hence,

   $$\exists_{x\geq 0}\, C = A + (B - A)x$$
   $$\Longrightarrow$$
   $$(C = A \text{ or } \exists_{x>0}\, C = A + (B - A)x).$$

   To establish the converse, note first that if $C = A$ then ($0 \geq 0$ and $C = A + (B - A)0$), so that

   $$C = A \Longrightarrow \exists_{x\geq 0}\, C = A + (B - A)x.$$

   Second, if ($a > 0$ and $C = A + (B - A)a$ then ($a \geq 0$ and $C = A + (B - A)a$), so that

   $$\exists_{x>0}\, C = A + (B - A)x \Longrightarrow \exists_{x\geq 0}\, C = A + (B - A)x.$$

   From these two results it follows that

   $$(C = A \text{ or } \exists_{x>0}\, C = A + (B - A)x)$$
   $$\Longleftrightarrow$$
   $$\exists_{x\geq 0}\, C = A + (B - A)x.$$

   Hence, for $A \neq B$, $C \in \overset{\bullet\!\!\longrightarrow}{AB} \Longleftrightarrow \exists_{x\geq 0}\, C = A + (B - A)x$ and, consequently, $\overset{\bullet\!\!\longrightarrow}{AB} = \{X: \exists_{x\geq 0}\, X = A + (B - A)x\}$.

   [We have gone into some detail in order to illustrate the rules for 'or' and '$\exists$'. Much less need be required of students.]

2. (a) Suppose that $C \in \overrightarrow{AB}$. It follows, by definition, that $C - A \in [B - A]^{+}$ and so, by Exercise 2(b) of Part B on page 301, that $[C - A]^{+} = [B - A]^{+}$. So, by definition $\overrightarrow{AC} = \overrightarrow{AB}$ and, by this, $\overleftrightarrow{AC} = \overleftrightarrow{AB}$. Hence, if $C \in \overrightarrow{AB}$ then $(\overrightarrow{AC} = \overrightarrow{AB}$ and $\overleftrightarrow{AC} = \overleftrightarrow{AB})$.

   (b) No. [It might be the case that $C = A = B$, in which case $\overrightarrow{AC} = \emptyset = \overrightarrow{AB}$ and $C \notin \overrightarrow{AB}$.]

   (c) If $C \in \overrightarrow{AB}$ then $A \neq B$.

3. By Theorem 7-16(b), $A + (B - A)2 \in \overleftrightarrow{AB}$. Now, $A + (B - A)2 = (A + (B - A)) + (B - A) = B + (A - B) \cdot -1$. So, since $-1 \not> 0$ it follows by Theorem 7-16(b) and Theorem 7-15 that, for $A \neq B$, $A + (B - A)2 \notin \overleftrightarrow{BA}$. Hence, for $A \neq B$, $\overleftrightarrow{AB} \neq \overleftrightarrow{BA}$. [Of course, in place of 2 one might use any real number greater than 1. The result of Exercise 3 is used in proving Theorem 7-18 according to which a ray has a unique vertex. This, despite its intuitive obviousness, is a rather important theorem. The corresponding theorem for half-lines [Theorem 7-19] is much more difficult to prove. The difficulties are concentrated in Exercise 4.]

4. Suppose that $A \neq C$. If $A = B$ then $B \neq C$ and $\overrightarrow{AB} = \emptyset \neq \overrightarrow{CB}$. Suppose, then, that $A \neq B$. If $C \notin \overleftrightarrow{AB}$ then $C \neq B$ and $\overleftrightarrow{CB} \cap \overleftrightarrow{AB} = \{B\}$. So, any point of $\overrightarrow{AB}$ other than B is a point of $\overleftrightarrow{AB}$ which, since it does not belong to $\overleftrightarrow{CB}$, does not belong to $\overrightarrow{CB}$. Hence, if $C \notin \overleftrightarrow{AB}$ then $\overrightarrow{AB} \neq \overrightarrow{CB}$. Suppose, then, that $C \in \overleftrightarrow{AB}$ [as well as that $B \neq A \neq C$]. Since $C \in \overleftrightarrow{AB}$ and $C \neq A$, $C - A = (B - A)c$ where $c \neq 0$. If $c > 0$ then $C = A + (C - A) = A + (B - A)c \in \overrightarrow{AB}$. Since $C \notin \overrightarrow{CB}$ it follows that $\overrightarrow{AB} \neq \overrightarrow{CB}$. To deal with the remaining case — that in which $c < 0$ — note that since $A - C = (A - B)c = [(C - B) + (A - C)]c$ it follows that $(A - C)(1 - c) = (C - B)c$ and that, for $c < 0$, $1 - c > 0$. Hence, if $c < 0$ then $A = C + (A - C) = C + (C - B) \cdot \frac{c}{1 - c} = C + (B - C) \cdot \frac{-c}{1 - c} \in \overrightarrow{CB}$. Since $A \notin \overrightarrow{AB}$ it follows that $\overrightarrow{AB} \neq \overrightarrow{CB}$. [The motivation for the main case — that in which $B \neq A \neq C$ and $C \in \overleftrightarrow{AB}$ is obvious enough. Since $A \neq C \in \overleftrightarrow{AB}$, C belongs either to $\overrightarrow{AB}$ or to its opposite. If $C \in \overrightarrow{AB}$ then $\overrightarrow{AB} \neq \overrightarrow{CB}$ because $C \notin \overrightarrow{CB}$; if $C \in -\overrightarrow{AB}$ then, although $A \notin \overrightarrow{AB}$, $A \in \overrightarrow{CB}$.]

5. No. It may be the case that $B = A \neq C = D$. [This exceptional case cannot occur if $D = B$; and, as shown in Exercise 4, if $D = B$ and $\overrightarrow{AB} = \overrightarrow{CD}$ then $A = C$. See, also, Exercise 3 of Part B, below.]

1. Prove Theorem 7 - 18. [*Hint*: Use Theorem 7 - 17 and Exercise 3 of Part A. Assuming that $\overset{\bullet\!\!\longrightarrow}{AB} = \overset{\bullet\!\!\longrightarrow}{CD}$ show that $A = C$ or $\overset{\bullet\!\!\longrightarrow}{CD} = \overset{\bullet\!\!\longrightarrow}{CA}$. In the latter case $\overset{\bullet\!\!\longrightarrow}{CA} = \overset{\bullet\!\!\longrightarrow}{AB}$. Repeat the argument to show that, in this case, $C = A$ or $\overset{\bullet\!\!\longrightarrow}{AC} = \overset{\bullet\!\!\longrightarrow}{CA}$.]
2. Prove the corollary of Theorem 7 - 18. [*Hint*: If $\overset{\bullet\!\!\longrightarrow}{AB} = \overset{\bullet\!\!\longrightarrow}{CD}$ then $B \in \overset{\bullet\!\!\longrightarrow}{CD}$.]
3. Prove Theorem 7 - 19. [*Hint*: Use Theorem 7 - 17 and Exercise 4 of Part A.]
4. Prove the corollary of Theorem 7 - 19.

Part C

In Chapter 1 we defined the *opposite* of a ray [or a half-line] to be the ray [or half-line] with the same vertex but the opposite sense:

Definition 7 - 11 (a) $-\overrightarrow{A[\vec{a}]^+} = \overrightarrow{A[-\vec{a}]^+}$
(b) $-\overset{\bullet\!\!\longrightarrow}{A[\vec{a}]^+} = \overset{\bullet\!\!\longrightarrow}{A[-\vec{a}]^+}$

What is the intersection of $\overrightarrow{AB}$ and $-\overrightarrow{AB}$? Of $\overset{\bullet\!\!\longrightarrow}{AB}$ and $-\overset{\bullet\!\!\longrightarrow}{AB}$?

Show that

1. $\overleftrightarrow{AB} = \overrightarrow{AB} \cup \{A\} \cup -\overrightarrow{AB}$ [*Hint*: Recall Theorem 7 - 14.]
2. $\overleftrightarrow{AB} = \overrightarrow{AB} \cup -\overset{\bullet\!\!\longrightarrow}{AB}$ [*Hint*: Show that $\{A\} \cup -\overrightarrow{AB} = -\overset{\bullet\!\!\longrightarrow}{AB}$.]
3. $\overleftrightarrow{AB} = \overset{\bullet\!\!\longrightarrow}{AB} \cup \overset{\bullet\!\!\longrightarrow}{BA}$ [*Hint*: Show that $-\overset{\bullet\!\!\longrightarrow}{AB} \subseteq \overset{\bullet\!\!\longrightarrow}{BA}$. $A + (A - B)c = B + (A - B)$___?___.]
4. $\overleftrightarrow{AB} = \overrightarrow{AB} \cup \overrightarrow{BA}$ [$A \neq B$].
5. Suppose that $P \neq Q$.
   (a) What kind of set is the set of all points of $\overrightarrow{PQ}$ which are not in $\overrightarrow{QP}$?
   (b) Give a simple name for the set described in part (a).
   *(c) Prove what you have conjectured in part (b).

Part D

Draw an arrow to describe a proper translation $\vec{a}$ such that $\vec{a} = B - A$.

1. Locate the point M such that $M - A = \vec{a}\frac{1}{2}$. Show that $B - M = \vec{a}\frac{1}{2}$. [*Hint*: By Postulate 3, $(M - A) + (B - M) = B - A$.]
2. Locate the point N such that $N - A = \vec{a}\frac{1}{3}$. Determine n such that $B - N = \vec{a}n$. What is $n + \frac{1}{3}$?
3. Locate P such that $P - A = \vec{a}2$. Determine p such that $B - P = \vec{a}p$. What is $p + 2$?
4. Locate Q such that $Q - A = \vec{a} \cdot -1$. Determine q such that $B - Q = \vec{a}q$.
5. Which of the following have the same sense as $\overset{\bullet\!\!\longrightarrow}{AB}$? Which have the sense of $-\overset{\bullet\!\!\longrightarrow}{AB}$?

$\overset{\bullet\!\!\longrightarrow}{AM}$ $\overset{\bullet\!\!\longrightarrow}{AN}$ $\overset{\bullet\!\!\longrightarrow}{AP}$ $\overset{\bullet\!\!\longrightarrow}{AQ}$ $\overset{\bullet\!\!\longrightarrow}{MB}$ $\overset{\bullet\!\!\longrightarrow}{NB}$
$\overset{\bullet\!\!\longrightarrow}{PB}$ $\overset{\bullet\!\!\longrightarrow}{QB}$ $\overset{\bullet\!\!\longrightarrow}{MN}$ $\overset{\bullet\!\!\longrightarrow}{MP}$ $\overset{\bullet\!\!\longrightarrow}{MQ}$ $\overset{\bullet\!\!\longrightarrow}{PQ}$

6. Given that $R - A = \vec{a}r$, determine s such that $B - R = \vec{a}s$.

Answers for Part B

1. Suppose that $\overset{\bullet\!\!\longrightarrow}{AB} = \overset{\bullet\!\!\longrightarrow}{CD}$. Since $A \in \overset{\bullet\!\!\longrightarrow}{AB}$ it follows that $A \in \overset{\bullet\!\!\longrightarrow}{CD}$. So, by Theorem 7-17, $A = C$ or $A \in \overrightarrow{CD}$. In the latter case $\overset{\bullet\!\!\longrightarrow}{CD} = \overset{\bullet\!\!\longrightarrow}{CA}$ [again, by Theorem 7-17] and since, by assumption, $\overset{\bullet\!\!\longrightarrow}{AB} = \overset{\bullet\!\!\longrightarrow}{CD}$, $\overset{\bullet\!\!\longrightarrow}{AB} = \overset{\bullet\!\!\longrightarrow}{CA}$. Since $C \in \overset{\bullet\!\!\longrightarrow}{CA}$ it follows that $C \in \overset{\bullet\!\!\longrightarrow}{AB}$ and [as before] $C = A$ or $C \in \overrightarrow{AB}$. In the latter case $\overset{\bullet\!\!\longrightarrow}{AC} = \overset{\bullet\!\!\longrightarrow}{AB} = \overset{\bullet\!\!\longrightarrow}{CA}$ and so, by Exercise 3 of Part A, $A = C$. Hence, if $\overset{\bullet\!\!\longrightarrow}{AB} = \overset{\bullet\!\!\longrightarrow}{CD}$ then [in any case] $A = C$.
2. Suppose that $\overset{\bullet\!\!\longrightarrow}{AB} = \overset{\bullet\!\!\longrightarrow}{CD}$. Since $B \in \overset{\bullet\!\!\longrightarrow}{AB}$, $B \in \overset{\bullet\!\!\longrightarrow}{CD}$. So, $B = C$ or $B - C \in [D - C]^+$. Since, by Exercise 1, $A = C$ it follows that $B = A$ or $B - A \in [D - C]^+$. In the latter case it follows from Exercise 2(b) of Part B on page 301 that $[B - A]^+ = [D - C]^+$. In the former case $\overset{\bullet\!\!\longrightarrow}{AB} = \{A\} = \{C\}$ and, so, $\overset{\bullet\!\!\longrightarrow}{CD} = \{C\}$ and $D = C$. So, in this case, $[B - A]^+ = \emptyset = [D - C]^+$. Hence, if $\overset{\bullet\!\!\longrightarrow}{AB} = \overset{\bullet\!\!\longrightarrow}{CD}$ then $[B - A]^+ = [D - C]^+$.
3. Suppose that $A \neq B$ and that $\overrightarrow{AB} = \overrightarrow{CD}$. It follows that $B \in \overrightarrow{AB}$ and, so, that $B \in \overrightarrow{CD}$. From this it follows that $\overrightarrow{CD} = \overrightarrow{CB}$ and, since $\overrightarrow{AB} = \overrightarrow{CD}$, that $\overrightarrow{AB} = \overrightarrow{CB}$. So, by Exercise 4 of Part A, $A = C$. Hence, for $A \neq B$, if $\overrightarrow{AB} = \overrightarrow{CD}$ then $A = C$. [Students who have not solved the optional Exercise 4 of Part A should still be able to apply it to the solution of the present exercise.]
4. Suppose that $\overrightarrow{AB} = \overrightarrow{CD}$. If $A = B$ then $\overrightarrow{CD} = \overrightarrow{AB} = \emptyset$ and, so, $C = D$ and $[B - A]^+ = \emptyset = [D - C]^+$. If $A \neq B$ then $B \in \overrightarrow{AB} = \overrightarrow{CD}$ and, so, $B - C \in [D - C]^+$. Since, by Exercise 3, $A = C$ it follows that $B - A \in [D - C]^+$ and, so, by an earlier result, that $[B - A]^+ = [C - D]^+$.

*

Another corollary of Theorem 7-19 is:

$$\overrightarrow{AB} = \overrightarrow{CD} \Longrightarrow \overset{\bullet\!\!\longrightarrow}{AB} = \overset{\bullet\!\!\longrightarrow}{CD} \quad [A \neq B]$$

The converse [without restriction] is a corollary of Theorem 7-18.

*

The notion of the opposite of a ray was introduced on page 28. Note that, by Definition 7-9(b), $-\overrightarrow{AB} = -\overrightarrow{A[B - A]^+}$ and so, by Definition 7-11(a) [and Theorem 3-5], $-\overrightarrow{AB} = \overrightarrow{A[A - B]^+}$. Similarly, $-\overset{\bullet\!\!\longrightarrow}{AB} = \overset{\bullet\!\!\longrightarrow}{A[A - B]^+}$. Since, by Exercise 1(f) of Part B on page 301, $[B - A]^+ \cap [A - B]^+ = \emptyset$ it follows that $\overrightarrow{AB} \cap -\overrightarrow{AB} = \emptyset$ and $\overset{\bullet\!\!\longrightarrow}{AB} \cap -\overset{\bullet\!\!\longrightarrow}{AB} = \{A\}$.

Note that Theorems 7-18 and 7-19 and their corollaries are needed to justify adopting Definition 7-11. For example, before adopting part (a) of this definition we must be sure that if $\overrightarrow{A[\vec{a}\,]^+} = \overrightarrow{C[\vec{c}\,]^+}$ then $\overrightarrow{A[-\vec{a}\,]^+} = \overrightarrow{C[-\vec{c}\,]^+}$. If this were not the case then the same half-line would, according to the proposed definition, have two opposites. While we would, then, speak of an opposite of a half-line, we could not speak of the opposite. Now, suppose that $\overrightarrow{A[\vec{a}\,]^+} = \overrightarrow{C[\vec{c}\,]^+}$. With [for abbreviation] B = A + $\vec{a}$ and D = C + $\vec{c}$, $\overrightarrow{A[\vec{a}]^+} = \overrightarrow{AB}$ and $\overrightarrow{C[\vec{c}]^+} = \overrightarrow{CD}$. So, $\overrightarrow{AB} = \overrightarrow{CD}$. By the Corollary to Theorem 7-19 it follows that $[B - A]^+ = [C - D]^+$ — that is, that $[\vec{a}]^+ = [\vec{c}]^+$. Hence, by Exercise 4(c) of Part B on page 302, $[-\vec{a}]^+ = [-\vec{c}]^+$. From the theorem itself it follows that, for $\vec{a} \neq \vec{0}$, A = C. So, $\overrightarrow{A[-\vec{a}]^+} = \overrightarrow{C[-\vec{c}]^+}$ at least in the case in which $\vec{a} \neq \vec{0}$. Hence, we have:

$$(*) \qquad \overrightarrow{A[\vec{a}]^+} = \overrightarrow{C[\vec{c}]^+} \Longrightarrow \overrightarrow{A[-\vec{a}]^+} = \overrightarrow{C[-\vec{c}]^+} \quad [\vec{a} \neq \vec{0}]$$

In case $\vec{a} = \vec{0}$, $\overrightarrow{A[\vec{a}]^+} = \emptyset$ and, so, if $\overrightarrow{A[\vec{a}]^+} = \overrightarrow{C[\vec{c}]^+}$ then $\overrightarrow{C[\vec{c}]^+} = \emptyset$ and $\vec{c} = \vec{0}$. So, in case $\vec{a} = \vec{0}$, it follows from the antecedent of (*) that $-\vec{a} = \vec{0} = -\vec{c}$ and, so, that $\overrightarrow{A[-\vec{a}]^+} = \emptyset = \overrightarrow{C[-\vec{c}]^+}$. So, (*) holds without restriction, and Definition 7-11(a) is acceptable. The acceptability of Definition 7-11(b) is slightly simpler to establish because Theorem 7-18 carries no restriction.

Answers for Part C

1. $C \in \overleftrightarrow{AB} \Longleftrightarrow C - A \in [B - A] = [B - A]^+ \cup \{\vec{0}\} \cup [-(B - A)]^+$
   $\Longleftrightarrow (C - A \in [B - A]^+$ or $C - A = \vec{0}$ or $C - A \in [-(B - A)]^+)$
   $\Longleftrightarrow (C \in \overrightarrow{A[B - A]^+}$ or $C = A$ or $C \in -\overrightarrow{A[B - A]^+})$
   $\Longleftrightarrow (C \in \overrightarrow{AB}$ or $C = A$ or $C \in -\overrightarrow{AB})$

   Hence, $\overleftrightarrow{AB} = \overrightarrow{AB} \cup \{A\} \cup -\overrightarrow{AB}$.

2. By Exercise 1, $\overleftrightarrow{AB} = (\{A\} \cup \overrightarrow{AB}) \cup (\{A\} \cup -\overrightarrow{AB})$. By Definition 7-10(b), $\{A\} \cup \overrightarrow{AB} = \overset{\bullet\!\!\longrightarrow}{AB}$. On the other hand, $\{A\} \cup -\overrightarrow{AB} = \{A\} \cup \overrightarrow{A[A - B]^+} = \overset{\bullet\!\!\longrightarrow}{A[A - B]^+} = -\overset{\bullet\!\!\longrightarrow}{AB}$. Hence, $\overleftrightarrow{AB} = \overset{\bullet\!\!\longrightarrow}{AB} \cup -\overset{\bullet\!\!\longrightarrow}{AB}$.

3. This follows from Exercise 2 once it has been shown that $-\overset{\bullet\!\!\longrightarrow}{AB} \subseteq \overset{\bullet\!\!\longrightarrow}{BA}$. To do so, suppose that $C \in -\overset{\bullet\!\!\longrightarrow}{AB} = -\overset{\bullet\!\!\longrightarrow}{A[B - A]^+} = \overset{\bullet\!\!\longrightarrow}{A[A - B]^+}$. It follows [by Theorem 7-16(b)] that, for some number — say, c — $c \geq 0$ and $C = A + (A - B)c = [B + (A - B)] + (A - B)c = B + (A - B)(c + 1) \in \overset{\bullet\!\!\longrightarrow}{BA}$, since $c + 1 \geq 0$. Hence, if $C \in -\overset{\bullet\!\!\longrightarrow}{AB}$ then $C \in \overset{\bullet\!\!\longrightarrow}{BA}$ — that is, $-\overset{\bullet\!\!\longrightarrow}{AB} \subseteq \overset{\bullet\!\!\longrightarrow}{BA}$.

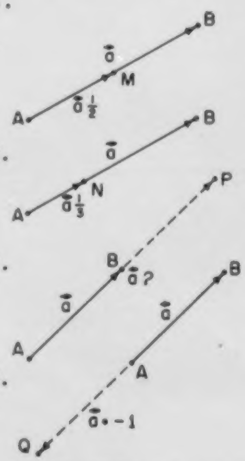
4. As in Exercise 3, but using Theorem 7-16(a), it follows that, for $A \neq B$, $-\overrightarrow{AB} \subseteq \overrightarrow{BA}$. [This inclusion also holds, trivially, in case A = B.] Also, for $A \neq B$, $A \in \overrightarrow{BA}$. [Here, the assumption that $A \neq B$ is essential.] So, by Exercise 1, for $A \neq B$, $\overleftrightarrow{AB} = \overrightarrow{AB} \cup \overrightarrow{BA}$. [Note that $\overset{\bullet\!\!\longrightarrow}{AA} = \{A\} \neq \emptyset = \overrightarrow{AA} \cup \overrightarrow{AA}$.]

5. (a) This set is a ray.
   (b) $-\overset{\bullet\!\!\longrightarrow}{QP}$
   (c) By the hint for Exercise 2, $-\overset{\bullet\!\!\longrightarrow}{QP} = \{Q\} \cup -\overrightarrow{QP}$. Since $Q \notin \overrightarrow{QP}$ and $\overrightarrow{QP} \cap -\overrightarrow{QP} = \emptyset$ it follows that $\overrightarrow{QP} \cap -\overset{\bullet\!\!\longrightarrow}{QP} = \emptyset$. By Exercise 1, $\overrightarrow{QP} \cup -\overset{\bullet\!\!\longrightarrow}{QP} = \overleftrightarrow{QP}$. Hence, $-\overset{\bullet\!\!\longrightarrow}{QP}$ consists of just those points of $\overleftrightarrow{QP}$ which are not in $\overrightarrow{QP}$. But, by Exercise 4, for $Q \neq P$, $\overleftrightarrow{QP} = \overrightarrow{QP} \cup \overset{\bullet\!\!\longrightarrow}{PQ}$. So, the points of $\overleftrightarrow{QP}$ which are not in $\overrightarrow{QP}$ are all in $\overset{\bullet\!\!\longrightarrow}{PQ}$. Since $\overset{\bullet\!\!\longrightarrow}{PQ} \subseteq \overleftrightarrow{PQ} = \overleftrightarrow{QP}$ it follows that the points of $\overset{\bullet\!\!\longrightarrow}{PQ}$ which are not in $\overrightarrow{QP}$ are precisely those of $\overleftrightarrow{QP}$ which are not in $\overrightarrow{QP}$ — that is, they are precisely the points of $-\overset{\bullet\!\!\longrightarrow}{QP}$.

There is another way of handling Exercise 5(c). This depends on Theorem 7-15 which asserts that, for $Q \neq P$, there is a one-to-one correspondence between the points of $\overleftrightarrow{QP}$ and those of $\mathcal{R}$ such that a point $C \in \overleftrightarrow{QP}$ and a number $c \in \mathcal{R}$ correspond if and only if $C - Q = (P - Q)c$. Since this correspondence is one-to-one we may deduce from Theorem 7-16(a) that a point C of $\overleftrightarrow{QP}$ belongs to $\overrightarrow{QP}$ if and only if the number corresponding with C is positive. So, a point of $\overleftrightarrow{QP}$ fails to belong to $\overrightarrow{QP}$ if and only if the corresponding number is not positive. Now, by Definition 7-11(b) and Theorem 7-16(b), $-\overset{\bullet\!\!\longrightarrow}{QP} = \{X\colon \exists_{x \leq 0}\ X = Q + (P - Q)x\}$. So, a point C of $\overleftrightarrow{QP}$ belongs to $-\overset{\bullet\!\!\longrightarrow}{QP}$ if and only if the corresponding number is not positive. Hence, a point of $\overleftrightarrow{QP}$ belongs to $-\overset{\bullet\!\!\longrightarrow}{QP}$ if and only if it fails to belong to $\overrightarrow{QP}$. The remainder of the argument proceeds, as in the given solution, via Exercise 4.

Exercise 4 can be dealt with in the same manner. Use Theorem 7-16(a) and Exercise 3(b) of the following Part D, together with the fact that each real number is either greater than 0 or less than [illegible].

Answers for Part D



1. $\{\, B - M = (A - M) + (B - A) = -(\vec{a}\tfrac{1}{2}) + \vec{a} = \vec{a}\tfrac{1}{2}$

2. $\Big\{ \begin{array}{l} B - N = (A - N) + (B - A) = -(\vec{a}\tfrac{1}{3}) + \vec{a} = \vec{a}\tfrac{2}{3}, \\ n = \tfrac{2}{3};\ n + \tfrac{1}{3} = 1. \end{array}$

3. $\Big\{ \begin{array}{l} B - P = (A - P) + (B - A) = -(\vec{a}2) + \vec{a} \\ = \vec{a}\cdot -1,\ p = -1;\ p + 2 = 1 \end{array}$

4. $\Big\{ \begin{array}{l} B - Q = (A - Q) + (B - A) = -(\vec{a}\cdot -1) + \vec{a} = \vec{a}2, \\ q = 2;\ [q + -1 = 1] \end{array}$

5. $\overset{\bullet\!\!\longrightarrow}{AM}$, $\overset{\bullet\!\!\longrightarrow}{AN}$, $\overset{\bullet\!\!\longrightarrow}{AP}$, $\overset{\bullet\!\!\longrightarrow}{MB}$, $\overset{\bullet\!\!\longrightarrow}{NB}$, $\overset{\bullet\!\!\longrightarrow}{QB}$, and $\overset{\bullet\!\!\longrightarrow}{MP}$ have the sense of $\overset{\bullet\!\!\longrightarrow}{AB}$; $\overset{\bullet\!\!\longrightarrow}{AQ}$, $\overset{\bullet\!\!\longrightarrow}{PB}$, $\overset{\bullet\!\!\longrightarrow}{MN}$, $\overset{\bullet\!\!\longrightarrow}{MQ}$, and $\overset{\bullet\!\!\longrightarrow}{PQ}$ have the sense of $-\overset{\bullet\!\!\longrightarrow}{AB}$.

6. $B - R = (A - R) + (B - A) = -(\vec{a}r) + \vec{a} = \vec{a}(1 - r)$. So, if $R - A = \vec{a}r$ then $B - R = \vec{a}s$, where $s = 1 - r$.

Part E

1. Suppose that $P \neq Q$.
   (a) Find a point in $\overrightarrow{PQ} \cap \overrightarrow{QP}$. [*Hint*: See Exercise 8, page 304, and Theorem 7-16.]
   (b) Complete:

$$\overrightarrow{PQ} \cap \overrightarrow{QP} = \{X: \exists_x\ (0 < x \_\_\_\_ \text{ and } X = P + \_\_\_\_)\}$$

2. Show that

$$B + (A - B)a = A + (B - A)(1 - a).$$

3. Show that
   (a) $\overrightarrow{\dot{B}A} = \{X: \exists_{x \leq 1}\ X = A + (B - A)x\}$, and
   (b) $\overrightarrow{BA} = \{X: \exists_{x < 1}\ X = A + (B - A)x\}$ $[A \neq B]$.
4. Use brace-notation to describe $\overrightarrow{\dot{A}B} \cap \overrightarrow{\dot{B}A}$ and $\overrightarrow{AB} \cap \overrightarrow{BA}$.

Intervals and Segments

In Chapter 1 we discussed intervals and segments. Intuitively, the *interval* $\overline{AB}$ with *endpoints* A and B is the set of all points "between A and B". We shall merely adopt the definitions we used in Chapter 1:

Definition 7-12 (a) $\overline{AB} = \overrightarrow{AB} \cap \overrightarrow{BA}$
(b) $\overline{\dot{A}\dot{B}} = \{A, B\} \cup \overline{AB}$

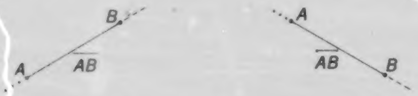


Fig. 7-8

For $A \neq B$ it is reasonable to read '$\overline{AB}$' as 'interval AB' and '$\overline{\dot{A}\dot{B}}$' as 'segment AB'. As in the cases of 'line', 'half-line' and 'ray', we shall use the words 'interval' and 'segment' only with reference to nondegenerate sets – that is, sets which contain more than one element. As you might suspect, we have a theorem:

Theorem 7-21
(a) $\overline{AB} = \{X: \exists_x\ (0 < x < 1 \text{ and } X = A + (B - A)x)\}$ $[A \neq B]$
(b) $\overline{\dot{A}\dot{B}} = \{X: \exists_x\ (0 \leq x \leq 1 \text{ and } X = A + (B - A)x)\}$

Answers for Part E

1. (a) By Theorem 7-16, $C \in \overrightarrow{PQ} \Longleftrightarrow \exists_{x > 0}\ C = P + (Q - P)x$ and $C \in \overrightarrow{QP} \Longleftrightarrow \exists_{x > 0}\ C = Q + (P - Q)x$. So, $C \in \overrightarrow{PQ} \cap \overrightarrow{QP}$ if and only if $C = P + (Q - P)a = Q + (P - Q)b$ where both a and b are positive. Now

$$P + (Q - P)a = Q + (P - Q)b$$

if and only if $(P - Q)(1 - a - b) = \vec{0}$ and, since $P - Q \neq \vec{0}$, the latter is the case if and only if $a + b = 1$. In order for a and b to be positive solutions of this equation we need only choose a between 0 and 1 and choose $b = 1 - a$. So, for example, $P + (Q - P)\frac{1}{3} \in \overrightarrow{PQ} \cap \overrightarrow{QP}$.

   (b) 1; $(Q - P)x$ [For, it has been shown in part (a) that, for $P \neq Q$, $C \in \overrightarrow{PQ} \cap \overrightarrow{QP}$ if and only if $\exists_x\ (0 < x < 1$ and $C = P + (Q - P)x)$]

2. $B + (A - B)a = [A + (B - A)] + (B - A) \cdot -a = A + (B - A)(1 - a)$

3. (a) Since $a \geq 0$ if and only if $1 - a \leq 1$ it follows by Exercise 2 that

$$(a \geq 0 \text{ and } C = B + (A - B)a)$$
$$\Longleftrightarrow$$
$$(1 - a) \leq 1 \text{ and } C = A + (B - A)(1 - a).$$

Hence,

$$\exists_{x \geq 0}\ C = B + (A - B)x$$
$$\Longleftrightarrow$$
$$\exists_{x \leq 1}\ C = A + (B - A)x.$$

Consequently, by Theorem 7-16(b), $\overrightarrow{\dot{B}A} = \{X: \exists_{x \leq 1}\ X = A + (B - A)x\}$.

   (b) [Like (a), but use Theorem 7-16(a) and note the need for the restriction.]

4. $\overleftrightarrow{\dot{A}B} \cap \overleftrightarrow{\dot{B}A} = \{X: \exists_x\ (0 \leq x \leq 1 \text{ and } X = A + (B - A)x)\}$;
   $\overrightarrow{AB} \cap \overrightarrow{BA} = \{X: \exists_x\ (0 < x < 1 \text{ and } X = A + (B - A)x)\}$ $[A \neq B]$

Exercises

Part A

1. Show that $\overset{\bullet\!-\!\bullet}{AB} = \overset{\bullet\!\!\rightarrow}{AB} \cap \overset{\bullet\!\!\rightarrow}{BA}$.
2. Prove Theorem 7 - 21. [*Hint*: Recall Exercise 3 of Part E on page 308.]
3. Show that $\overline{PQ} = \overline{QP}$ and that $\overset{\bullet\!-\!\bullet}{PQ} = \overset{\bullet\!-\!\bullet}{QP}$.
4. What is $\overline{AA}$? $\overset{\bullet\!-\!\bullet}{AA}$?
5. If $\overset{\bullet\!-\!\bullet}{AB} = \overset{\bullet\!-\!\bullet}{CD}$ does it follow that $A = C$ and $B = D$? If not, what does follow about A, B, C, and D? What follows if $\overline{AB} = \overline{CD}$?
6. If $\{A, B\} = \{C, D\}$ does it follow that $\overset{\bullet\!-\!\bullet}{AB} = \overset{\bullet\!-\!\bullet}{CD}$? That $\overline{AB} = \overline{CD}$?

Part B

As in the case of vertices of rays and half-lines, we need an "enabling theorem" before we are justified in speaking of *the* endpoints of a given segment or interval. It should, for example, be possible to justify your answers for Exercise 5 of Part A.

Theorem 7-22 (a) $\overline{AB} = \overline{CD} \Longrightarrow \{A, B\} = \{C, D\}$ $[A \neq B]$

(b) $\overset{\bullet\!-\!\bullet}{AB} = \overset{\bullet\!-\!\bullet}{CD} \Longrightarrow \{A, B\} = \{C, D\}$

The proof of Theorem 7-22(a) is rather long, and we omit it. But, you should be able to fill in the details of the following proof of part (b). Better yet, when you get an idea, follow it through by trying to complete the proof on your own.

The case in which $A = B$ is trivial. [Discuss it.] Suppose, then, that $A \neq B$, and that $\overset{\bullet\!-\!\bullet}{AB} = \overset{\bullet\!-\!\bullet}{CD}$. There are numbers—say, a, b, c, and d—all of which are in $\{x: 0 \leq x \leq 1\}$, such that $A = C + (D - C)a$, $B = C + (D - C)b$, $C = A + (B - A)c$, and $D = A + (B - A)d$. It follows that $B - A = (D - C)(b - a)$ and $D - C = (B - A)(d - c)$, and that $-1 \leq b - a \leq 1$ and $-1 \leq d - c \leq 1$. Since $A \neq B$, $(b - a)(d - c) = 1$. So, if $|b - a| < 1$ then $|d - c| > 1$. Since $|d - c| \leq 1$ it follows that $|b - a| \geq 1$. So, $|b - a| = 1$. We now know that either $b - a = 1$ or $a - b = 1$ and that $0 \leq a \leq 1$ and $0 \leq b \leq 1$. What choices does this permit for a and b?

Part C

Draw a picture of $\mathscr{R}$ and below it draw a picture of a line $\overleftrightarrow{PQ}$. [Mark the point P directly below the mark for 0 and mark Q directly below the mark for 1.] Consider the following subsets of $\overleftrightarrow{PQ}$.

$$p_1 = \{X: \exists_x\, (\tfrac{1}{4} \leq x \leq \tfrac{3}{4} \text{ and } X = P + (Q - P)x)\}$$
$$p_2 = \{X: \exists_x\, (-1 < x < \tfrac{1}{4} \text{ and } X = P + (Q - P)x)\}$$
$$p_3 = \{X: \exists_{x \geq 2}\, X = P + (Q - P)x\}$$
$$p_4 = \{X: \exists_{x < 3}\, X = P + (Q - P)x\}$$

1. Draw lightly four lines below your picture of $\overleftrightarrow{PQ}$ and on each line graph a different one of the sets p_1, p_2, p_3, p_4.

Part A is a reasonable homework assignment. Part B is best as a class activity. Give the students an opportunity to "predict what comes next" at each step of the derivation. Exercise 4 of Part C should be discussed in class also. The other exercises of Part C may be used either as class or homework exercises.

Answers for Part A

1. By definition, $\overset{\bullet\!-\!\bullet}{AB} = \{A, B\} \cup (\overrightarrow{AB} \cap \overrightarrow{BA})$. Since $\{A, B\} \subseteq \overset{\bullet\!\!\rightarrow}{AB}$ and $\overrightarrow{AB} \cap \overrightarrow{BA} \subseteq \overrightarrow{AB} \subseteq \overset{\bullet\!\!\rightarrow}{AB}$ it follows that $\overset{\bullet\!-\!\bullet}{AB} \subseteq \overset{\bullet\!\!\rightarrow}{AB}$. Similarly, $\overset{\bullet\!-\!\bullet}{AB} \subseteq \overset{\bullet\!\!\rightarrow}{BA}$. Hence, $\overset{\bullet\!-\!\bullet}{AB} \subseteq \overset{\bullet\!\!\rightarrow}{AB} \cap \overset{\bullet\!\!\rightarrow}{BA}$. On the other hand, if $C \in \overset{\bullet\!\!\rightarrow}{AB}$ and is not A then $C \in \overrightarrow{AB}$, while if $C \in \overset{\bullet\!\!\rightarrow}{BA}$ and is not B then $C \in \overrightarrow{BA}$. So, if $C \in \overset{\bullet\!\!\rightarrow}{AB} \cap \overset{\bullet\!\!\rightarrow}{BA}$ and is neither A nor B then $C \in \overrightarrow{AB} \cap \overrightarrow{BA}$ Hence $\overset{\bullet\!\!\rightarrow}{AB} \cap \overset{\bullet\!\!\rightarrow}{BA} \subseteq \overset{\bullet\!-\!\bullet}{AB}$. Consequently, $\overset{\bullet\!-\!\bullet}{AB} = \overset{\bullet\!\!\rightarrow}{AB} \cap \overset{\bullet\!\!\rightarrow}{BA}$.
2. (a) If $C = A + (B - A)c$ where $0 < c < 1$ then, for $A \neq B$, $C \in \overrightarrow{AB}$ [by Theorem 7-16(a)] and $C \in \overrightarrow{BA}$ [by Exercise 3(b) of Part E on page 308]. Consequently, for $A \neq B$,
   $$\{X: \exists_x\, (0 < x < 1 \text{ and } X = A + (B - A)x)\} \subseteq \overline{AB}.$$
   On the other hand, suppose that $C \in \overline{AB} = \overrightarrow{AB} \cap \overrightarrow{BA}$. It follows for $A \neq B$, that $C = A + (B - A)a$ where $a > 0$ and that $C = A + (B - A)b$ where $b < 1$. By Theorem 7-15 it follows that $a = b$ and, so, $0 < a < 1$. Hence, if $C \in \overline{AB}$ then $\exists_x\, (0 < x < 1 \text{ and } C = A + (B - A)x)$. Consequently,
   $$\overline{AB} \subseteq \{X: \exists_x\, (0 < x < 1 \text{ and } X = A + (B - A)x\}.$$
   Hence, Theorem 7-21(a).

   (b) [Theorem 7-21(b) can be proved just as Theorem 7-21(a) has been, using Exercise 1, above, and Exercise 3(a) of Part E. The restriction that $A \neq B$ is needed only in the proof of the analogue of the second of the displayed inclusions. But, since $\overline{AA} = \emptyset$, this inclusion is trivial if $A = B$.

   Alternatively, Theorem 7-21(b) can be derived from Theorem 7-21(a) and Definition 7-12(b), in case $A \neq B$. The case in which $A = B$ is trivial.]
3. $\overline{PQ} = \overrightarrow{PQ} \cap \overrightarrow{QP} = \overrightarrow{QP} \cap \overrightarrow{PQ} = \overline{QP}$; $\overset{\bullet\!-\!\bullet}{PQ} = \{P, Q\} \cup \overline{PQ} = \{Q, P\} \cup \overline{QP} = \overset{\bullet\!-\!\bullet}{QP}$
4. $\overline{AA} = \emptyset$; $\overset{\bullet\!-\!\bullet}{AA} = \{A\}$
5. No. But, if $\overset{\bullet\!-\!\bullet}{AB} = \overset{\bullet\!-\!\bullet}{CD}$ then either $(A = C$ and $B = D)$ or $(A = D$ and $B = C)$ — in short $\{A, B\} = \{C, D\}$. If $\overline{AB} = \overline{CD}$ then $\{A, B\} = \{C, D\}$ or $(A = B$ and $C = D)$.
6. Yes.; Yes. [If $A = C$ and $B = D$ then, by definition, $\overset{\bullet\!-\!\bullet}{AB} = \overset{\bullet\!-\!\bullet}{CD}$ and $\overline{AB} = \overline{CD}$. If, on the other hand, $A = D$ and $B = C$ then $\overset{\bullet\!-\!\bullet}{AB} = \overset{\bullet\!-\!\bullet}{DC} = \overset{\bullet\!-\!\bullet}{CD}$ and $\overline{AB} = \overline{DC} = \overline{CD}$.]

Answers for Part B

Suppose that $\overset{\bullet\!-\!\bullet}{AB} = \overset{\bullet\!-\!\bullet}{CD}$. If $A = B$ then $\overset{\bullet\!-\!\bullet}{CD} = \overset{\bullet\!-\!\bullet}{AB} = \{A\}$ and, so $C = D = A = B$. Hence, for $A = B$, if $\overset{\bullet\!-\!\bullet}{AB} = \overset{\bullet\!-\!\bullet}{CD}$ then $\{A, B\} = \{C, D\}$. Suppose, then, that $A \neq B$. The existence of numbers such as a, b, c, and d follows from Theorem 7-21(b) and the fact that $\{A, B\} \subseteq \overset{\bullet\!-\!\bullet}{AB} = \overset{\bullet\!-\!\bullet}{CD}$ and $\{C, D\} \subseteq \overset{\bullet\!-\!\bullet}{CD} = \overset{\bullet\!-\!\bullet}{AB}$. Since $0 \leq a \leq 1$ and $-1 \leq -b \leq 0$ it follows that $-1 \leq a - b \leq 1$. Similarly, $-1 \leq d - c \leq 1$. By substitution, $B - A = [(B - A)(d - c)](b - a) = (B - A)[(d - c)(b - a)]$. So, since $B - A \neq \vec{0}$, $(d - c)(b - a) = 1$. It follows that $|d - c| \cdot |b - a| = 1$ and, from facts about products of nonnegative numbers, that if $|b - a| < 1$ then $|d - c| > 1$. But, we already know that $-1 \leq d - c \leq 1$ — that is, that $|d - c| \not> 1$. So, $|b - a| \not< 1$ and, since $|b - a| \leq 1$, $|b - a| = 1$. [The use of absolute values may be avoided by using properties of reciprocals.] Since $0 \leq b \leq 1$ it follows that if $b - a = 1$ then $0 \leq a + 1 \leq 1$ — that is, $-1 \leq a \leq 0$. Since $0 \leq a \leq 1$ it follows that if $b - a = 1$ then $a = 0$ and $b = 1$. Hence, in this case, $A = C$ and $B = D$. Similarly, if $a - b = 1$ then $A = D$ and $B = C$. Hence, in any case, if $\overset{\bullet\!-\!\bullet}{AB} = \overset{\bullet\!-\!\bullet}{CD}$ then $\{A, B\} = \{C, D\}$.

*

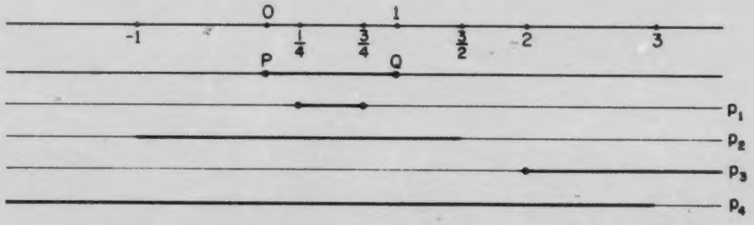
The proof of Theorem 7-22(a) is not extremely difficult and, since it illustrates the notions developed in Part C on page 307 and gives practice in using inequations, you may wish to work through it in class. We assume that $\overline{AB} = \overline{CD}$ and that $A \neq B$. It follows at once that $\overline{AB} \neq \emptyset$ and, so, that $\overline{CD} \neq \emptyset$ and $C \neq D$. It would be helpful, now, if we could find numbers like the a, b, c, and d of the proof of part (a). This is a little more difficult here since, for example, $A \notin \overline{CD}$. We do know, however, that $\overline{AB} \subseteq \overleftrightarrow{AB}$, that $\overline{CD} \subseteq \overleftrightarrow{CD}$ and that $\overline{AB}$ [which is $\overline{CD}$] contains at least two points. From the last it follows that $\overleftrightarrow{AB} = \overleftrightarrow{CD}$ and so, for example, that $\{C, D\} \subseteq \overleftrightarrow{AB}$. So there are numbers — say, c and d — such that $C = A + (B - A)c$ and $D = A + (B - A)d$. [As it turns out, we shall not use the similar consequence of the fact that $\{A, B\} \subseteq \overleftrightarrow{CD}$.] Let's concentrate, now, on locating C. Since $A \neq B$, $\overleftrightarrow{AB} = \overrightarrow{AB} \cup \overrightarrow{BA}$ and, since $C \in \overleftrightarrow{AB}$ and $C \notin \overline{CD} = \overline{AB} = \overrightarrow{AB} \cap \overrightarrow{BA}$, either ($C \in \overrightarrow{AB}$ and $C \notin \overrightarrow{BA}$) or ($C \in \overrightarrow{BA}$ and $C \notin \overrightarrow{AB}$). By Exercise 5 of Part C [page 307] it follows that $C \in -\overleftrightarrow{BA}$ or $C \in -\overleftrightarrow{AB}$. In the second case we should be able to show that $C = A$. If we can do so then we can certainly show that in the first case $C = B$. So, we shall concentrate on the second case, and assume that $C \in -\overleftrightarrow{AB}$. In this case the number c introduced previously is less than or equal to 0. Now, in any event, $D \in \overleftrightarrow{AB} = \overrightarrow{AB} \cup -\overleftrightarrow{AB}$. Suppose that $D \in -\overleftrightarrow{AB}$. [With both C and D in $-\overleftrightarrow{AB}$ it would seem that $\overline{CD}$ is outside of $\overrightarrow{AB}$ — an impossibility. We aim to show that this is the case.] Since $C \in -\overleftrightarrow{AB}$ and $D \in -\overleftrightarrow{AB}$, both $c \leq 0$ and $d \leq 0$.

Answers for Part B [cont.]

Consider the point $C + (D - C)\frac{1}{2}$ which certainly belongs to $\overline{CD}$. Since $C = A + (B - A)c$ and $D - C = (B - A)(d - c)$, this point is $A + (B - A)[c + \frac{d - c}{2}]$. Now, $c + (d - c)/2 = (d + c)/2 \leq 0$. So [since $A \neq B$], the point in question cannot belong to $\overline{AB}$. Since it does belong to $\overline{CD}$, we have a contradiction and $D \notin -\overleftrightarrow{AB}$. So, $D \in \overrightarrow{AB}$ and $d > 0$. [In this case, since $C \in -\overleftrightarrow{AB}$, we should have $C = A$ or $A \in \overline{CD}$. After checking on this we may conclude that $C = A$ — thus, essentially, finishing the proof.] Since $C = A + (B - A)c$, $A = C + (B - A) \cdot -c = C + (D - C) \cdot \frac{c}{c - d}$. Since $c \leq 0$ and $d > 0$ it follows that $c/(c - d) \geq 0$ and, $0 \leq -c < d - c$, $c/(c - d) < 1$. So, $A = C$ [if $c = 0$] or $A \in \overline{CD} = \overline{AB}$. Since $A \notin \overline{AB}$, $A = C$. Hence, if $C \in -\overleftrightarrow{AB}$ then $C = A$. Similarly, if $C \in -\overleftrightarrow{BA}$ then $C = B$. Since this covers all cases it follows that $C \in \{A, B\}$. Similarly, $D \in \{A, B\}$. Since $C \neq D$ it follows that $\{A, B\} = \{C, D\}$.

Answers for Part C

1.



Answers for Part C [cont.]

2. p_1 is a segment; p_2 is an interval; p_3 is a ray; p_4 is a half-line.
3. $P + (Q - P)\frac{1}{4}$ and $P + (Q - P)\frac{3}{4}$; $P + (Q - P) \cdot -1$ and $P + (Q - P)\frac{3}{2}$; $P + (Q - P)2$; $P + (Q - P)3$
4. Since $A = P + (Q - P)a$ and $B = P + (Q - P)b$, we have that $B - A = (Q - P)(b - a)$. So,

$$\begin{aligned} P + (Q - P)r &= [P + (Q - P)a] + (Q - P)(r - a) \\ &= A + (Q - P)(r - a) \\ &= A + (B - A)\frac{r - a}{b - a}. \end{aligned}$$

[Since $a < b$, $b - a \neq 0$.] Also,

$$\begin{aligned} a \leq r \leq b &\Longleftrightarrow 0 \leq r - a \leq b - a \\ &\Longleftrightarrow 0 \leq \frac{r - a}{b - a} \leq 1. \end{aligned}$$

2. For each of the sets tell whether you think it is a half-line, ray, interval, or segment.
3. For each of the sets, name its vertex or its endpoints [whichever is appropriate].
4. Consider the set p_1. This set appears to be the segment $\overset{\bullet\!-\!\bullet}{AB}$ where $A = P + (Q - P)\frac{1}{4}$ and $B = P + (Q - P)\frac{3}{4}$. That is, it appears that

$$(*)\ \{X: \exists_x (\tfrac{1}{4} \leq x \leq \tfrac{3}{4} \text{ and } X = P + (Q - P)x)\}$$

$$= \{X: \exists_x (0 \leq x \leq 1 \text{ and } X = A + (B - A)x)\}.$$

Let's show that this is so.

To begin with, we are interested in points $P + (Q - P)r$, where $\frac{1}{4} \leq r \leq \frac{3}{4}$ – that is, where $0 \leq r - \frac{1}{4} \leq \frac{1}{2}$. This, and the suggested choice of A suggests noting that

$$\begin{aligned} P + (Q - P)r &= [P + (Q - P)\tfrac{1}{4}] + (Q - P)(r - \tfrac{1}{4}) \\ &= A + (Q - P)(r - \tfrac{1}{4}). \end{aligned}$$

Since we have guessed that $B = P + (Q - P)\frac{3}{4}$, as well as that $A = P + (Q - P)\frac{1}{4}$, $B - A = (Q - P)\frac{1}{2}$. This suggests noting that

$$A + (Q - P)(r - \tfrac{1}{4}) = A + [(Q - P)\tfrac{1}{2}]\frac{r - \frac{1}{4}}{\frac{1}{2}}$$

Combining our results we see that

$$P + (Q - P)r = A + (B - A)\frac{r - \frac{1}{4}}{\frac{1}{2}}.$$

Finally, we note that

$$\begin{aligned} \tfrac{1}{4} \leq r \leq \tfrac{3}{4} &\Longleftrightarrow 0 \leq r - \tfrac{1}{4} \leq \tfrac{1}{2}. \\ &\Longleftrightarrow 0 \leq \frac{r - \frac{1}{4}}{\frac{1}{2}} \leq 1. \end{aligned}$$

This completes the proof of (*).
Now *you* show that, for $a < b$,

$$\{X: \exists_x (a \leq x \leq b \text{ and } X = P + (Q - P)x)\} = \overset{\bullet\!-\!\bullet}{AB},$$

where $A = P + (Q - P)a$ and $B = P + (Q - P)b$.

5. You can use the work you did for Exercise 4 to show that

$$\{X: \exists_{x > a}\, X = P + (Q - P)x\} = \overrightarrow{AB},$$

where $A = P + (Q - P)a$ and $B = P + (Q - P)(a + 1)$. Do so.

*6. Show that

$$\{X: \exists_{x \leq a}\, X = P + (Q - P)x\} = \overset{\bullet\!\longrightarrow}{AB},$$

where $A = P + (Q - P)a$ and B is properly chosen. [Exercise 5 may suggest a simple way to choose B.]

[Since a < b, b - a > 0.] So, for a < b,

$$(a \leq r \leq b \text{ and } C = P + (Q - P)r)$$
$$\Longleftrightarrow$$
$$(0 \leq \frac{r - a}{b - a} \leq 1 \text{ and } C = A + (B - A)\frac{r - a}{b - a})$$

Hence, the desired result.

5. Taking b = a + 1, we have, by Exercise 4,

$$P + (Q - P)r = A + (B - A)(r - a).$$

Also, $r > a \Longleftrightarrow r - a > 0$

So:

$$(r > a \text{ and } C = P + (Q - P)r)$$
$$\Longleftrightarrow$$
$$(r - a > 0 \text{ and } C = A + (B - A)(r - a))$$

Hence, the desired result.

6. Let B = P + (Q - P)(a - 1). As in the preceding exercises,

$$P + (Q - P)r = A + (B - A)(a - r)$$
$$r \leq a \Longleftrightarrow a - r \geq 0$$

Hence:

$$\exists_{x \leq a}\, C = P + (Q - P)x$$
$$\Longleftrightarrow$$
$$\exists_{x \geq 0}\, C = A + (B - A)x$$

Sample Quiz

1. Suppose that $\vec{a} \neq \vec{0}$ and that A, B, and R are points such that, for some a and b, $R - A = \vec{a}a$ and $B - R = \vec{a}b$. Show that if $B - A = \vec{a}$ then $a + b = 1$.
2. Given the conditions in Exercise 1, assume that $B - A = \vec{a}$ and that $0 < a < 1$. Which of the following are true and which are false.
   (a) $\overset{\bullet\!-\!\bullet}{AR} \subseteq \overset{\bullet\!-\!\bullet}{RB}$
   (b) $\overset{\bullet\!-\!\bullet}{RB} \subseteq \overset{\bullet\!-\!\bullet}{AB}$
   (c) $\overset{\bullet\!-\!\bullet}{RA} \cup \overset{\bullet\!-\!\bullet}{RB} = \overset{\bullet\!-\!\bullet}{AB}$
   (d) $\overset{\bullet\!-\!\bullet}{AR}$ and $\overset{\bullet\!-\!\bullet}{RB}$ have the same sense.
   (e) $\overset{\bullet\!-\!\bullet}{RA}$ and $\overset{\bullet\!-\!\bullet}{RB}$ have the same sense.
3. In the directions for Exercise 2, replace '$0 < a < 1$' by '$a < 0$'. Now, answer parts (a) through (e) with this new condition.

Key for Sample Quiz

1. Suppose that $B - A = \vec{a}$. By Postulate 3, $B - A = (R - A) + (B - R) = \vec{a}a + \vec{a}b = \vec{a}(a + b)$. Since $\vec{a}1 = \vec{a}$, it follows that $a + b = 1$.
2. (a) False. (b) True. (c) True. (d) True. (e) False.
3. (a) False. (b) False. (c) False. (d) False. (e) True.

Parallelism

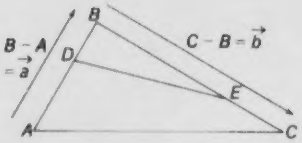
We have defined parallel lines as lines with the same direction. Intuitively speaking, each nondegenerate subset of a line [that is, a subset containing at least two points] "inherits" the direction of that line [Why 'nondegenerate'?]. This suggests that we agree that, for example, parallel rays are rays which are subsets of parallel lines and, to continue the example, parallel segments are segments which are subsets of parallel lines. In general, let us agree that a first set is parallel to a second set if and only if they are nondegenerate subsets of parallel lines.

Does it follow from this agreement that two segments in parallel lines are parallel? How about two segments in the same line? How about two rays with opposite senses?

Exercises

Part A

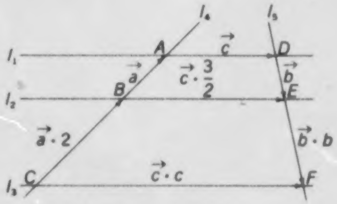
Consider three noncollinear points A, B, and C, where $B - A = \vec{a}$ and $C - B = \vec{b}$. Let $D = \vec{a}\frac{1}{3}$ and $E = B + \vec{b}\frac{1}{3}$. These conditions are illustrated in the diagram at the right.



1. From the data given, it follows that
   (a) $E - D =$ ________ and (b) $C - A =$ ________
2. Using the results of Exercise 1, show that $\overline{DE}$ is not parallel to $\overline{AC}$.
3. (a) Locate a point F on $\overline{AC}$ such that $\overline{DF} \parallel \overline{BC}$. How many such points are there?
   (b) Since $F \in \overline{AC}$, $F = A + (C - A)f$, where $0 \leq f \leq 1$. Determine f.
   (c) Express $F - A$ in terms of $\vec{a}$ and $\vec{b}$.
   (d) Since $\overline{DF} \parallel \overline{BC}$, it follows that $F - D$ is in the direction of ____. What is a relation between $\overline{DF}$ and $\overline{BE}$? Between $\overline{DF}$ and $\overline{EC}$? Between $F - D$ and $E - B$? Between $F - D$ and $C - E$?

Part B

Consider five lines l_1, l_2, l_3, l_4, and l_5 which intersect as indicated in the figure at the right. Suppose that $l_1 \parallel l_2 \parallel l_3$ and that $\vec{a}$ and $\vec{c}$ are linearly independent translations, as indicated in the figure.



A degenerate set is either Ø or a singleton — say, {A}. In any case, such a set is a subset of many lines with differing directions and, so, cannot be thought of as having the direction of the line which contains it.

Since we have reserved the use of 'half-line', 'ray', 'interval', and 'segment' for nondegenerate sets, we may speak of any two sets, each of which is of one of these kinds as being parallel — supposing of course, that they are subsets of parallel lines.

Note, however, that to be sure, for example, that $\overrightarrow{AB} \parallel \overrightarrow{BA}$, we must make sure that $A \neq B$. [In the contrary case, neither $\overrightarrow{AB}$ nor $\overrightarrow{BA}$ is a ray.]

Parts A and B illustrate important applications of our knowledge of translations. All students should attempt these exercises.

Answers for Part A

1. (a) $\vec{a}\frac{1}{3} + \vec{b}\frac{2}{3}$ (b) $\vec{a} + \vec{b}$
2. If $\overline{DE}$ and $\overline{AC}$ were parallel then $E - D$ and $C - A$ would have the same direction and there would be a number — say, c — such that
$$\vec{a}\frac{1}{3} + \vec{b}\frac{2}{3} = (\vec{a} + \vec{b})c = \vec{a}c + \vec{b}c.$$
Since $(\vec{a}, \vec{b})$ is linearly independent, this would require that $c = \frac{1}{3}$ and $c = \frac{2}{3}$. So, there is no such number as c, and $\overline{DE}$ and $\overline{AC}$ are not parallel.
3. (a) $F = A + (\vec{a} + \vec{b})a$, where $0 \leq a \leq 1$. So, $F - D = \vec{a}(a - \frac{2}{3}) + \vec{b}a$ while $C - B = \vec{b}$. So, $\overline{DF} \parallel \overline{BC}$ if and only if there is a number — say, c — such that
$$\vec{a}(a - \tfrac{2}{3}) + \vec{b}a = \vec{b}c.$$
Since $(\vec{a}, \vec{b})$ is linearly independent, there is such a number if and only if $a - \frac{2}{3} = 0$. Hence, $F = A + (\vec{a} + \vec{b})\frac{2}{3}$.
   (b) Since $C - A = \vec{a} + \vec{b}$, $f = \frac{2}{3}$.
   (c) $F - A = \vec{a}\frac{2}{3} + \vec{b}\frac{2}{3}$
   (d) $F - D$ is in the direction of $\vec{b}$ [or: of $\overline{BC}$].; $\overline{DF} \parallel \overline{BE}$; $\overline{DF} \parallel \overline{EC}$; $[F - D] = [E - B]$; $[F - D] = [C - E]$

1. Determine b and c such that $F = C + \overrightarrow{c}c$ and $F = E + \overrightarrow{b}b$.
2. (a) Locate the point M such that $M \in \overleftrightarrow{BE} \cap \overleftrightarrow{AF}$.
   (b) Show that $(M - A)2 = F - M$.
3. Locate a point R such that $R + \overrightarrow{a}a = A$ and $R + \overrightarrow{b}d = D$ for some a and d. Is there more than one such point? Justify your answer.

7.09 Ratios of Translations

If $\overrightarrow{a}$ and $\overrightarrow{b}$ are proper translations which have the same direction then either is the product of the other by some unique nonzero real number. We shall call this real number *the ratio of the first translation to the second*. It can be defined by:

Definition 7-13 $\overrightarrow{a} : \overrightarrow{b} = c \Longleftrightarrow \overrightarrow{a} = \overrightarrow{b}c \; [[\overrightarrow{a}] = [\overrightarrow{b}] \neq \{\overrightarrow{0}\}]$

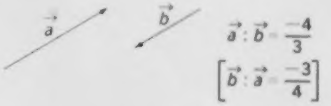

$\overrightarrow{a}$
$\overrightarrow{b}$
$\overrightarrow{a} : \overrightarrow{b} = \frac{-4}{3}$
$\left[\overrightarrow{b} : \overrightarrow{a} = \frac{-3}{4}\right]$

Fig. 7-9

Note that ratios are never 0 and are defined only for non-$\overrightarrow{0}$ vectors which are linearly dependent. What do you know about $\overrightarrow{a}$ and $\overrightarrow{b}$ if $\overrightarrow{a} : \overrightarrow{b} > 0$? If $\overrightarrow{a} : \overrightarrow{b} < 0$? If $\overrightarrow{a} : \overrightarrow{b} = 1$? Also recall that, by earlier theorems, $\overrightarrow{a}$ and $\overrightarrow{b}$ are non-$\overrightarrow{0}$ vectors with the same direction if and only if $\overrightarrow{a} \neq \overrightarrow{0}$ and $\overrightarrow{a} \in [\overrightarrow{b}]$. Explain.

The following theorem is equivalent to Definition 7-13:

Theorem 7-23 (a) $\overrightarrow{a} = \overrightarrow{b}(\overrightarrow{a} : \overrightarrow{b}) \; [[\overrightarrow{a}] = [\overrightarrow{b}] \neq \{\overrightarrow{0}\}]$
(b) $\overrightarrow{a} = \overrightarrow{b}c \Longrightarrow \overrightarrow{a} : \overrightarrow{b} = c \; [\overrightarrow{a} \neq \overrightarrow{0}]$

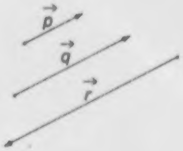
Corollary (a) $\overrightarrow{a} : \overrightarrow{b} = a/b \Longrightarrow \overrightarrow{a}b = \overrightarrow{b}a \; [[\overrightarrow{a}] = [\overrightarrow{b}] \neq \{\overrightarrow{0}\}, b \neq 0]$
(b) $\overrightarrow{a}b = \overrightarrow{b}a \Longrightarrow \overrightarrow{a} : \overrightarrow{b} = a/b \; [\overrightarrow{a} \neq \overrightarrow{0}, b \neq 0]$

[The corollary follows from the theorem and the fact that, for $b \neq 0$, $\overrightarrow{a}b = \overrightarrow{b}a$ if and only if $\overrightarrow{a} = \overrightarrow{b}(a/b)$.]

Exercises

Part A

1. Consider the translations $\overrightarrow{p}$, $\overrightarrow{q}$, and $\overrightarrow{r}$ pictured at the right, where $\overrightarrow{q} = \overrightarrow{p}2$ and $\overrightarrow{r} = \overrightarrow{p} \cdot -3$. Compute:
   (a) $\overrightarrow{p} : \overrightarrow{q}$ (b) $\overrightarrow{q} : \overrightarrow{p}$ (c) $\overrightarrow{p} : \overrightarrow{r}$
   (d) $\overrightarrow{r} : \overrightarrow{p}$ (e) $\overrightarrow{q} : \overrightarrow{r}$ (f) $\overrightarrow{r} : \overrightarrow{q}$



Answers for Part B

1. $(B - C) + (E - B) + (F - E) + (C - F) = \overrightarrow{0}$; $(A - B) + (D - A) + (E - D) + (B - E) = \overrightarrow{0}$. So,

$$\overrightarrow{a}2 + \overrightarrow{c}\frac{3}{2} + \overrightarrow{b}b + \overrightarrow{c} \cdot -c = \overrightarrow{0} \text{ and } \overrightarrow{a} + \overrightarrow{c} + \overrightarrow{b} + \overrightarrow{c} \cdot -\frac{3}{2} = \overrightarrow{0}.$$

From the latter it follows that $\overrightarrow{b} = \overrightarrow{a} \cdot -1 + \overrightarrow{c}\frac{1}{2}$ and, substituting into the former and collecting terms yields:

$$\overrightarrow{a}(2 - b) + \overrightarrow{c}\left(\frac{3}{2} + \frac{b}{2} - c\right) = \overrightarrow{0}$$

Since $(\overrightarrow{a}, \overrightarrow{c})$ is linearly independent it follows that $b = 2$ and $c = \frac{5}{2}$.

2. $M = B + \overrightarrow{c}m = A + [\overrightarrow{c} + \overrightarrow{b}(b + 1)]n$, for some m and n, where, as determined in Exercise 1, $b = 2$. Since $A - B = \overrightarrow{a}$, it follows that

$$\overrightarrow{a} + \overrightarrow{c}(n - m) + \overrightarrow{b}(3n) = \overrightarrow{0}$$

and, since $\overrightarrow{b} = \overrightarrow{a} \cdot -1 + \overrightarrow{c}\frac{1}{2}$,

$$\overrightarrow{a}(1 - 3n) + \overrightarrow{c}\left(n - m + \frac{3n}{2}\right) = \overrightarrow{0}.$$

Since $(\overrightarrow{a}, \overrightarrow{c})$ is linearly independent, $n = \frac{1}{3}$ and $m = \frac{5}{6}$. So,

$$M = B + \overrightarrow{c} \cdot \frac{5}{6}.$$

3. $A - \overrightarrow{a}a = D - \overrightarrow{b}d$; $\overrightarrow{a}a - \overrightarrow{b}d + \overrightarrow{c} = \overrightarrow{0}$; $\overrightarrow{a}a + \overrightarrow{c} = \overrightarrow{a} \cdot -d + \overrightarrow{c}\frac{d}{2}$; $a = -d$, $d = 2$; $R = A + \overrightarrow{a} \cdot -2 = D + \overrightarrow{b} \cdot 2$.

Because of the linear independence of $(\overrightarrow{a}, \overrightarrow{c})$ there is no other solution. Alternatively, because $\overrightarrow{c} \neq \overrightarrow{0}$, $A \neq D$ and $\ell_4 \neq \ell_5$; so, there is no more than one point common to ℓ_4 and ℓ_5.

In introducing the ratio of proper translations with a common direction we are approaching the notion of distance. Intuitively, each translation moves all points the same distance, and the ratio of two proper translations in the same direction is the quotient of the respective distances, with due regard to the senses of the translations. [This quotient is, of course, independent of the choice of unit for measuring distance.] The notion of distance in $\mathcal{E}$ will be introduced formally only in volume 2, and it is not until then that we shall be able to compare the distances through which points are moved by translations in different directions or, for that matter, to deal formally with distances between points. Until then, squares are indistinguishable from other rectangles and, in fact, rectangles are not distinguishable from other parallelograms.

As has been pointed out in the commentary for page 180 [TC 180(1, 2)] the notion of the ratio of two lengths is both logically and psychologically prior to the notion of computing such a ratio as the quotient of the measures of the lengths with respect to an arbitrary unit length. So it is far from unreasonable to deal with ratios of translations before introducing the notion of distance. To say that $\overrightarrow{a} : \overrightarrow{b} = -2$ is to say that $\overrightarrow{a}$ moves points twice as far as $\overrightarrow{b}$ does, but in the opposite sense. Recognition that one wall of a room is twice as far away from one, to one's left, as the opposite wall is, to one's right, does not require a prior knowledge of yardsticks or of their use in assigning measures to lengths.

Since $\mathcal{R}$ is a vector space in which all nonzero vectors have the same direction, the theory of ratios of real numbers is a special case of the theory developed in this section. For this, see the Background Topic on page 319.

If $\vec{a} : \vec{b} > 0$ then $\vec{a}$ and $\vec{b}$ have the same sense. If $\vec{a} : \vec{b} < 0$ then $\vec{a}$ and $\vec{b}$ have opposite senses. If $\vec{a} : \vec{b} = 1$ then $\vec{a} = \vec{b}$.

If $\vec{a}$ and $\vec{b}$ are non-$\vec{0}$ vectors with $[\vec{a}] = [\vec{b}]$ then, since $\vec{a} \in [\vec{a}]$, $\vec{a} \in [\vec{b}]$ and, of course, $\vec{a} \neq \vec{0}$. On the other hand, suppose that $\vec{a} \neq \vec{0}$ and that $\vec{a} \in [\vec{b}]$. From the latter, $\vec{a} = \vec{b}a$ where, since $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$ and $a \neq 0$. So, $\vec{b} = \vec{a} \cdot /a \in [\vec{a}]$. Since $\vec{a} \in [\vec{b}]$ and $\vec{b} \in [\vec{a}]$, $[\vec{a}] \subseteq [\vec{b}]$ and $[\vec{b}] \subseteq [\vec{a}]$. Hence, if $\vec{a} \neq \vec{0}$ and $\vec{a} \in [\vec{b}]$ then $\vec{a} \neq \vec{0} \neq \vec{b}$ and $[\vec{a}] = [\vec{b}]$.

In view of the result just established, the restriction on Definition 7-13 may be replaced by '$\vec{0} \neq \vec{a} \in [\vec{b}]$'. [See proof, below, for Theorem 7-23(b).]

Proof of Theorem 7-23(a). By Definition 7-13, for $[\vec{a}] = [\vec{b}] \neq \{\vec{0}\}$, if $\vec{a} : \vec{b} = \vec{a} : \vec{b}$ then $\vec{a} = \vec{b}(\vec{a} : \vec{b})$. So, $\vec{a} = \vec{b}(\vec{a} : \vec{b})$. [On the other hand, assuming that $\vec{a} : \vec{b} = c$ it follows by Theorem 7-23(a) that $\vec{a} = \vec{b}c$. So, Theorem 7-13(a) might serve as a substitute for the if-part of the definition.]

Proof of Theorem 7-23(b). Suppose that $\vec{a} = \vec{b}c$. Then, by definition $\vec{a} \in [\vec{b}]$ and, for $\vec{a} \neq \vec{0}$, $[\vec{a}] = [\vec{b}] \neq \{\vec{0}\}$. So, by Definition 7-14, $\vec{a} : \vec{b} = c$. Hence, for $\vec{a} \neq \vec{0}$, if $\vec{a} = \vec{b}c$ then $\vec{a} : \vec{b} = c$. [On the other hand, if $[\vec{a}] = [\vec{b}] \neq \{\vec{0}\}$ then $\vec{a} \neq \vec{0}$. So, Theorem 7-23(b) might serve as a substitute for the only if-part of the definition.]

Proof of Corollary. (a) If $\vec{a} : \vec{b} = a/b$ then, by the theorem [or the definition], for $[\vec{a}] = [\vec{b}] \neq \{\vec{0}\}$, $\vec{a} = \vec{b}(a/b)$ and, for $b \neq 0$, $\vec{a}b = \vec{b}a$.

(b) If $\vec{a}b = \vec{b}a$ then, for $b \neq 0$, $\vec{a} = \vec{b}(a/b)$ and, by the theorem, for $\vec{a} \neq \vec{0}$, $\vec{a} : \vec{b} = a/b$.

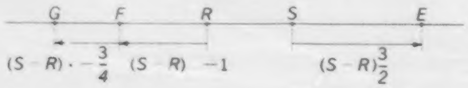
To be sure that students understand Definition 7-13, we recommend that you discuss Part A in class. Parts B and C make a reasonable homework assignment. Part D lends itself to an in-class activity. Parts E and F are important and should be considered by each student, followed by a class discussion in which the "geometric" implications are emphasized.

Answers for Part A

1. (a) 1/2 (b) 2 (c) −1/3 (d) −3 (e) −2/3 (f) −3/2

2. Here are some sentences about non-$\vec{0}$ translations. For each sentence, draw an appropriate figure and write an equivalent sentence about ratios of translations. [When possible, draw figures so that not all points are collinear.]
   - **(a)** $(R - A)3 = (P - Q)2$ **(b)** $(R - S)2 = (P - Q)3$
   - **(c)** $(M - K)2 = (L - M)3$ **(d)** $(K - L)3 = (L - M)2$
   - **(e)** $(M - K)2 = (M - L)3$ **(f)** $(M - K)2 = (L - M) \cdot -3$
   - **(g)** $(M - K) \cdot -2 = (L - M)3$ **(h)** $(M - K)3 = (L - M) \cdot -2$
   - **(i)** $M - K = (L - K) \cdot -2$ **(j)** $M - K = (L - M) \cdot -1$ $[K \neq L]$
   - **(k)** $M - K = (L - P) \cdot -1$ **(l)** $M - K = L - M$

3. Here is a picture of a line $\overleftrightarrow{RS}$. As indicated in the picture, $E - S = (S - R)\frac{3}{2}$, $F - R = (S - R) \cdot -1$, and $G - F = (S - R) \cdot -\frac{3}{4}$.



Complete these sentences.
   - **(a)** $E - R = (S - R)$____. So, $(E - R) : (S - R) =$ ____.
   - **(b)** $(F - R) : (G - F) =$ ____. So, $F - R = (G - F)$____.
   - **(c)** $(F - S) : (E - S) =$ ____.
   - **(d)** $(G - S) : (E - G) =$ ____.

4. Suppose that P and Q are two points of a line l, that $\vec{q} = Q - P$, and that A, B, and C are points such that $A = P + \vec{q}3$, $B = P + \vec{q} \cdot -1$, and $C = P + \vec{q}\frac{3}{2}$.
   - **(a)** Draw an appropriate picture for these conditions.
   - **(b)** Express each of the translations $B - A$, $C - A$, and $C - B$ as a linear combination of $\vec{q}$.
   - **(c)** Compute these ratios:
     - **(i)** $(B - A) : (C - A)$ **(ii)** $(C - A) : (C - B)$
     - **(iii)** $(C - B) : (B - A)$ **(iv)** $(C - B)5 : (B - A)5$

5. **(a)** Given, for $\vec{a} \neq \vec{0}$, that $\vec{a} = \vec{b}\frac{7}{2}$, what is $\vec{a} : \vec{b}$? What is $\vec{b} : \vec{a}$? What is $\vec{a}5 : \vec{b}3$?
   (b) Given, for $\vec{p} \neq \vec{0}$, that $\vec{p}3 = \vec{q}2$, what is $\vec{p} : \vec{q}$? What is $\vec{p}2 : \vec{q}3$? What is $\vec{p} \cdot -2 : \vec{q}3$?

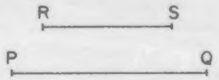
Part B

Suppose that $\vec{a}$ and $\vec{b}$ are proper translations with the same direction and that $\vec{a} : \vec{b} = c$.

1. **(a)** Making use of Definition 7–13, express $\vec{a}$ as a linear combination of $\vec{b}$.
   (b) Now, using Theorem 7–23(a), express $\vec{a}$ as a linear combination of $\vec{b}$.

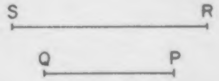
Answers for Part A [cont.]

2. (a)



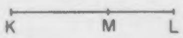
$(R - S):(P - Q) = \frac{2}{3}$

(b)



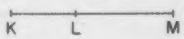
$(R - S):(P - Q) = \frac{3}{2}$

(c)



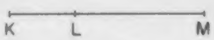
$(M - K):(L - M) = \frac{3}{2}$

(d)



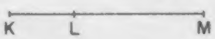
$(K - L):(L - M) = \frac{2}{3}$

(e)



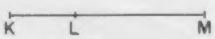
$(M - K):(M - L) = \frac{3}{2}$

(f)



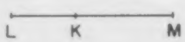
$(M - K):(L - M) = -\frac{3}{2}$

(g)



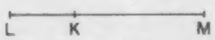
$(M - K):(L - M) = -\frac{3}{2}$

(h)



$(M - K):(L - M) = -\frac{2}{3}$

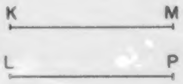
(i)



$(M - K):(L - K) = -2$

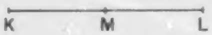
(j) [impossible]

(k)



$(M - K):(L - P) = -1$

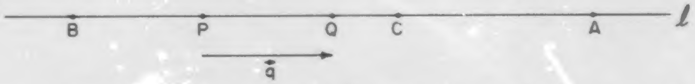
(ℓ)



$(M - K):(L - M) = 1$

3. (a) 5/2; 5/2 (b) 4/3 : 4/3 (c) −4/3 (d) −11/17

4. (a)



(b) $B - A = \vec{q} \cdot -4$; $C - A = \vec{q} \cdot -\frac{3}{2}$; $C - B = \vec{q} \cdot \frac{5}{2}$

(c) (i) 8/3 (ii) −3/5 (iii) −5/8 (iv) −5/8

5. (a) 7/2; 2/7; 35/6
   (b) 2/3; 4/9; −4/9

Answers for Part B

1. (a) $\vec{a} = \vec{b}c$ (b) $\vec{a} = \vec{b}(\vec{a} : \vec{b})$

2. **(a)** Since $\vec{a}$ is a linear combination of $\vec{b}$, so is $\vec{a}3$. Making use of Theorem 7-23(a), express $\vec{a}3$ as a linear combination of $\vec{b}$.

(b) Complete: $\vec{a}3 : \vec{b} = (\vec{a} : \vec{b})$____

(c) Complete these sentences. Be prepared to justify your answers.

(i) $(\vec{a}7) : \vec{b} = (\vec{a} : \vec{b})$____ **(ii)** $(\vec{a} \cdot -4) : \vec{b} = (\vec{a} : \vec{b})$____

(iii) $(\vec{a} \cdot -\frac{1}{3}) : \vec{b} = (\vec{a} : \vec{b})$____ **(iv)** $\vec{a} : (\vec{b} \cdot -2) = (\vec{a} : \vec{b})$____

3. **(a)** Since $\vec{a}$ is a linear combination of $\vec{b}$ then $\vec{a}$ is a linear combination of $\vec{b}5$. In particular, $\vec{a} = \vec{b}(\vec{a} : \vec{b}) = (\vec{b}5)[(\vec{a} : \vec{b})\frac{1}{5}]$. So, $\vec{a} : (\vec{b}5) = (\vec{a} : \vec{b})$____ [Complete.]

(b) Complete these sentences. Be prepared to justify your answers.

(i) $\vec{a} : (\vec{b}12) = (\vec{a} : \vec{b})$____ **(ii)** $\vec{a} : (\vec{b} \cdot -3) = (\vec{a} : \vec{b})$____

(iii) $\vec{a} : (\vec{b}\frac{1}{5}) = (\vec{a} : \vec{b})$____ **(iv)** $\vec{a} : (\vec{b} \cdot -\frac{1}{7}) = (\vec{a} : \vec{b})$____

4. **(a)** The results in Exercise 2 suggest that, for $a \neq 0$, $(\vec{a}a) : \vec{b}$ = ________. [Complete.]

(b) The results in Exercise 3 suggest that, for $b \neq 0$, $\vec{a} : (\vec{b}b)$ = ________. [Complete.]

(c) Make use of the generalizations in (a) and (b) to compute, for $a \neq 0 \neq b$, the ratio $(\vec{a}a) : (\vec{b}b)$ in terms of the ratio $\vec{a} : \vec{b}$.

Part C

1. The following theorem is suggested by the exercises of Part B:

Theorem 7-24 $(\vec{a}a) : (\vec{b}b) = (\vec{a} : \vec{b})(a/b)$
$[[\vec{a}] = [\vec{b}] \neq \{\vec{0}\},\ a \neq 0 \neq b]$

Supply the details of the following proof of Theorem 7-24:

Suppose that $[\vec{a}] = [\vec{b}] \neq \{\vec{0}\}$ and $a \neq 0 \neq b$. It follows that $[\vec{a}a] = [\vec{b}b] \neq \{\vec{0}\}$. So, by definition, $(\vec{a}a) : (\vec{b}b) = (\vec{a} : \vec{b})(a/b)$ if [and only if]

$$\vec{a}a = (\vec{b}b)[(\vec{a} : b)(a/b)].$$

But, $(\vec{b}b)[(\vec{a} : \vec{b})(a/b)] = [(\vec{b}b)(a/b)](\vec{a} : \vec{b}) = (\vec{b}a)(\vec{a} : \vec{b}) = [\vec{b}(\vec{a} : \vec{b})]a = \vec{a}a$.

2. Prove:

Theorem 7-25 $(\vec{a}a + \vec{b}b) : \vec{c} = (\vec{a} : \vec{c})a + (\vec{b} : \vec{c})b$
$[[\vec{a}] = [\vec{b}] = [\vec{c}] \neq \{\vec{0}\},\ \vec{a}a + \vec{b}b \neq \vec{0}]$

Part D

Prove each of the following theorems.

1. $\vec{a} : \vec{a} = 1$ $[\vec{a} \neq \vec{0}]$

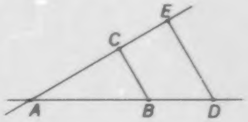
2. $(\vec{a} : \vec{b})(\vec{b} : \vec{c}) = \vec{a} : \vec{c}$ $[[\vec{a}] = [\vec{b}] = [\vec{c}] \neq \{\vec{0}\}]$ [Theorem 7-26]

3. $\vec{b} : \vec{a} = /(\vec{a} : \vec{b})$ $[[\vec{a}] = [\vec{b}] \neq \{\vec{0}\}]$ [*Hint*: Use Exercises 1 and 2.]

4. $\vec{a} : \vec{b} = \vec{c} : \vec{d} \longleftrightarrow \vec{a} : \vec{c} = \vec{b} : \vec{d}$ $[[\vec{a}] = [\vec{b}] = [\vec{c}] = [\vec{d}] \neq \{\vec{0}\}]$
[*Hint*: Use two instances of Exercise 2.]

Part E

Suppose that $\{A, B, C\}$ is noncollinear, $D \in \overleftrightarrow{AB}$ and $D \neq A$, and $E \in \overleftrightarrow{AC}$ and $E \neq A$.



Answers for Part B [cont.]

2. (a) $\vec{a}3 = (\vec{b}(\vec{a}:\vec{b}))3 = \vec{b}(3(\vec{a}:\vec{b}))$

(b) 3

(c) (i) 7 (ii) −4 (iii) −1/3 (iv) (−1/2)

3. (a) (1/5)

(b) (i) (1/12) (ii) (−1/3) (iii) 5 (iv) (−7)

4. (a) $(\vec{a}:\vec{b})a$ (b) $(\vec{a}:\vec{b})\cdot /b$

(c) $(\vec{a}a):(\vec{b}b) = (\vec{a}:\vec{b})(a/b)$

Answers for Part C

1. In any case, since $\vec{a}a \in [\vec{a}]$, $[\vec{a}a] \subseteq [\vec{a}]$. For $a \neq 0$, $\vec{a} = (\vec{a}a)\cdot /a \in [\vec{a}a]$. So, for $a \neq 0$, $[\vec{a}a] = [\vec{a}]$. [Similarly, for $\vec{b} \neq \vec{0}$, $[\vec{b}b] = [\vec{b}]$.] The last line of the proof can be justified by referring to Postulate 4_8 [page 191] and properties of real numbers.

2. For $\vec{a}a + \vec{b}b \neq \vec{0}$ it follows by Theorem 7-23(b) that $(\vec{a}a + \vec{b}b):\vec{c} = (\vec{a}:\vec{c})a + (\vec{b}:\vec{c})b$ if $\vec{a}a + \vec{b}b = \vec{c}[(\vec{a}:\vec{c})a + (\vec{b}:\vec{c})]b$. By Postulates 4_6 and 4_8, $\vec{c}[(\vec{a}:\vec{c})a + (\vec{b}:\vec{c})b] = [\vec{c}(\vec{a}:\vec{c})]a + [\vec{c}(\vec{b}:\vec{c})b]$. By Theorem 7-23(a), for $[\vec{a}] = [\vec{c}] \neq \{0\}$, $\vec{c}(\vec{a}:\vec{c}) = \vec{a}$ and, for $[\vec{c}] = [\vec{b}] \neq \{\vec{0}\}$, $\vec{c}(\vec{b}:\vec{c}) = \vec{b}$. So, $\vec{c}[(\vec{a}:\vec{c})a + (\vec{b}:\vec{c})b] = \vec{a}a + \vec{b}b$. Hence, for $[\vec{a}] = [\vec{b}] = [\vec{c}] \neq \{\vec{0}\}$ and $\vec{a}a + \vec{b}b \neq \vec{0}$, $(\vec{a}a + \vec{b}b):\vec{c} = (\vec{a}:\vec{c})a + (\vec{b}:\vec{c})b$.

[Theorems 7-24 and 7-25 are, of course, analogous of real number theorems which are used to justify the usual algorithms for "multiplying and adding fractions".]

Answers for Part D

1. Since $\vec{a} = \vec{a}1$ it follows, by Theorem 7-23(b), that, for $\vec{a} \neq \vec{0}$, $\vec{a}:\vec{a} = 1$.

2. $\vec{c}[(\vec{a}:\vec{b})(\vec{b}:\vec{c})] = [\vec{c}(\vec{b}:\vec{c})](\vec{a}:\vec{b}) = \vec{b}(\vec{a}:\vec{b}) = \vec{a}$, for $[\vec{b}] = [\vec{c}] \neq \{\vec{0}\}$ and $[\vec{a}] = [\vec{b}] \neq \{\vec{0}\}$, by Theorem 7-23(a). So, for $\vec{a} \neq \vec{0}$, $\vec{a}:\vec{c} = (\vec{a}:\vec{b})(\vec{b}:\vec{c})$, by Theorem 7-23(b). [Since $[\vec{0}] = \{\vec{0}\}$, $\vec{a} \neq \vec{0}$ if $[\vec{a}] \neq \{\vec{0}\}$.]

3. By Exercises 1 and 2, $(\vec{a}:\vec{b})(\vec{b}:\vec{a}) = 1$ for $[\vec{a}] = [\vec{b}] \neq \{\vec{0}\}$. Hence [under this restriction], $\vec{b}:\vec{a} = /(\vec{a}:\vec{b})$. [It is a real number theorem that if $ab = 1$ then $b = /a$.]

4. By Exercise 2, $(\vec{a}:\vec{b})(\vec{b}:\vec{c}) = \vec{a}:\vec{c}$ and $(\vec{b}:\vec{c})(\vec{c}:\vec{d}) = \vec{b}:\vec{d}$. Hence, if $\vec{a}:\vec{b} = \vec{c}:\vec{d}$ then $\vec{a}:\vec{c} = \vec{b}:\vec{d}$. [The converse is the instance of this obtained by interchanging '$\vec{b}$' and '$\vec{c}$'.]

1. **(a)** Show that $D - A \in [B - A]$ and $D - A \neq \vec{0}$.
   (b) It follows from (a) and Definition 7–13 that there exists a real number x such that $(D - A) : (B - A) = x$. Assume that

$$(D - A) : (B - A) = d \text{ and } (E - A) : (C - A) = e.$$

   Express $C - B$ and $E - D$ in terms of '$C - A$' and '$B - A$'.
   (c) Complete: $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$ if and only if there exists a real number x such that $E - D =$ _______.
   (d) Show that $E - D = (C - B)c$ if and only if $e = c = d$.
   (e) Draw a conclusion concerning $\overleftrightarrow{DE}$, $\overleftrightarrow{BC}$, and the ratios introduced in part (b).
   (f) Complete: $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC} \longrightarrow (E - D) : (C - B) = (D -$ ____.
2. Suppose that $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$.
   (a) Let l be a line through A, other than $\overleftrightarrow{AB}$, which intersects $\overleftrightarrow{BC}$ at a point P and $\overleftrightarrow{DE}$ at a point Q. Show that $(Q - D) : (E - D) = (P - B) : (C - B)$. [*Hint*: What does the generalization proved in Exercise 1(f) tell you about $Q - D$ and $P - B$?]
   (b) Show that if $P = B + (C - B)t$ and $Q = D + (E - D)t$ then $\{A, P, Q\}$ is collinear.

Part F

1. **(a)** Picture a segment $\overline{AB}$ and, if possible, locate a point M such that $M - A = B - M$.
   (b) Is there more than one point such as M?
2. If M, A, and B are points such that $M - A = B - M$ then $\{M, A, B\}$ is collinear [Why?]. So, assuming that $A \neq B$, $M \in \overleftrightarrow{AB}$. It follows that there is a number—say, p—such that $M = A + (B - A)p$. Find all values of 'p' such that $M - A = B - M$. [*Hint*: Express $M - B$ in terms of 'A', 'B', and 'p'.]

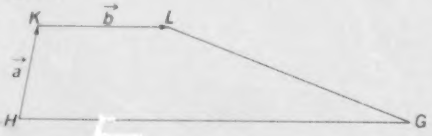
*

Definition 7–14 M is the midpoint of $\overline{AB} \Longleftrightarrow M - A = B - M$.

Theorem 7–27 The midpoint of $\overline{AB} = A + (B - A)\frac{1}{2}$.

*

3. Exercise 2 contains the proof of Theorem 7–27 in case $A \neq B$. Discuss the case $A = B$.
4. Given that $\overleftrightarrow{KL}$ and $\overleftrightarrow{GH}$ are parallel, that $(\vec{a}, \vec{b})$ is linearly independent, and that $(G - H) : (L - K) = 3$, as shown in the diagram below.



Answers for Part E

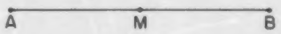
1. (a) By definition, $D \in \overleftrightarrow{AB}$ if and only if $D - A \in [B - A]$. And $D - A \neq \vec{0}$ if and only if $D \neq A$. So, by assumption, $D - A \in [B - A]$ and $D - A \neq \vec{0}$.
   (b) $C - B = (C - A) - (B - A)$;
   $E - D = (E - A) - (D - A) = (C - A)e - (B - A)d$
   (c) $(C - B)x$
   (d) $E - D = (C - B)c$ if and only if

$$\begin{aligned}(C - A)e - (B - A)d &= [(C - A) - (B - A)]c \\ &= (C - A)c - (B - A)c\end{aligned}$$

   Since $\{A, B, C\}$ is noncollinear, $(C - A, B - A)$ is linearly independent. So, $(C - A)e - (B - A)d = (C - A)c - (B - A)c$ if and only if $e = c$ and $d = c$. Hence, $E - D = (C - B)c$ if and only if $e = c = d$.
   (e) By part (d) there exists a real number x such that $E - D = (C - B)x$ if and only if $e = d$. So, by part (c)
   $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$ if and only if $(E - A):(C - A) = (D - A):(B - A)$.
   (f) $(D - A):(B - A)$
2. (a) We can apply the result of Exercise 1(f) with 'P' for 'C' and 'Q' for 'E'. So, $(Q - D):(P - B) = (D - A):(B - A)$. So, by Exercise 1(f), itself, $(Q - D):(P - B) = (E - D):(C - B)$. Hence, by Exercise 4 of Part D, $(Q - D):(E - D) = (P - B):(C - B)$.
   (b) Suppose that $P = B + (C - B)t$ and $Q = D + (E - D)t$. It follows that $P - A = (B - A) + (C - B)t$ and $Q - A = (D - A) + (E - D)t$. By Exercise 1(f), $E - D$ and $D - A$ are multiples of $C - B$ and $B - A$, respectively, by the same real number. So, $Q - A$ is the product of $P - A$ by this same number. Hence, $(P - A, Q - A)$ is linearly dependent, and $\{A, P, Q\}$ is collinear.

Answers for Part F

1. (a)



   (b) No.
2. If $M - A = B - M$ then $(A - M)1 + (B - M)1 = \vec{0}$. So, $(A - M, B - M)$ is linearly dependent and $\{M, A, B\}$ is collinear. Suppose that $M = A + (B - A)p$. It follows that $M - B = (A - B) + (B - A)p = (B - A)(p - 1)$. Since $M - A = (B - A)p$ and $B - M = (B - A)(1 - p)$ it follows that $M - A = B - M$ if and only if $p = 1/2$.
3. $M - A = A - M$ if and only if $M = A$. Also, $A = A + (A - A)\frac{1}{2}$.
   So, by Definition 7-14, the midpoint of $\overline{AA}$ is A; and Theorem 7-27 holds in case $B = A$.

(a) Let M and N be the midpoints of $\overline{HK}$ and $\overline{LG}$, respectively. Express the translation $N - M$ as a linear combination of $\vec{a}$ and $\vec{b}$.

(b) Show that $\overleftrightarrow{MN}$ is parallel to $\overleftrightarrow{KL}$.

(c) What is $(N - M) : (L - K)$? $(N - M) : (G - H)$?

*(d)** Determine whether the midpoints of $\overline{KL}$, $\overline{MN}$, and $\overline{HG}$ are or are not collinear.

7.10 Chapter Summary

Vocabulary Summary

collinear
half-line
interval
direction
  of a vector
  of a line
vertex
  of a half-line, or: a ray
midpoint
parallel

line
ray
segment
sense
  of a vector
  of a half-line, or: a ray
endpoints
  of an interval, or: a segment
ratio
skew

Definitions

7-1. $\{A, B, C\}$ is collinear if and only if $(B - A, C - A)$ is linearly dependent.

7-2. l is a line if and only if (a) l is a subset of $\mathscr{E}$ which contains at least two points, and (b) $\forall_X \forall_Y\, [(\{X, Y\} \subseteq l \text{ and } X \neq Y) \Longrightarrow \forall_Z\, [Z \in l \longleftrightarrow \{X, Y, Z\} \text{ is collinear}]]$

7-3. $\overleftrightarrow{AB} = \{X : \exists_x\, X = A + (B - A)x\}$ $[= \{X : X - A \in [B - A]\}]$

7-4. $[l] = \{\vec{x} : \exists_Y \exists_Z\, (Y \in l \text{ and } Z \in l \text{ and } \vec{x} = Z - Y)\}$ [Read '$[l]$' as 'the direction of l'. Also, read '$[\vec{a}]$' as 'the direction of arrow a'.]

7-5. **(a)** $\overleftrightarrow{A[\vec{a}]} = \{X : X - A \in [\vec{a}]\}$

(b) $\overleftrightarrow{A[l]} = \{X : X - A \in [l]\}$

7-6. $l \parallel m$ if and only if $[l] = [m]$.

7-7. For $\mathscr{K} \subseteq \mathscr{E}$, $\mathscr{K} + \vec{a} = \{X : \exists_Y\, (Y \in \mathscr{K} \text{ and } X = Y + \vec{a})\}$.

7-8. $[\vec{a}]^+ = \{\vec{x} : \vec{x} \neq \vec{0} \text{ and } \exists_{x > 0}\, \vec{x} = \vec{a}x\}$ [Read '$[\vec{a}]^+$' as 'the sense of $\vec{a}$'.]

7-9. **(a)** $\overrightarrow{A[\vec{a}]^+} = \{X : X - A \in [\vec{a}]^+\}$

(b) $\overrightarrow{AB} = \overrightarrow{A[B - A]^+}$ [Read '$\overrightarrow{AB}$' as 'arrow AB'; when $A \neq B$, it is proper to read '$\overrightarrow{AB}$' as 'half-line AB'.]

7-10. **(a)** $\overrightarrow{A[\vec{a}]^+} = \{A\} \cup \overrightarrow{A[\vec{a}]^+}$

(b) $\overrightarrow{AB} = \{A\} \cup \overrightarrow{AB}$

7-11. **(a)** $-\overrightarrow{A[\vec{a}]^+} = \overrightarrow{A[-\vec{a}]^+}$

(b) $-\overrightarrow{A[\vec{a}]^+} = \overrightarrow{A[-\vec{a}]^+}$

Answers for Part F [cont.]

4. (a) $M = H + \vec{a}\frac{1}{2}$. Since $L = H + (\vec{a} + \vec{b})$ and $G - L = \vec{b}3 - (\vec{a} + \vec{b})$, $N = H + (\vec{a} + \vec{b}) + (\vec{b}2 - \vec{a})\frac{1}{2} = H + \vec{a}\frac{1}{2} + \vec{b}2$. So, $N - M = \vec{b}2$.

(b) $[N - M] = [\vec{b}2] = [\vec{b}] = [L - K]$. So, $\overleftrightarrow{MN} \parallel \overleftrightarrow{KL}$.

(c) $(N - M):(L - K) = (\vec{b}2):\vec{b} = 2$; $(N - M):(G - H) = (\vec{b}2):(\vec{b}3) = 2/3$.

(d) The midpoints are A, B, and C, where $A = K + \vec{b}\frac{1}{2}$, $B = M + \vec{b}$, and $C = H + \vec{b}\frac{3}{2}$. So, $B - A = (M - K) + \vec{b}\frac{1}{2} = \vec{a} \cdot -\frac{1}{2} + \vec{b}\frac{1}{2}$ and $C - A = (H - K) + \vec{b} = \vec{a} \cdot -1 + \vec{b}$. Hence, $C - A = (B - A)2$ and, in particular, $(B - A, C - A)$ is linearly dependent and $\{A, B, C\}$ is collinear.

7-12. **(a)** $\overline{AB} = \overrightarrow{AB} \cap \overrightarrow{BA}$
(b) $\overset{\bullet\!-\!\!-\!\bullet}{AB} = \{A, B\} \cup \overline{AB}$

7-13. $\vec{a} : b = c \Longleftrightarrow \vec{a} = \vec{b}c \; [[\vec{a}] = [\vec{b}] \neq \{\vec{0}\}]$

7-14. M is the midpoint of $\overset{\bullet\!-\!\!-\!\bullet}{AB} \Longleftrightarrow M - A = B - M$

Other Theorems

7-1. For $A \neq B$, $\overleftrightarrow{AB}$ is the line which contains A and B.

Corollary. There is one and only one line which contains two given points.

Corollary. Two points are contained in at most one line.

Corollary. Two lines have at most one point in common.

7-2. $(\{C, D\} \subseteq \overleftrightarrow{AB}$ and $C \neq D) \Longrightarrow \overleftrightarrow{AB} = \overleftrightarrow{CD}$

7-3. $(\{A, B\} \subseteq l$ and $A \neq B) \Longrightarrow [l] = [B - A]$

7-4. $(A \in l$ and $\vec{a} \in [l]) \Longrightarrow A + \vec{a} \in l$

7-5. **(a)** For $\vec{a} \neq \vec{0}$, $\overleftrightarrow{A[\vec{a}]}$ is the line through A in the direction of $\vec{a}$.
(b) $\overleftrightarrow{A[l]}$ is the line through A in the direction of l.

Corollary. $A \in l \Longleftrightarrow l = \overleftrightarrow{A[l]}$

7-6. There is one and only one line through a given point and parallel to a given line.

7-7. **(a)** $l \parallel l$ **(b)** $l \parallel m \Longrightarrow m \parallel l$
(c) $(l \parallel m$ and $m \parallel n) \Longrightarrow l \parallel n$

7-8. A translation maps any line onto a parallel line.

7-9. $\mathscr{K} + \vec{a} = \{X \colon X - \vec{a} \in \mathscr{K}\} \; [\mathscr{K} \subseteq \mathscr{E}]$

7-10. $\overleftrightarrow{A[l]} + \vec{a} = \overleftrightarrow{(A + \vec{a})[l]}$

7-11. $\overleftrightarrow{A[B - C]} \cap \overleftrightarrow{B[A - C]} = \{A + (B - C)\}$
$[\{A, B, C\}$ noncollinear$]$

7-12. $\overleftrightarrow{D[l]} \cap m \neq \varnothing \Longleftrightarrow \overleftrightarrow{D[m]} \cap l \neq \varnothing \; [l \cap m \neq \varnothing]$

7-13. **(a)** $[\vec{a}]^+ = \{\vec{x} \colon \exists_{x > 0}\, \vec{x} = \vec{a}x\} \; [\vec{a} \neq \vec{0}]$
(b) $[\vec{0}]^+ = \varnothing$

7-14. $[\vec{a}] = [\vec{a}]^+ \cup \{\vec{0}\} \cup [-\vec{a}]^+$

7-15. $A + (B - A)a = A + (B - A)b \Longleftrightarrow a = b \; [A \neq B]$

7-16. **(a)** $\overrightarrow{AB} = \{X \colon \exists_{x > 0}\, X = A + (B - A)x\} \; [A \neq B]$
(b) $\overset{\bullet\!\longrightarrow}{AB} = \{X \colon \exists_{x \geq 0}\, X = A + (B - A)x\}$

7-17. $C \in \overrightarrow{AB} \Longrightarrow (\overrightarrow{AC} = \overrightarrow{AB}$ and $\overset{\bullet\!\longrightarrow}{AC} = \overset{\bullet\!\longrightarrow}{AB})$

7-18. $\overset{\bullet\!\longrightarrow}{AB} = \overset{\bullet\!\longrightarrow}{CD} \Longrightarrow A = C$

Corollary. $\overset{\bullet\!\longrightarrow}{AB} = \overset{\bullet\!\longrightarrow}{CD} \Longrightarrow [B - A]^+ = [D - C]^+$

7-19. $\overrightarrow{AB} = \overrightarrow{CD} \Longrightarrow A = C \; [A \neq B]$

Corollary. $\overrightarrow{AB} = \overrightarrow{CD} \Longrightarrow [B - A]^+ = [D - C]^+$

7-20. $\overleftrightarrow{AB} = \overset{\bullet\!\longrightarrow}{AB} \cup \overset{\bullet\!\longrightarrow}{BA}$

7-21. **(a)** $\overline{AB} = \{X \colon \exists_x\, (0 < x < 1$ and $X = A + (B - A)x)\}$
$[A \neq B]$
(b) $\overset{\bullet\!-\!\!-\!\bullet}{AB} = \{X \colon \exists_x\, (0 \leq x \leq 1$ and $X = A + (B - A)x)\}$

7-22. **(a)** $\overline{AB} = \overline{CD} \Longrightarrow \{A, B\} = \{C, D\} \; [A \neq B]$
(b) $\overset{\bullet\!-\!\!-\!\bullet}{AB} = \overset{\bullet\!-\!\!-\!\bullet}{CD} \Longrightarrow \{A, B\} = \{C, D\}$

7-23. **(a)** $\vec{a} = \vec{b}(\vec{a} : \vec{b}) \; [[\vec{a}] = [\vec{b}] \neq \{\vec{0}\}]$
(b) $\vec{a} = \vec{b}c \Longrightarrow \vec{a} : \vec{b} = c \; [\vec{a} \neq \vec{0}]$

Corollary. **(a)** $\vec{a} : \vec{b} = a/b \Longrightarrow \vec{a}b = \vec{b}a \; [[\vec{a}] = [\vec{b}] \neq \{\vec{0}\}, b \neq 0]$
(b) $\vec{a}b = \vec{b}a \Longrightarrow \vec{a} : \vec{b} = a/b \; [\vec{a} \neq \vec{0}, b \neq 0]$

7-24. $(\vec{a}a) : (\vec{b}b) = (\vec{a} : \vec{b})(a/b) \; [[\vec{a}] = [\vec{b}] \neq \{\vec{0}\}, a \neq 0 \neq b]$

7-25. $(\vec{a}a + \vec{b}b) : \vec{c} = (\vec{a} : \vec{c})a + (\vec{b} : \vec{c})b \; [[\vec{a}] = [\vec{b}] = [\vec{c}] \neq \{\vec{0}\}$, $\vec{a}a + \vec{b}b \neq \vec{0}]$

7-26. $(\vec{a} : \vec{b})(\vec{b} : \vec{c}) = \vec{a} : \vec{c} \; [[\vec{a}] = [\vec{b}] = [\vec{c}] \neq \{\vec{0}\}]$

7-27 The midpoint of $\overset{\bullet\!-\!\!-\!\bullet}{AB} = A + (B - A)\tfrac{1}{2}$.

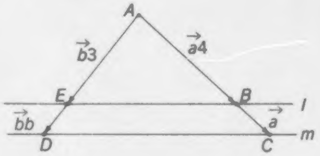
Other Rules of Logic

$$\frac{(\text{not } p \text{ and } q) \Longrightarrow r}{(\text{not } r \text{ and } q) \Longrightarrow p} \; [\text{page } 285]$$

$$\frac{p \Longrightarrow q}{p \Longleftrightarrow (p \text{ and } q)} \quad \frac{p \Longrightarrow q}{(p \text{ or } q) \Longleftrightarrow q} \quad \frac{p \Longleftrightarrow q}{(p \text{ or } q) \Longleftrightarrow (p \text{ and } q)} \; [\text{page } 302]$$

Chapter Test

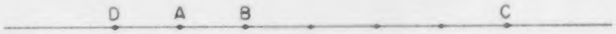
1. Suppose that $A \neq B$ and that $C = A + (B - A)5$ and $D = A + (B - A) \cdot -1$.
   (a) Make a sketch of this situation.
   (b) Let $P = C + (D - C)\tfrac{1}{3}$. Find p such that $P = A + (B - A)p$.
   (c) Let $Q = A + (B - A)2$. Find q such that $Q = C + (D - C)q$.
   (d) Give an argument to show that P, Q, C, and D are collinear.
2. Suppose that $K \neq L$ and that $M = K + (L - K)\tfrac{1}{3}$ and $N = K + (L - K)2$. Determine the ratio $L - K : N - M$.
3. Recall that $P \in \overleftrightarrow{AB}$ if and only if $P = A + (B - A)p$, for some p. Determine all values for 'p' such that:
   (a) $P \in \overset{\bullet\!\longrightarrow}{AB}$ (b) $P \in \overline{AB}$
   (c) $P \in \overleftrightarrow{AB}$ and $P \in \overset{\bullet\!-\!\!-\!\bullet}{AB}$ (d) $P = B$
4. Suppose that $l \parallel m$ and that $(\vec{a}, \vec{b})$ is linearly independent, as illustrated in the figure at the right. Also, from the figure, we see that $E - A = \vec{b}3$, $B - A = \vec{a}4$, $C - B = \vec{a}$, and $D - E = \vec{b}b$, for some b.



   (a) Express $B - E$ and $C - D$ as linear combinations of $\vec{a}$ and $\vec{b}$.
   (b) Determine all values for 'b'.
   (c) Determine 'c' such that $C - D = (B - E)c$.
   (d) Determine the ratio $(E - A) : (D - A)$.

Answers for Chapter Test

1. (a) The students should have drawings something like this:



(b) From the picture, it is clear that $p = 3$. Here is one way to determine the value of 'p' using the algebra of points and translations:

$$P = C + (D - C)\tfrac{1}{3} = [A + (B - A)5] + [(B - A)\cdot -6]\tfrac{1}{3}$$
$$= A + (B - A)3. \text{ So, } p = 3.$$

(c) From the picture, it is clear that $q = \frac{1}{2}$. Here is a proof that this is the case:

$$Q = A + (B - A)2 = [C + (B - A)\cdot -5] + (B - A)2$$
$$= C + (B - A)\cdot -3. \text{ Since } D - C = (B - A)\cdot -6,$$
$$Q = C + (D - C)\cdot -\tfrac{1}{6}\cdot -3 = C + (D - C)\tfrac{1}{2}. \text{ So, } q = \tfrac{1}{2}.$$

(d) From the information given about the points P, Q, C, D, it is the case that each belongs to the line $\overleftrightarrow{AB}$. So, the given points are collinear.

2. From the given information $N - M = (L - K)\frac{5}{3}$. More conveniently, $L - K = (N - M)\frac{3}{5}$. So, by definition, $(L - K):(N - M) = \frac{3}{5}$.

3. (a) $\{x: x \geq 0\}$ (b) $\{x: 0 < x < 1\}$

(c) $\{x: x < 0 \text{ or } x > 1\}$ (d) $\{1\}$

4. (a) $B - E = \vec{a}4 - \vec{b}3$ [or: $\vec{a}4 + \vec{b}\cdot -3$]

$C - D = \vec{a}5 - \vec{b}(3 + b)$ [or: $\vec{a}5 + \vec{b}\cdot -(3 + b)$]

(b) $b = \frac{3}{4}$. One solution: Since $\ell \parallel m$, $C - D = (B - E)t$, for some t. By the results of (a), $\vec{a}5 - \vec{b}(3 + b) = (\vec{a}4 - \vec{b}3)t$, so that $\vec{a}(5 - 4t) + \vec{b}(3t - (3 + b)) = \vec{0}$. Since $(\vec{a}, \vec{b})$ is linearly independent, $5 - 4t = 0$ and $3t - (3 + b) = 0$. So, $t = \frac{5}{4}$ and $b = \frac{3}{4}$.

Another solution: Since $(\vec{a}, \vec{b})$ is linearly independent and $-\vec{b}3 + \vec{a}4 + (E - B) = \vec{0}$, it follows, by Theorem 6-12, that $(-\vec{b}3)d + (\vec{a}4)e + (E - B)f = \vec{0}$ if and only if $d = e = f$. From the figure [or, the given information], and since, for some t, $D - C = (E - B)t$, it follows that

$$(-\vec{b}3)\left(\frac{3 + b}{3}\right) + (\vec{a}4)\frac{5}{4} + (E - B)t = \vec{0}.$$

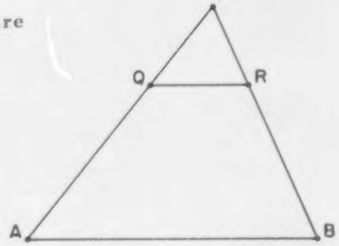
So, by Theorem 6-12, $\frac{3 + b}{3} = \frac{5}{4} = t$. That is, $t = \frac{5}{4}$ and $b = \frac{3}{4}$.

(c) From either solution given in (b), it follows that $c = \frac{5}{4}$.

(d) 4/5 [This follows from the fact that $E - A = \vec{b}3 = (\vec{b}\frac{15}{4})\frac{4}{5} = (D - A)\frac{4}{5}$.]

Answers for Chapter Test [cont.]

5. (a) Here is an appropriate picture of the given situation:



(b) $[\overleftrightarrow{AB}] = [B - A] = [(Q - P)4 - (R - P)4] = [(Q - R)4] = [R - Q] = [\overleftrightarrow{QR}]$. So, $\overleftrightarrow{AB} \parallel \overleftrightarrow{QR}$.

5. Suppose that P, Q, and R are noncollinear points and that $A = P + (Q - P)4$ and $B = P + (R - P)4$.
   (a) Draw a diagram to illustrate this situation.
   (b) Show that $\overleftrightarrow{AB} \parallel \overleftrightarrow{QR}$.

Background Topic

Recall that $\mathscr{R}$ – as well as $\mathscr{T}$ – is a vector space over $\mathscr{R}$. It follows that real numbers can be thought of as vectors, and that we may speak of ratios of real numbers. According to our definitions of direction and sense for vectors, all nonzero real numbers have the same direction and each nonzero real number has one of two senses – the positive sense or the negative sense. [Explain.] According, then, to Definition 7 - 13,

$$a : b = c \longleftrightarrow a = bc \; [a \neq 0 \neq b].$$

Since, for $b \neq 0$, $a = bc$ if and only if $c = a/b$ it follows that, for nonzero real numbers a and b, $a : b = a/b$. In consequence we may, for example, read 'a/b' as 'the ratio of a to b [when $a \neq 0 \neq b$] and, since $a = a/1$, we may also read 'a' as 'the ratio of a to 1' [when $a \neq 0$]. In the next chapter we shall find these readings convenient.

When the vectors under consideration are the real numbers the notion of ratio leads to that of *proportion*. By definition, nonzero real numbers a, b, c, and d are said to be *in proportion* if and only if

$$(*) \qquad a : b = c : d.$$

[Alternatively, one says that the ordered pairs (a, b) and (c, d) are *proportional*. Sometimes (*) is read as 'a is to b as c is to d'.]

1. Determine all values for 't' such that the following are true. If no value for 't' will satisfy a given sentence, say so.
   (a) $5 : t = 10 : 3$
   (b) $|t - 2| : 4 = 3 : 6$
   (c) $(t - 2) : 6 = 6 : 12$
   (d) $|5 + t| : 7 = -12 : 14$
   (e) $4 : t = t : 16$
   (f) $5 : (t - 20) = t : 25$

Here are some theorems about proportion. Think about what they say, and relate each of them to things you already know about ratios of vectors and about multiplication and division of real numbers. [All variables have the set of all nonzero real numbers for domain.]

2. $a : b = c : d \longleftrightarrow ad = bc$
3. $a : b = c : d \longleftrightarrow a : c = b : d$
4. $(ac) : (bd) = (a : b)(c : d)$
5. $a : b = c : d \longleftrightarrow \exists_{x \neq 0}\,(a = cx \text{ and } b = dx)$
6. $a : b = b : c \longleftrightarrow |b| = \sqrt{ac}$ [When $a : b = b : c$, b is said to be a *mean proportional* between a and c.]

Answers for Background Exercises

1. (a) 3/2 (b) 0, 4 (c) 5 (d) no values (e) 8, −8 (f) 25, −5

2. Since $a:b = a/b$ and $c:d = c/d$, the theorem in question is equivalent to:

$$\frac{a}{b} = \frac{c}{d} \Longleftrightarrow ad = bc,$$

a familiar theorem concerning real numbers. [Since $b \neq 0 \neq d$ it follows that $bd \neq 0$ and, so, $a/b = c/d$ if and only if $(a/b)(bd) = (c/d)(bd)$ — that is, if and only if $ad = bc$.] Notice, also, that

$$ad = bd \Longleftrightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0.$$

So [page 273], $ad = bc$ if and only if there exist numbers x and y, not both 0, such that

$$ax + by = 0 \quad \text{and} \quad cx + dy = 0.$$

In other words, the 2-dimensional measure vectors [page 206] (a, c) and (b, d) are linearly dependent if and only if $ad = bc$. [This result is independent of the restriction to nonzero real numbers.]

On the other hand, by applying Theorem 7-23(a) to the vector space $\mathscr{R}$ we see that $a = b(a:b)$ and $c = d(c:d)$. So, $ad = (bd)(a:b)$ and $bc = (bd)(c:d)$. Hence, if $a:b = c:d$ then $ad = bc$. Conversely, by Theorem 7-23(b), if $ad = bc$ then $a:b = c/d = c:d$. [The result of Exercise 2 does not generalize completely to vector spaces other than $\mathscr{R}$ because, in higher dimensional vector spaces there is no entirely satisfactory analogue of multiplication in $\mathscr{R}$.]

3. This follows from the result in Exercise 2 since $ad = bc$ if and only if $ad = cb$. On the other hand it is a consequence of Exercise 4 of Part D on page 314.

4. $(ac):(bd) = (ac)/(bd) = (a/b)(c/d) = (a:b)(c:d)$, by the familiar real number theorem for "multiplying fractions". On the other hand, by applying Theorem 7-24 to the vector space $\mathscr{R}$ we see that $(ac):(bd) = (a:b)(c/d) = (a:b)(c:d)$.

5. $a:b = c:d$ if and only if $a/b = c/d$. So, if $a:b = c:d$ then $a = (c/d)b = c(b/d)$ and, since $b = d(b/d)$ and $b/d \neq 0$ it follows that $\exists_{x \neq 0}\,(a = cx \text{ and } b = dx)$. And, if $a = ck$ and $b = dk$ with $k \neq 0$ then $a/b = (ck)/(dk) = c/d$ and, so, $a:b = c:d$.

On the other hand, by Theorem 7-23(a), $a = c(a:c)$ and $b = d(b:d)$. By Exercise 3, if $a:b = c:d$ then $a:c = b:d$ and it follows, since ratios are nonzero, that $\exists_{x \neq 0}\,(a = cx \text{ and } b = dx)$. Conversely, if $a = ck$ and $b = dk$ where $k \neq 0$ then it follows by Theorem 7-24 that $a:b = (ck):(dk) = (c:d)(k/k) = c:d$.

6. By Exercise 1, $a:b = b:c$ if and only if $ac = b^2$ — that is, if and only if $\sqrt{ac} = \sqrt{b^2} = |b|$.

Chapter Eight
Triangles and Quadrilaterals

8.01 Ratios and Parallel Segments

Consider two noncollinear parallel segments $\overline{AC}$ and $\overline{BD}$. [Since $\overline{AC}$ is a segment, $C \neq A$. Can A and B be the same point? How many points are in $\{A, B, C, D\}$?] Since $\overline{AC} \parallel \overline{BD}$ we know that $[D - B] = |C - A|$ It may happen, as in Figure 8–1(a), that $D - B = C - A$.

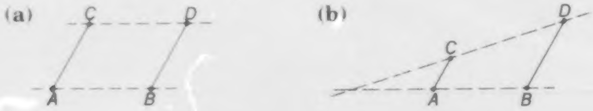


Fig. 8–1

What can you say in this case about $D - C$ and $B - A$? About $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$? The more likely case—that in which $D - B \neq C - A$—is illustrated in Figure 8–1(b). [In the figure, $(D - B) : (C - A) > 1$. Draw a figure for which this ratio is between 0 and 1. Draw another for which the same ratio is negative.] It seems intuitively likely that in case $(D - B) : (C - A) \neq 1$ the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at some point. Let's try to show that this is the case.

To begin with, we know that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect if and only if there are numbers—say, p and q—such that

$$C + (D - C)q = A + (B - A)p \tag{1}$$

—that is, such that

$$C - A = (B - A)p + (C - D)q. \qquad \text{[Explain.]} \tag{2}$$

Our problem, then, is to show that $C - A$ is a linear combination of $B - A$ and $C - D$. With this hint, Figure 8–1(a) should remind us that

$$C - A = (B - A) + (C - B)$$

and, so, that

$$C - A = (B - A) + (D - B) + (C - D). \tag{3}$$

[Explain.] If we recall that $D - B = (C - A)[(D - B) : (C - A)]$, and that we are assuming that $(D - B) : (C - A) \neq 1$, then it is easy to find numbers p and q which satisfy (1). What are they? Are there any other solutions of (1)?

We have proved two theorems. The first [see (2)] is:

> **Theorem 8–1** $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} \neq \varnothing \Longleftrightarrow C - A \in [B - A, C - D]$

Note that this is not restricted to the situation we have been discussing—that in which $\overline{AC}$ and $\overline{BD}$ are noncollinear parallel segments. The second is:

> **Theorem 8–2** If $\overline{AC}$ and $\overline{BD}$ are noncollinear parallel segments and $(D - B) : (C - A) = r$ then
> $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ if $r = 1$ and
> $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{A + (B - A) \cdot /(1 - r)\}$ if $r \neq 1$.

Exercises

Part A

1. Draw figures to illustrate Theorem 8–2 in case
   (a) $r = \frac{1}{2}$ **(b)** $r = 2$ **(c)** $r = -1$ **(d)** $r = -\frac{1}{2}$
2. In each of your figures, label the point of intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ with 'P'.
   - **(i)** In each case, estimate $(P - A) : (B - P)$.
   - **(ii)** Make a guess as to how the value of '$(P - A) : (B - P)$' depends on that of 'r'.
   - **(iii)** Check your guess by drawing other figures.
3. Your guess in Exercise 2 might be put in the form:

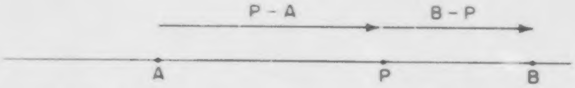
$$P = A + (B - A) \cdot / (1 - r) \longrightarrow (P - A) : (B - P) = \underline{\qquad\qquad} \quad [A \neq P \neq B, r \neq 1]$$

Complete this theorem and prove it.

Part B

1. Explain why $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect if and only if equation (1) has a solution (p, q).
2. Show that equations (1) and (2) are equivalent.
3. What postulate yields equation (3)?
4. Explain how equation (3) is used to find a solution of (2).
5. **(a)** What assumption about $\overline{AC}$ and $\overline{BD}$ tells you that equation (1) has at most one solution?

In this chapter the theory of ratios of translations is applied to the study of triangles and quadrilaterals. The basic results are, for the most part, implicit in some of the exercises in the preceding chapter — particularly those near the end of section 7.09. These results are presented more formally in section 8.01. The notion of the ratio in which a point divides an interval is basic for most of the theorems and problems of this chapter. This notion is introduced in section 8.02 and is used repeatedly thereafter, especially in sections 8.04, 8.07, and 8.08. Briefly, for $A \neq B$, any point $P \in \overleftrightarrow{AB}$ [other than A and B] is said to divide the interval from A to B in $(P - A):(B - P)$. Intuitively, this ratio of translations is, also, the ratio of the distance from A to P



to that from P to B, due regard being paid to sense. As is pointed out in Part D on page 363, it is possible at this stage to introduce such "sensed distances" for ordered pairs of points of any given line, even though the postulates of the present volume do not furnish a basis for comparing distances between points situated, in pairs, on nonparallel lines. Such comparisons will be possible in volume 2; but at present we cannot, for example, single out for special considerations isosceles triangles, rhombuses, or rectangles. In spite of this there are many interesting results to be obtained. Some of these are listed as theorems and others occur as exercises.

As you may have noticed, the notion of the ratio in which a point divides an interval includes both "interior division", where the point belongs to the interval and the ratio is positive, and "exterior division", where the point is elsewhere on the line containing the interval and the ratio is negative. [The end points of the interval are excluded in order to avoid zero-difficulties.] As a result of this generality, many theorems about triangles turn out to be special cases of more general theorems which apply also to trapezoids. [Incidentally, we find it convenient to define 'trapezoid' in such a way that parallelograms are included among trapezoids. This is in line with the usual tendency to consider squares as special rectangles, rectangles as special parallelograms, and equilateral triangles as special isosceles triangles.]

By this time students have at their disposal numerous techniques which can be applied to solve problems of the type considered here. Consequently, you may expect — and should welcome — a variety of solutions for any given problem.

Some of the details of the proofs of Theorems 8-1 and 8-2, concerning which questions are asked in the text, are reverted to in the exercises of Part B on page 321. The questions in the text may be answered as follows:

$A \neq B$; for if $A = B$ then $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$, being parallel, would be collinear.

$\{A, B, C, D\}$ consists of four points.

If $D - B = C - A$ then $D - C = B - A$ and $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

A figure for which $0 < (D - B):(C - A) < 1$ can be obtained from Figure 8-1(b) by interchanging the labels 'A' and 'B', 'C' and 'D'. One for which the ratio in question is negative is obtained by interchanging the labels 'B' and 'D' and redrawing the dashed lines.

The explanation of (2) might be that, by a "bargain theorem", if $C + \vec{a} = A + \vec{b}$ then $C - A = \vec{b} - \vec{a}$. In more detail, if $C + \vec{a} = A + \vec{b}$ then $(C + \vec{a}) - A = (A + \vec{b}) - A$ and, so, $(C - A) + \vec{a} = \vec{b}$ and $C - A = \vec{b} + -\vec{a}$. Also, $-[(D - C)q] = -(D - C)q = (C - D)q$.

Equation (3) is obtained by two uses of Postulate 3, separated by a use of the associative principle for addition of translations.

Let $(D - B):(C - A) = r$. Then, $D - B = (C - A)r$ and, by (3),

$$C - A = (B - A) + (C - A)r + (C - D).$$

Since $r \neq 1$,

$$C - A = (B - A) \cdot /(1 - r) + (C - D) \cdot /(1 - r).$$

Comparing this with (2) — which is equivalent to (1) — it follows that (1) is satisfied if

$$p = /(1 - r) = q.$$

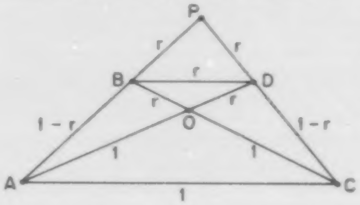
If there were two solutions of (1) with different values of 'p' then, since $A \neq B$, there would be two points of $\overleftrightarrow{AB}$ belonging to $\overleftrightarrow{CD}$. From this it would follow that $\overleftrightarrow{AC} = \overleftrightarrow{BD}$. But, $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are noncollinear. Similarly, there cannot be two solutions of (1) with different values of 'q'. Hence, (1) has at most one solution.

Note that, as is pointed out in Exercise 6 of Part B, it is also the case that, for $r \neq 1$,

$$\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{C + (D - C) \cdot /(1 - r)\}.$$

[This follows from the fact that q, as well as p, is $/(1 - r)$. It also follows from the theorem by way of the hint given for Exercise 6.]

Note that interchanging 'B' and 'D' in Theorem 8-2 and replacing 'r' by '−r' yields an instance of the theorem whose antecedent is equivalent to that of the theorem but whose consequent gives information about $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ — rather than about $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$. The following diagram summarizes these two results in case $0 < r < 1$:



In the figure, the 'r's, '1's, and '1 - r's indicate the ratios of pairs of translations with the same direction. For example, since $(1 - r) + r = 1$, the figure shows that

$$(P - A):(B - A) = 1/(1 - r).$$

It also shows that

$$(O - A):(D - O) = 1/r \text{ and } (O - A):(D - A) = 1/(1 + r).$$

[The fact that 'r' is associated with, for example, both $\overleftrightarrow{PD}$ and $\overleftrightarrow{PB}$ indicates nothing concerning the relative lengths of these segments.] Although, as remarked, the figure illustrates the case in which $0 < r < 1$, the formulas read from it hold in any case.

Some, at least, of the preceding [including the figure] should be included in classroom discussion of Theorem 8-2. Following the answers given below there is a more extended discussion whose purpose is to better acquaint you with some of its implications.

Following the discussion of pages 320 - 321 we recommend Part A as a class activity, followed by Part B as a homework assignment. Part C is probably best treated in class due to the lengthy discussion following Exercise 3. Parts D and E together would make a rather long homework assignment. Perhaps Part D could be homework for all students, with a team of students assigned to each exercise of Part E.

Theorem 8-2 is of considerable importance, and many of the exercises in this chapter are rather thinly disguised repetitions of it. Some discussion of its ramifications is in order here.

In the first place, assuming that $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are noncollinear parallel segments, it follows from the theorem that if $\overleftrightarrow{AB} \nparallel \overleftrightarrow{CD}$ then $(D - B):(C - A) \neq 1$ and $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ consists of a single point — say, P. [Reference to the preceding figure will be helpful.] In this case, letting $(D - B):(C - A)$ be r, $(P - A):(B - A) = /(1 - r)$. Equivalently, $(A - B):(P - A) = r - 1$ [a fact which, also, is easily read from the figure] and, since $(P - A):(P - A) = 1$ and $(A - B) + (P - A) = P - B$, $(P - B):(P - A) = r$. In short, in case $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{P\}$,

$$(P - B):(P - A) = (D - B):(C - A). \qquad (1)$$

The result (1) can be restated as:

$$(P - B):(P - A) = r \Longleftrightarrow (D - B):(C - A) = r$$

Revising the algebraic steps in the preceding paragraph, we see that

$$P = A + (B - A)\cdot/(1 - r) \Longleftrightarrow (D - B):(C - A) = r.$$

It follows that, if $r \neq 1$, then

$$\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{A + (B - A)\cdot/(1 - r)\} \Longleftrightarrow (D - B):(C - A) = r.$$

Only the if-part of this is explicit in Theorem 8-2; but, as we have just seen, the only if-part is a consequence of the theorem and some elementary algebra.

More simply, the theorem implies that, $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ being noncollinear parallel segments,

$$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD} \text{ if and only if } D - B = C - A. \qquad (2)$$

The if-part is explicit; the only if-part follows from the fact that, by the second of the two conclusions of the theorem, if $D - B \neq C - A$ then $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect in a single point and, so, are not parallel.

As pointed out in discussing the preceding figure, interchanging 'B' and 'D' in the theorem yields results, like the preceding, concerning $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$. In case $(D - B):(C - A) \neq -1$ these lines intersect at a point — say, O — and, by (1),

$$(O - D):(O - A) = (B - D):(C - A).$$

More conveniently written:

$$(O - D):(A - O) = (D - B):(C - A) \qquad (3)$$

Assuming that $|(D - B):(C - A)| \neq 1$, both (1) and (3) hold.

Interchanging 'A' and 'C', and 'B' and 'D' we have [since $(B - D):(A - C) = (D - B):(C - A)$] that

$$(P - D):(P - C) = (D - B):(C - A) \qquad (1')$$

and

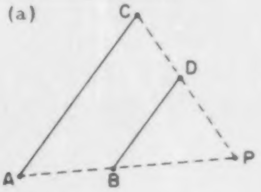
$$(O - B):(C - O) = (D - B):(C - A). \qquad (3')$$

The preceding results [which are essentially only interpretations of Theorem 8-2] imply many familiar theorems. For example, (1) yields a theorem on the proportionality of corresponding sides of triangles whose corresponding sides are parallel [Theorem 8-11]. [Without a way of comparing measures of nonparallel intervals, this is as close as we can easily come to the AAA similarity theorem.] The case of (3) in which $(D - B):(C - A) > 0$ yields a theorem concerning the ratio in which the point of intersection of the diagonals of a trapezoid divides each of them — this ratio is the same as the ratio of the bases. [This includes the theorem according to which the diagonals of a parallelogram bisect each other.] The converse — that a quadrilateral whose diagonals intersect in a point which divides them in the same ratio is a trapezoid — follows from Theorem 8-3. The result (1) can also be thought of as a considerable generalization of the familiar theorem concerning the relation between the interval joining the midpoints of two sides of a triangle and the third side. Similarly (3) generalizes the theorem concerning the existence and location of the intersection of the medians of a triangle. Finally, (2) may be interpreted as that parallel lines are "everywhere equidistant", along whatever direction — other than that of the lines themselves — one chooses to measure distance. [This is the only if-part of (2). The if-part yields a familiar characterization of parallelograms.]

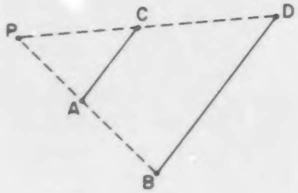
Between them, Theorems 8-2 and 8-3 [on page 324] include a large amount of that part of the geometry of triangles and trapezoids which deals with ratio and similarity. Because conventional geometry courses may have accustomed you to deal with special cases of these theorems [with midpoints and medians rather than arbitrary points of division and intervals related to them] and because, with stronger postulates at hand, these courses can deal at an earlier stage than ours with angle measure and stronger similarity theorems, you may tend to underrate the present development. If so, try not to let it show. It is not unlikely that once you have mastered it you will be impressed by the unity among unlike topics which this treatment reveals as well as by the fact that so much comes out of so little.

Answers for Part A

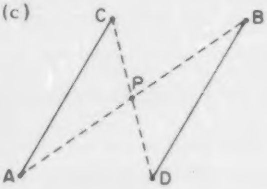
1. (a)



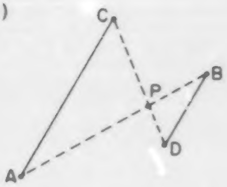
(b)



(c)



(d)



2. (i) −2; −1/2; 1; 2 (ii) (P - A):(B - P) = −/r

(iii) [various]

3. −/r; If P = A + (B - A)·/(1 - r) then P - A = [(P - A) + (B - P)]·/(1 - r) and, so,

$$(P - A)\left(1 - \frac{1}{1 - r}\right) = (B - P)\frac{1}{1 - r}.$$

From this last it follows that (P - A):(B - P) = −1/r.

Answers for Part B

1. By definition, the points of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are the values, for the various values of 'p' and 'q' of 'A + (B - A)p' and 'C + (D - C)q', respectively. So, a point belongs to both lines if and only if it is a value of both expressions. In particular, there is a point which belongs to both lines if and only if (1) has a solution.

2. C + (D - C)q = A + (B - A)p if and only if [C + (D - C)q] - A = [A + (B - A)p] - A. But, [C + (D - C)q] - A = (C - A) + (D - C)q and [A + (B - A)p] - A = (B - A)p. Furthermore, (C - A) + (D - C)q = (B - A)p if and only if C - A = (B - A)p + −[(D - C)q]. But, −[(D - C)q] = −(D - C)q = (C - D)q.

3. Postulate 3 [and Postulate 4_1].

4. From (3), with D - B = (C - A)r, it follows that C - A = (B - A)·/(1 - r) + (C - D)·/(1 - r). Comparing this with (2) shows immediately that (2) is satisfied if p = /(1 - r) = q.

5. (a) The assumption that $\overline{AC}$ and $\overline{BD}$ are noncollinear segments implies that $\overleftrightarrow{AB} \neq \overleftrightarrow{CD}$ and, so, that (1) has at most one solution.

(b) What can you conclude about $(B - A, C - D)$ in the case in which (2) has exactly one solution?

6. In your proof of the second part of Theorem 8–2 it turned out that $p = q$. So, for $r \neq 1$, $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{C + (D - C) \cdot / (1 - r)\}$. This result is not included explicitly in the theorem. Show that, nevertheless, it follows from the theorem itself. [*Hint*: Rewrite the theorem, interchanging 'A' and 'C' as well as 'B' and 'D'.]

7. The second part of the theorem has a corollary:

> If $\overline{AC}$ and $\overline{BD}$ are noncollinear parallel segments
> and $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ then $D - B = C - A$.

(a) Show that this corollary follows from the theorem [and a theorem concerning the intersection of parallel lines].

(b) Give another proof of the corollary by showing that, for $\{A, B, C\}$ noncollinear, if $[D - B] = [C - A]$ and $[D - C] = [B - A]$ then $D - B = C - A$. [*Hint*: Suppose that $D - B = (C - A)b$ and $D - C = (B - A)c$. Express '$C - B$' in two ways in terms of '$A - B$' and '$C - A$'.]

Part C

1. Suppose that $\overline{AC}$ and $\overline{BD}$ are parallel noncollinear segments, that $\overline{AE}$ and BF are parallel noncollinear segments, and that $(F - B) : (E - A) = (D - B) : (C - A) \neq 1$.
   - (a) Show that $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \overleftrightarrow{AB} \cap \overleftrightarrow{EF}$.
   - (b) Does it follow from these assumptions that $\overline{CE}$ and $\overline{DF}$ are noncollinear?
   - (c) Show that $[E - C] = [F - D]$.
   - (d) Under what conditions can you conclude that $\overline{CE} \parallel \overline{DF}$?

2. Suppose that $\overline{PQ}$ and $\overline{RS}$ are parallel noncollinear segments. We know, by Theorem 8–2, that either (i) $\overleftrightarrow{PR} \parallel \overleftrightarrow{QS}$ or (ii) $\overleftrightarrow{PR} \cap \overleftrightarrow{QS}$ consists of a single point–say, T. Let $L = P + (Q - P)2$, $M = R + (S - R)3$, and $N = R + (S - R)2$.
   - (a) Draw an appropriate picture in the case $\overleftrightarrow{PR} \parallel \overleftrightarrow{QS}$.
   - (b) Draw an appropriate picture in the case $\overleftrightarrow{PR} \cap \overleftrightarrow{QS} = \{T\}$.
   - (c) In part (a), is either of the lines $\overleftrightarrow{LM}$ or $\overleftrightarrow{LN}$ parallel to $\overleftrightarrow{QS}$? If so, which one? Prove your answer.
   - (d) In part (b), what appears to be the case about lines $\overleftrightarrow{LN}$ and $\overleftrightarrow{QS}$? Prove that what you say is the case.

3. Suppose that $\overline{AC}$ and $\overline{BD}$ are parallel noncollinear segments. By Theorem 8–2 there are two cases:
   - (i) $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$,
   - (ii) $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ consists of a single point–say, P.

   Suppose, now, that $Q \in \overleftrightarrow{AC}$ and $R \in \overleftrightarrow{BD}$. In particular, suppose that

$$Q = A + (C - A)s \text{ and } R = B + (D - B)t.$$

   (a) Show in case (i) that $s = t$ if and only if $\overleftrightarrow{QR} \parallel \overleftrightarrow{AB}$.

(b) That (1) has exactly one solution implies that $(B - A, C - D)$ is linearly independent.

6. By the suggested instance of the theorem it follows that, for $r \neq 1$, if $\overline{CA}$ and $\overline{DB}$ are noncollinear parallel segments and $(B - D):(A - C) = r$ then $\overleftrightarrow{CD} \cap \overleftrightarrow{AB} = \{C + (D - C) \cdot /(1 - r)\}$. The desired result follows from this and the fact that $\overleftrightarrow{CA} = \overleftrightarrow{AC}$, $\overleftrightarrow{DB} = \overleftrightarrow{BD}$, $(B - D):(A - C) = (D - B):(C - A)$, and $\overleftrightarrow{CD} \cap \overleftrightarrow{AB} = \overleftrightarrow{AB} \cap \overleftrightarrow{CD}$.

7. (a) If $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ then $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ does not consist of a single point. So, by the theorem, if $\overline{AC}$ and $\overline{BD}$ are noncollinear parallel segments and $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ then it is not the case that $r \neq 1$, where $r = (D - B):(C - A)$. So, $(D - B):(C - A) = 1$ — that is, $D - B = C - A$.

   (b) Suppose that $D - B = (C - A)b$ and $D - C = (B - A)c$. Since $C - B = (D - B) + (C - D)$ it follows that $C - B = (C - A)b + (A - B)c$. But, also, $C - B = (A - B) + (C - A)$. Assuming that $\{A, B, C\}$ is noncollinear it follows that $(A - B, C - A)$ is linearly independent. So, $b = c = 1$. Hence, $D - B = C - A$ [and $D - C = B - A$].

Answers for Part C

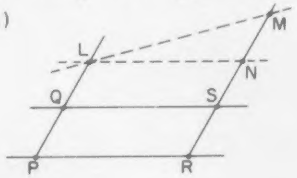
1. (a) Let $(F - B):(E - A) = (D - B):(C - A) = r$, where $r \neq 1$. It follows from Theorem 8-2 that $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{A + (B - A) \cdot /(1 - r)\} = \overleftrightarrow{AB} \cap \overleftrightarrow{EF}$.

   (b) No. [E and F may belong to $\overleftrightarrow{CD}$.]

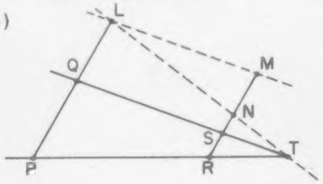
   (c) $F - D = (F - B) - (D - B) = (E - A)r - (C - A)r = (E - C)r$. So, $F - D \in [E - C]$. Since, as a ratio, $r \neq 0$ it follows, also, that $E - C \in [F - D]$. Hence, $[E - C] = [F - D]$.

   (d) $\overleftrightarrow{CD} \parallel \overleftrightarrow{DF}$ if [and only if] $E \neq C$ [or, equivalently, $F \neq D$].

2. (a)



(b)



   (c) $\overleftrightarrow{LN} \parallel \overleftrightarrow{QS}$ For,

$$\begin{aligned} N - L &= (R + (S - R)2) - (P + (Q - P)2) \\ &= (R - P) + ((S - R) - (Q - P))2 \\ &= (R - P) + ((S - Q) - (R - P))2. \end{aligned}$$

   Since we know by Theorem 8-2 that in case (a) $S - Q = R - P$ it follows that $N - L = R - P = S - Q$. Hence, $[\overleftrightarrow{LN}] = [N - L] = [S - Q] = [\overleftrightarrow{QS}]$.

   (d) $\overleftrightarrow{PR} \cap \overleftrightarrow{LN} = \{T\}$ Since $L - P = (Q - P)2$ and $N - R = (S - R)2$ it follows that $(N - R):(L - P) = (S - R):(Q - P)$ and so, by Theorem 8-2 [or Exercise 1], that $\overleftrightarrow{PR} \cap \overleftrightarrow{LN} = \overleftrightarrow{PR} \cap \overleftrightarrow{QS} = \{T\}$.

(b) Show in case (ii) that $s = t$ if and only if $P \in \overleftrightarrow{QR}$.
[*Hint*: If $s \neq 0 \neq t$ then, in any case, '$s = t$' can be translated into a sentence about ratios and both parts (a) and (b) can be solved easily by using things you already know. Having done this, consider the case $s = 0$ and the case $t = 0$.]

*

Suppose that A, B, C, D, and P are five points such that $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{P\}$.

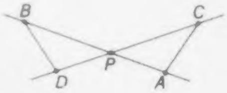


Fig. 8–2

We know from Theorem 8–2 that, under these circumstances, if $\overset{\bullet\!-\!\bullet}{AC} \parallel \overset{\bullet\!-\!\bullet}{BD}$ then $(D - B) : (C - A) = r \neq 1$ and that $P = A + (B - A) \cdot / (1 - r) = C + (D - C) \cdot /(1 - r)$. It follows that

$$P = B + (B - A)\left[\frac{1}{1 - r} - 1\right]$$
$$= B + (B - A)\frac{r}{1 - r}.$$

So, $(P - B) : (B - A) = \dfrac{r}{1 - r}$ and, since $(P - A) : (B - A) = \dfrac{1}{1 - r}$,

$$(P - B) : (P - A) = r.$$

Similarly,

$$(P - D) : (P - C) = r,$$

and, so,

$$(4) \qquad (P - D) : (P - C) = (P - B) : (P - A).$$

Hence, if $\overset{\bullet\!-\!\bullet}{AC} \parallel \overset{\bullet\!-\!\bullet}{BD}$ then

$$(P - D) : (P - C) = (D - B) : (C - A) = (P - B) : (P - A).$$

Suppose, now, that (4) holds [in the situation pictured in Figure 8–2], and let r be the common value of the ratios. It follows that

$$P - D = (P - C)r \text{ and } P - B = (P - A)r.$$

Answers for Part C [cont.]

3. Suppose that $s \neq 0 \neq t$. It follows that $\overset{\bullet\!-\!\bullet}{AQ}$ and $\overset{\bullet\!-\!\bullet}{BR}$ are parallel noncollinear segments. It also follows, by hypothesis, that in this case $s = (Q - A):(C - A)$ and $t = (R - B):(D - B)$. So, $s = t$ if and only if $(Q - A):(C - A) = (R - B):(D - B)$. That is, $s = t$ if and only if $(R - B):(Q - A) = (D - B):(C - A)$. Now, in case (i) it follows from Theorem 8-2 that $s = t$ if and only if $(R - B):(Q - A) = 1$. So, by applying Theorem 8-2 again, $s = t$ if and only if $\overleftrightarrow{AB} \parallel \overleftrightarrow{QR}$. Turning to case (ii), it follows from Theorem 8-2 that, in this case, $s = t$ if and only if $\overleftrightarrow{AB} \cap \overleftrightarrow{QR} = \{P\}$. Since $P \in \overleftrightarrow{AB}$ and $\overleftrightarrow{QR} \neq \overleftrightarrow{AB}$, $\overleftrightarrow{AB} \cap \overleftrightarrow{QR} = \{P\}$ if and only if $P \in \overleftrightarrow{QR}$.

In case $s = 0$ or $t = 0$, $s = t$ if and only if $s = 0$ and $t = 0$ — that is, if and only if $Q = A$ and $R = B$. Now, if $Q = A$ and $R = B$ then $\overleftrightarrow{QR} = \overleftrightarrow{AB}$ and, since $P \in \overleftrightarrow{AB}$, $\overleftrightarrow{QR} \parallel \overleftrightarrow{AB}$ and $P \in \overleftrightarrow{QR}$. [So, we have established the only if-parts of (a) and (b).] On the other hand, assuming, as we are, that one of s and t is 0, it follows that $Q = A$ or $R = B$. Suppose that $Q = A$. If either $\overleftrightarrow{QR} \parallel \overleftrightarrow{AB}$ or $P \in \overleftrightarrow{QR}$ it follows that $\overleftrightarrow{QR} = \overleftrightarrow{AB}$ and, so, that $R \in \overleftrightarrow{AB}$. Since $R \in \overleftrightarrow{BD}$ and $\overleftrightarrow{AB} \cap \overleftrightarrow{BD} = \{B\}$ it follows that $R = B$. Similarly, supposing that $R = B$ it follows that if $\overleftrightarrow{QR} \parallel \overleftrightarrow{AB}$ or $P \in \overleftrightarrow{QR}$ then $Q = A$. Hence, for $s = 0$ or $t = 0$, if either $\overleftrightarrow{QR} \parallel \overleftrightarrow{AB}$ or $P \in \overleftrightarrow{QR}$ then $Q = A$ and $R = B$ [and, so, $s = t$].

Combining the results of the preceding paragraph, we have accomplished both (a) and (b) in case $s = 0$ or $t = 0$. The contrary case having been settled previously, this completes the argument.

[It may seem that, in view of the triviality of the case in which $s = 0$ or $t = 0$, this case might best be simply ignored. Doing so would, however, burden us with unnecessary and troublesome restrictions to an important theorem. It is reasonable, though, in view of the complexity of the argument, to consider this case as intuitively obvious or to replace the argument given above by a less complex one.]

As is brought out in a later exercise, the results established in Exercise 3 are basic for a useful theorem concerning the proportionality of intervals intercepted on parallel lines by parallel or concurrent transversals. [See Theorem 8-6(a) and part of Theorem 8-6(b) on page 335.]

So,

$$D - B = (P - A)r - (P - C)r = (C - A)r,$$

and it follows that $AC \parallel \overleftrightarrow{BD}$. Hence,

$$\text{if } (P - D) : (P - C) = (P - B) : (P - A) \text{ then } \overleftrightarrow{AC} \parallel \overleftrightarrow{BD}.$$

These results are summarized in the next theorem.

> **Theorem 8-3** If A, B, C, D, and P are five points such that $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{P\}$ then
> (a) $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD} \longrightarrow (P - D) : (P - C) = (D - B) : (C - A) = (P - B) : (P - A)$, and
> (b) $(P - D) : (P - C) = (P - B) : (P - A) \longrightarrow \overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$.

> **Corollary** Under the conditions specified in the theorem, $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$ if and only if $(P - D) : (P - C) = (P - B) : (P - A)$

Part D

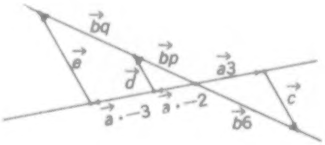
1. Prove that if A, B, and P are three collinear points and $0 \neq t \neq 1$ then each two of the following are equivalent.
   (a) $(P - A) : (B - A) = t$ **(b)** $(P - B) : (A - B) = 1 - t$
   (c) $(P - A) : (B - P) = \dfrac{t}{1 - t}$ **(d)** $(P - B) : (P - A) = \dfrac{t - 1}{t}$
2. Show that, for three collinear points A, B, and P, and nonzero numbers a and b such that $a + b \neq 0$,

$$(P - A) : (B - P) = \frac{a}{b} \longleftrightarrow P = A + (B - A)\frac{a}{a + b}$$

   and, for any point O,

$$(P - A) : (B - P) = \frac{a}{b} \longleftrightarrow P - O = (A - O)\frac{b}{a + b} + (B - O)\frac{a}{a + b}.$$

3. Given seven points on two lines [as shown in the figure] and the indicated translations. Assume that $[\vec{c}] = [\vec{d}] = [\vec{e}]$. Make use of the results in Theorem 8-3 to compute the following.



As noted in the proof of Theorem 8-3, part (a) is a restatement of the second conclusion of Theorem 8-2, together with the instance of the latter obtained by interchanging 'A' and 'C', and 'B' and 'D'. [See equations (1) and (1′) in the discussion of Theorem 8-2 in the commentary for page 320.] The proof of part (a) does involve some additional manipulation of ratios. Skill in such manipulation is very useful, and the exercises in Parts D and E which follow and in Parts A and C on page 322 should help students develop it. See, also, Part E on pages 330 and 331.

Part (b) of Theorem 8-3 is not a consequence of Theorem 8-2. Rather, it complements the latter. Converses of theorems deriving from Theorem 8-2 can often be established by using Theorem 8-3(b).

Answers for Part D

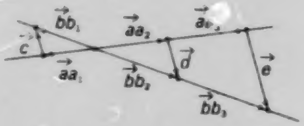
1. (a)$\Longleftrightarrow$(b): $(P - A):(B - A) = t \Longleftrightarrow P = A + (B - A)t$ $[t \neq 0]$
$\Longleftrightarrow P = (B + (A - B)) + (B - A)t$
$\Longleftrightarrow P = B + (A - B)(1 - t)$
$\Longleftrightarrow (P - B):(A - B) = 1 - t$ $[t \neq 1]$
(b)$\Longleftrightarrow$(c): $(P - B):(A - B) = 1 - t \Longleftrightarrow P - B = (A - B)(1 - t)$ $[t \neq 1]$
$\Longleftrightarrow P - B = ((P - B) + (A - P))(1 - t)$
$\Longleftrightarrow (P - B)t = (A - P)(1 - t)$
$\Longleftrightarrow (P - A):(B - P) = \dfrac{t}{1 - t}$ $[0 \neq t \neq 1]$
(c)$\Longleftrightarrow$(d): $(P - A):(B - P) = \dfrac{t}{1 - t} \Longleftrightarrow (P - A):(P - B) = -\dfrac{t}{1 - t}$
$\Longleftrightarrow (P - B):(P - A) = \dfrac{t - 1}{t}$

2. The first can be established directly by the techniques in Exercise 1. Alternatively, use the equivalence of (a) and (c) of Exercise 1, choosing t such that $t/(1 - t) = a/b$. The second comes from the first together with the fact that

$$P = A + (B - A)r \Longleftrightarrow P - O = (A - O) + ((B - O) - (A - O))r$$

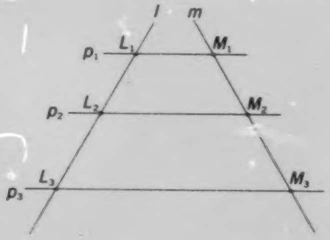
(a) p (b) q (c) $\vec{c} : \vec{d}$
(d) $\vec{d} : \vec{e}$ (e) $\vec{e} : \vec{c}$ (f) $\vec{b6} : \vec{bp}$

4. Given seven points on two lines [as shown in the figure] and the indicated translations. Show that



(a) $[\vec{c}] = [\vec{d}] \Longleftrightarrow a_1b_2 = a_2b_1$
(b) $[\vec{d}] = [\vec{e}] \Longleftrightarrow a_2b_3 = a_3b_2$
(c) $[\vec{c}] = [\vec{d}] \Longrightarrow \vec{c} : \vec{d} = a_1/a_2$
(d) $[\vec{d}] = [\vec{e}] \Longrightarrow \vec{d} : \vec{e} = b_2/(b_2 + b_3)$

5. Suppose that p_1, p_2, and p_3 are three parallel lines which intersect lines l and m as shown in the figure. Show that $(L_3 - L_2) : (L_2 - L_1) = (M_3 - M_2) : (M_2 - M_1)$. [*Hint*: By Theorem 8-2, either l is parallel to m or l intersects m at a point P. The first case occurs when $M_2 - L_2 = M_1 - L_1$. In the second case $(L_2 - L_1) : (P - L_2) = (M_2 - M_1) : (P - M_2)$ [Why?].]



6. Add a fourth parallel line, p_4, to the figure for Exercise 5 and show that $(L_4 - L_3) : (L_2 - L_1) = (M_4 - M_3) : (M_2 - M_1)$. [*Hint*: Use Exercise 5.]

Part E

Suppose that A, B, and P are three collinear points and that a and b are nonzero numbers such that $a + b \neq 0$. You proved in Exercise 2 of Part D that

$$\text{(i)}\ (P - A) : (B - P) = \frac{a}{b} \Longleftrightarrow P = A + (B - A)\frac{a}{a + b}$$
$$[a \neq 0 \neq b, a + b \neq 0].$$

Taking $b = 1$ and $a = s$ it follows that

$$\text{(ii)}\ (P - A) : (B - P) = s \Longleftrightarrow P = A + (B - A)\frac{s}{s + 1}$$
$$[-1 \neq s \neq 0].$$

Answers for Part D [cont.]

3. (a) −4 (b) −6 (c) −3/2 (d) 2/5 (e) −5/3 (f) −3/2

4. (a) $[\vec{c}] = [\vec{d}] \Longleftrightarrow (\vec{a}a_2):(\vec{a}a_1) = (\vec{b}b_2):(\vec{b}b_1)$
$\Longleftrightarrow a_2/a_1 = b_2/b_1$
$\Longleftrightarrow a_1b_2 = a_2b_1$
(b) $[\vec{d}] = [\vec{e}] \Longleftrightarrow (\vec{a}a_2):(\vec{a}a_2 + \vec{a}a_3) = (\vec{b}b_2):(\vec{b}b_2 + \vec{b}b_3)$
$\Longleftrightarrow a_2/(a_2 + a_3) = b_2/(b_2 + b_3)$
$\Longleftrightarrow a_2(b_2 + b_3) = b_2(a_2 + a_3)$
$\Longleftrightarrow a_2b_3 = a_3b_2$
(c) $[\vec{c}] = [\vec{d}] \Longrightarrow \vec{c}:\vec{d} = (\vec{a}a_1):(\vec{a}a_2)$
$\Longrightarrow \vec{c}:\vec{d} = a_1/a_2$
(d) $[\vec{d}] = [\vec{e}] \Longrightarrow \vec{d}:\vec{e} = (\vec{b}b_2):(\vec{b}b_2 + \vec{b}b_3)$
$\Longrightarrow \vec{d}:\vec{e} = b_2/(b_2 + b_3)$

5. By Theorem 8-2, in case $\ell \parallel m$, $M_3 - L_3 = M_2 - L_2 = M_1 - L_1$ and, so, $M_3 - M_2 = L_3 - L_2$ and $M_2 - M_1 = L_2 - L_1$. Hence, in this case, $(L_3 - L_2):(L_2 - L_1) = (M_3 - M_2):(M_2 - M_1)$.

In case $\ell \nparallel m$ then, by Theorem 8-2, $\ell \cap m = \{P\}$ where $P = L_2 + (L_1 - L_2)\cdot/(1 - r)$ and $P = M_2 + (M_1 - M_2)\cdot/(1 - r)$ with $r = (M_1 - L_1):(M_2 - L_2)$. It follows that $(L_2 - L_1):(P - L_2) = r - 1 = (M_2 - M_1):(P - M_3)$. Similarly, $(L_2 - L_3):(P - L_2) = (M_2 - M_3):(P - M_2)$. From these results it follows that $(L_3 - L_2):(L_2 - L_1) = (M_3 - M_2):(M_2 - M_1)$.

6. $(L_4 - L_3):(L_3 - L_2) = (M_4 - M_3):(M_3 - M_2)$ and $(L_3 - L_2):(L_2 - L_1) = (M_3 - M_2):(M_2 - M_1)$. So ["multiplying corresponding sides"], $(L_4 - L_3):(L_2 - L_1) = (M_4 - M_3):(M_2 - M_1)$. [These results are formulated in Theorem 8-6(b) on page 335.]

Taking $a + b = 1$ and $a = t$ it follows that

$$\text{(iii)}\ (P - A) : (B - P) = \frac{t}{1 - t} \longleftrightarrow P = A - (B - A)t\ [0 \neq t \neq 1].$$

[Compare this last formula with a result obtained in Exercise 1 of Part D.]

1. In terms of the restrictions on 'a' and 'b' in (i), explain the restrictions on 's' in (ii) and those on 't' in (iii).
2. Check the consistency of (ii) and (iii) by showing that

$$s = \frac{t}{1 - t} \longleftrightarrow t = \frac{s}{s + 1}\ [t \neq 1, s \neq -1].$$

3. Sentences (ii) and (iii) were obtained as instances of (i). Show that, conversely, (i) is an instance of (ii) and that it is also an instance of (iii).
4. The right sides of (i), (ii), and (iii) show different ways of describing a point P of the line $\overleftrightarrow{AB}$, in terms of 'a' and 'b', or of 's', or of 't'. The left sides show corresponding ways of describing P by using the ratio of $P - A$ to $B - P$. For example, (ii) tells us that this ratio is 1 if and only if $P = A + (B - A)\frac{1}{2}$. For another example, (iii) tells us that $P \in \overline{AB}$ if and only if this ratio is $t/(1 - t)$ where $0 < t < 1$. Show that

   (a) if $0 < t < 1$ then $\frac{t}{1 - t} > 0$, and

   (b) if $\frac{t}{1 - t} > 0$ then $0 < t < 1$. [*Hint*: By Exercise 2, you can establish (b) by showing that if $s > 0$ then $0 < \frac{s}{s + 1} < 1$.]

5. In Exercise 4 you showed that

$$(*) \qquad P \in \overline{AB} \longleftrightarrow (P - A) : (B - P) > 0. \text{ [Explain.]}$$

It follows from this that

$$P \in -\overrightarrow{AB} \cup -\overrightarrow{BA} \longleftrightarrow -1 \neq (P - A) : (B - P) < 0. \text{ [Explain.]}$$

   (a) Draw a figure to illustrate the case in which $(P - A) : (B - P) < -1$ and another to illustrate the case in which $-1 < (P - A) : (B - P) < 0$. Guess two theorems like (*). [*Hint*: $P \in -\overrightarrow{AB} \longleftrightarrow$?]

   (b) Prove the theorems you have guessed. [*Hint*: Using (iii), one of your guesses probably amounts to showing that, for $t \neq 1$, $t < 0$ if and only if $-1 < \frac{t}{1 - t} < 0$. The key to this problem is that, since $t \neq 1$, $t < 1$ or $t > 1$. So,

$$-1 < \frac{t}{1 - t} < 0 \longleftrightarrow ((t < 1 \text{ and } -1 < \frac{t}{1 - t} < 0) \text{ or}$$

Answers for Part E

1. For $b = 1$ and $a = s$, $(a \neq 0 \neq b$ and $a + b \neq 0)$ if and only if $(s \neq 0$ and $1 + s \neq 0)$.

   For $a + b = 1$ and $a = t$, $(a \neq 0 \neq b$ and $a + b \neq 0)$ if and only if $(t \neq 0$ and $1 - t \neq 0)$.

2. $$\begin{aligned} s = \frac{t}{1 - t} &\Longleftrightarrow s - st = t && [t \neq 1] \\ &\Longleftrightarrow s = t(s + 1) \\ &\Longleftrightarrow t = \frac{s}{s + 1} && [s \neq -1] \end{aligned}$$

3. To show that (i) is an instance of (ii), take $s = a/b$ and note that it follows that $s/(s + 1) = a/(a + b)$.

   To show that (i) is an instance of (iii), take $t = a/(a + b)$, etc.

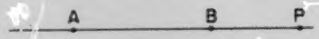
4. (a) Suppose that $0 < t < 1$. It follows that $t > 0$ and $1 - t > 0$. Since a quotient of positive numbers is positive it follows that $t/(1 - t) > 0$.

   (b) Suppose that $s > 0$. It follows that $0 < s < s + 1$ and, so, that

$$\frac{0}{s + 1} < \frac{s}{s + 1} < \frac{s + 1}{s + 1}.$$

   Hence, if $s > 0$ then $0 < s/(s + 1) < 1$. Substituting '$t/(1 - t)$' for 's' and using Exercise 2 yields the desired result.

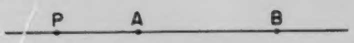
5. (a)



$(P - A) : (B - P) < -1$

$\Longleftrightarrow$

$P \in -\overrightarrow{BA}$



$-1 < (P - A) : (B - P) < 0$

$\Longleftrightarrow$

$P \in -\overrightarrow{AB}$

   (b) $\boxed{P \in -\overrightarrow{BA} \Longleftrightarrow (P - A) : (B - P) < -1}$

   Proof. We know that $P \in -\overrightarrow{BA}$ if and only if $P = A + (B - A)t$ for some $t > 1$. So, by (iii), it follows that $-P \in \overrightarrow{BA}$ if and only if $(P - A) : (B - P) = t/(1 - t)$ for some $t > 1$. Hence, we need to show that [for $t \neq 1$]

$$(1) \qquad \frac{t}{1 - t} < -1 \Longleftrightarrow t > 1.$$

   Now, as in the hint,

$$\begin{aligned} (2)\ \frac{t}{1 - t} < -1 \\ \Longleftrightarrow ((t < 1 \text{ and } \frac{t}{1 - t} < -1) \text{ or } (t > 1 \text{ and } \frac{t}{1 - t} < -1)). \end{aligned}$$

   Taking the first alternative we note that

$$\begin{aligned} (t < 1 \text{ and } \frac{t}{1 - t} < -1) &\Longleftrightarrow (t < 1 \text{ and } t < -1(1 - t)) \\ &\Longleftrightarrow (t < 1 \text{ and } 0 < -1). \end{aligned}$$

Answers for Part E [cont.]

Since $0 \not< -1$, the first alternative can never hold. Taking the second alternative,

$$\begin{aligned}\left(t > 1 \text{ and } \frac{t}{1 - t} < -1\right) &\iff (t > 1 \text{ and } t > -1(1 - t))\\ &\iff (t > 1 \text{ and } 0 > -1)\\ &\iff t > 1.\end{aligned}$$

Hence, by (2), (1).

$$\boxed{P \in -\overrightarrow{AB} \iff -1 < (P - A):(B - P) < 0}$$

Proof. [The proof of this second theorem is like that of the first, and is outlined in the hint. We give here only the reductions of the two alternatives.]

$$\begin{aligned}\left(t < 1 \text{ and } -1 < \frac{t}{1 - t} < 0\right) &\iff (t < 1 \text{ and } -1(1 - t) < t < 0)\\ &\iff (t < 1 \text{ and } -1 < 0 \text{ and } t < 0)\\ &\iff t < 0.\end{aligned}$$

$$\begin{aligned}\left(t > 1 \text{ and } -1 < \frac{t}{1 - t} < 0\right) &\iff (t > 1 \text{ and } -1(1 - t) > t > 0)\\ &\iff (t > 1 \text{ and } -1 > 0 \text{ and } t > 0)\end{aligned}$$

The second alternative being clearly impossible it follows that

$$-1 < \frac{t}{1 - t} < 0 \iff t < 0.$$

*

The results obtained in Exercises 4 and 5 are formulated as Theorem 8-5 on page 329. Another proof of the results in Exercise 5 is obtained in Exercises 2 and 3 of Part C on page 330.

$(t > 1 \text{ and } -1 < \frac{t}{1-t} < 0))$.

You will find that one of these alternatives is impossible and that the other holds if and only if $t < 0$.]

8.02 Points of Division

Consider two points, A and B, and a point $P \in \overline{AB}$. Intuitively, P

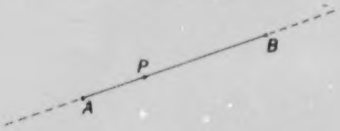


Fig. 8-3

divides the interval $\overline{AB}$ into two intervals, $\overline{AP}$ and $\overline{PB}$, the ratio of whose lengths is $(P - A) : (B - P)$. As yet we have no formal notion of length; but, even without such a notion, it makes sense to talk about dividing an interval into halves or into thirds, or to speak of the midpoint of a given interval.

For example, suppose that P is "one third of the way from A to B", so that

$$(P - A) : (B - P) = 1/2.$$

In this case we shall say that

the ratio in which P divides the interval *from A to B* is 1/2

or that

P divides the interval from A to B in the ratio of 1 to 2.

We shall also say 'segment' rather than 'interval'; but, due to the restrictions in Definition 7-13, P cannot be either A or B if '$(P - A) : (B - P)$' is to make sense.

Note that the ratio in which a point P of an interval $\overline{AB}$ divides the interval from B to A is $(P - B) : (A - P)$ and, so, is the reciprocal of the ratio in which P divides the same interval from A to B. [Explain.]

As noted in Exercise 5 of Part E, $P \in \overline{AB}$ if and only if $(P - A) : (B - P) > 0$. If P is somewhere else on $\overleftrightarrow{AB}$ [but $A \neq P \neq B$] then the ratio is negative [but, since $A \neq B$, the ratio is never -1]. As you were asked to show in Exercise 5, the possibilities are as indicated in Fig. 8-4(a) and Fig. 8-4(b). Even though in these cases $P \notin \overline{AB}$ it is

There are some general remarks on TC 320, 321(1) concerning division of an interval by a point. Also, students should recognize '$(P - A):(B - P)$' from recent exercises. Ratios — particularly in connection with "points of division" are basic for most of what follows in this volume.

The phrase 'the interval from A to B' may confuse some, since it appears to refer to a "sensed interval" and this notion has not been defined. The complete phrase:

P divides the interval from A to B

is best thought of as an elision for:

P divides the interval $\overline{AB}$ in the sense of $B - A$

So, in the example given on page 327, perhaps the best word translation for the sentence:

$$(P - A):(B - P) = 1/2$$

is this:

P divides the interval $\overline{AB}$ in the sense
of $B - A$ in the ratio of 1 to 2.

Intuitively, of course, the concept under discussion should be clear enough, at least when $P \in \overline{AB}$; and the difference in meaning brought about by interchanging 'A' and 'B' should be easily grasped.

Given that $(P - A):(B - P)$ is r/s, it follows that neither r nor s is zero and that $(P - B):(A - P)$ is s/r. Since r/s and s/r are reciprocals, so are $(P - A):(B - P)$ and $(P - B):(A - P)$.

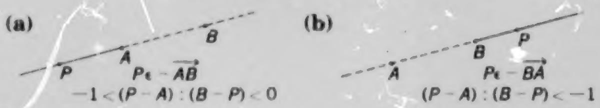


Fig. 8–4

still customary to say that P divides the interval from A to B in $(P - A) : (B - P)$. [Sometimes these cases are called "cases of exterior division" of the interval from A to B.] So, we shall adopt:

> **Definition 8–1** For $P \in \overleftrightarrow{AB}$, $A \neq P \neq B$, and $a \neq 0 \neq b$, P divides the interval from A to B in $a : b$
>
> if and only if
>
> $$(P - A) : (B - P) = a/b.$$

[As previously noted we shall allow ourselves to substitute 'segment' for 'interval' in Definition 8–1.] Note that to say that $P \in \overleftrightarrow{AB}$ and $A \neq P \neq B$ amounts to saying that A, B, and P are three collinear points. [Explain why it follows that $A \neq B$.] This is sufficient to insure that $P - A$ has a ratio to $B - P$, and that this ratio is not -1. [Explain.] In particular, if $(P - A) : (B - P) = a/b$ then $a + b \neq 0$.

From the definition and (i) of Part E we have:

> **Theorem 8–4** For $A \neq P \neq B$,
>
> (a) P divides the interval from A to B in $a : b$
>
> $$\longrightarrow P = A + (B - A)\frac{a}{a+b} \quad [P \in \overleftrightarrow{AB},\ a \neq 0 \neq b]$$
>
> (b) $P = A + (B - A)\dfrac{a}{a+b}$
>
> $\longrightarrow$ P divides the interval from A to B in $a : b$ $[a + b \neq 0]$.

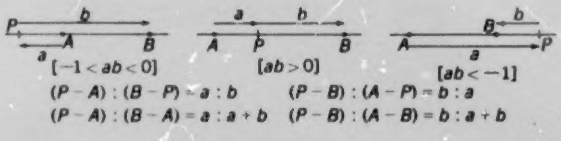


Fig. 8–5

As (ii) and (iii) were inferred from (i) in Part E, so we may infer

> **Corollary** For $A \neq P \neq B$,
>
> (a) P divides the interval from A to B in $s : 1$
>
> $$\longrightarrow P = A + (B - A)\frac{s}{s+1} \quad [P \in \overleftrightarrow{AB},\ s \neq 0]$$

The explanations asked for in the text following Definition 8-1 are as follows:

> Assuming that $P \in \overleftrightarrow{AB}$ it follows that if $A = B$ then $P = A = B$. So, if $A \neq P \neq B$ then $A \neq B$. In case $(P - A):(B - P) = -1$ it follows that $P - A = P - B$ and, so, that $A = B$. Since, as just noted, $A \neq B$, the ratio is not -1.

These remarks justify, for Theorem 8-4, replacing the restrictions on (i) of Part E by the restrictions which appear in the statement of the theorem.

In reading the definition, note that 'a : b' is read as 'the ratio of a to b'.

Figure 8-6 on page 329 illustrates Theorem 8-5, but the 's's and 't's also correlate with those of the corollary to Theorem 8-4.

(b) $P = A + (B - A)t$
$\longrightarrow P$ divides the interval from A to B in $t : 1 - t$.

Finally, from your work in Part E it follows that

Theorem 8-5 For $P \in \overleftrightarrow{AB}$ and $A \neq P \neq B$,

$$P \in -\overrightarrow{AB} \qquad \text{or } P \in \overline{AB} \qquad \text{or } P \in -\overrightarrow{BA}$$

according as the ratio in which P divides the interval from A to B is

between -1 and 0 or positive or less than -1.

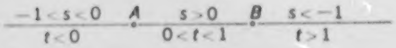


Fig. 8-6

Exercises

Part A

1. Picture a line $\overleftrightarrow{MN}$ and points P, Q, R, S, T, and U such that

(a) $(P - M) : (N - P) = 2$
(b) $(Q - M) : (N - Q) = -3$
(c) $(R - M) : (N - R) = -1 : 3$
(d) $S = M + (N - M)\frac{1}{2}$
(e) $T = M + (N - M) \cdot -\frac{3}{4}$
(f) $(U - M)4 = (N - U) \cdot -3$

2. Complete the following sentences about the points in Exercise 1.

(a) $P = M + (N - M) \cdot$ ____
(b) $Q = M + (N - M) \cdot$ ____
(c) $R = M + (N - M) \cdot$ ____
(d) $(S - M) : (N - S) =$ ____
(e) $(T - M) : (N - T) =$ ____
(f) $U = M + (N - M) \cdot$ ____

Part B

An alternative form of Definition 8-1 is suggested by the theorem:

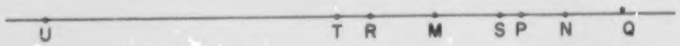
For $P \in \overleftrightarrow{AB}$, $A \neq P \neq B$, and $b \neq 0$,
$(P - A) : (B - P) = a/b \longleftrightarrow (P - A)b = (B - P)a$.

Prove this theorem.

Parts A and B provide class exercises to insure a facility with the various ways of expressing ratios. Parts C and D together make a reasonable homework assignment. Part E, when used as a class activity, can stimulate lively discussion. Part F can then be a homework activity. Be sure the discussion at the bottom of page 332 and top of page 331 is clear before assigning Part F.

Answers for Part A

1.



2. (a) 2/3 (b) 3/2 (c) $-1/2$ (d) 1 (e) $-3:7$ (f) -3

Answers for Part B

[The alternative form of the definition is obtained by replacing its last line (page 328) by '$(P - A)b = (B - P)a$'.]

Suppose that $P \in \overleftrightarrow{AB}$ and $A \neq P \neq B$. It follows that $[P - A] = [B - P] \neq \{\vec{0}\}$ and so, by Definition 7-14, that

$$(P - A):(B - P) = a/b \iff P - A = (B - P)(a/b).$$

Assuming that $b \neq 0$, $P - A = (B - P)(a/b) \iff (P - A)b = (B - P)a$.

TC 330 (1)

Answers for Part C

1. (a) Since $(P - A):(B - P) = s$, $P - A = (B - P)s = [(A - P) + (B - A)]s$. Hence, $(P - A)(s + 1) = (B - A)s = (A - B) \cdot -s$. Consequently, $(A - B):(P - A) = (s + 1):-s$.

[Note that $s + 1 \neq 0$ because $A \neq B$ and $(P - A):(B - P) = s$.] [Alternatively, this exercise can be solved by starting with part (a) of the corollary to Theorem 8-5.]

(b) [This can be established by the same technique used for (a). It is worthwhile, however, for students to note that it can be obtained from (a), itself. Here's how:

By (a),

$$(P - A):(B - P) = s \implies (A - B):(P - A) = -\frac{s + 1}{s}.$$

So, by a cyclic permutation which replaces 'P' by 'A', 'A' by 'B' and 'B' by 'P',

$$(A - B):(P - A) = r \implies (B - P):(A - B) = -\frac{r + 1}{r}.$$

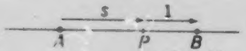
Substituting '$-\frac{s + 1}{s}$' for 'r' yields the expected results.]

2. $A \in \overline{BP}$ if and only if $P \in -\overleftrightarrow{AB}$. So, $A \in \overline{BP}$ and $P \neq A$ if and only if $P \in -\overrightarrow{AB}$. So, we could establish the first case of Theorem 8-5 by showing that $-1 < s < 0$ if and only if $(s + 1)/s < 0$. This is easy to show by considering the two cases [$s + 1 > 0$ and $s < 0$, $s + 1 < 0$ and $s > 0$] in which $(s + 1)/s < 0$. The second case holds if and only if $-1 < s < 0$; the first case is impossible.

3. By (b) and the second case of Theorem 8-5, $B \in \overline{AP}$ if and only if $s + 1 < 0$. So, as in Exercise 2, $P \in -\overrightarrow{BA}$ if and only if $s < -1$. This reestablishes the third case of Theorem 8-5.

Part C

$(P - A) : (B - P) = s$



$(A - B) : (P - A) = \frac{-(s + 1)}{s}$

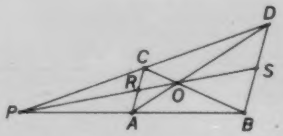
$(B - P) : (A - B) = \frac{-1}{s + 1}$

1. Suppose that P, A, and B are three collinear points, $s \neq 0$, and that P divides the interval from A to B in the ratio $s : 1$. Show that
   (a) A divides the interval from B to P in the ratio $s + 1 : -s$, and that
   (b) B divides the interval from P to A in the ratio $-1 : s + 1$.
2. From Exercise 1(a) and the second of the three cases of Theorem 8–5 it follows that $A \in \overline{BP}$ if and only if $(s + 1)/s < 0$. Relate this result to the first case of the theorem.
3. Relate the result in Exercise 1(b) to the third case of Theorem 8–5.

Part D

1. Show that if $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are noncollinear parallel segments such that $D - B \neq C - A$ then $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at a point which divides both the interval from A to B and the interval from C to D in $(C - A) : (B - D)$. [*Hint*: Most of the work has been done in proving an earlier theorem.]
2. Illustrate Exercise 1 for a case in which the ratio in question is negative and for a case in which this ratio is positive.
3. Suppose that $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are noncollinear parallel segments, that $Q \in \overrightarrow{AC}$, $R \in \overrightarrow{BD}$, $A \neq Q \neq C$, and $B \neq R \neq D$. Show that Q and R divide the intervals from A to C and from B to D, respectively, in the same ratio if and only if
   (i) $\overleftrightarrow{QR} \parallel \overleftrightarrow{AB}$ [in case $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$] or
   (ii) $P \in \overleftrightarrow{QR}$ [in case $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{P\}$]. [*Hint*: Again, most of the work was done earlier in this chapter.]
4. Restate the corollary to Theorem 8–3 in terms of the ratios in which points divide intervals.

Part E

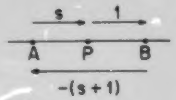


Suppose that $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are noncollinear parallel segments and that $(D - B) : (C - A) = r \neq -1$. Suppose that $R \in \overrightarrow{AC}$, $S \in \overrightarrow{BD}$, $A \neq R \neq C$, and $B \neq S \neq D$.

1. The figure illustrates the case in which $r > 1$. Draw other figures illustrating other cases.
2. Show that $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ intersect at a point O which divides both the interval from A to D and the interval from C to B in $1 : r$.
3. What ratios must be the same in order that $\{R, S, O\}$ be collinear?
4. (a) Suppose that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at P. Can you choose R and S so that both O and P belong to $\overleftrightarrow{RS}$?
   (b) Suppose that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Can you choose R and S so that $O \in \overleftrightarrow{RS}$ and $\overleftrightarrow{RS} \parallel \overleftrightarrow{AB}$?

Students should by now have at least a temporary acquaintance with the transformations of ratios dealt with in Part E on page 326, Theorem 8-4 and its corollary, and Exercise 1, above. Having established the validity of these transformations it is well worthwhile to bring out the fact that they can be easily recalled by sketching a figure showing a particular case. The figure for Exercise 1, for example, illustrates the case in which $(P - A):(B - P) > 0$. But, as shown in the exercise, the expressions for $(A - B):(P - A)$ and $(B - P):(A - B)$, which the figure makes obvious in this case, are valid in all cases:

Exercise 1.

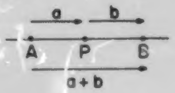


$(P - A):(B - P) = s$

$(A - B):(P - A) = -(s + 1)/s$

$(B - P):(A - B) = 1/-(s + 1) = -1/(s + 1)$

Theorem 8-4.

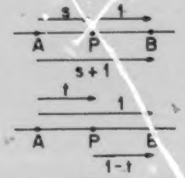


$(P - A):(A - B) = a : b$

$(P - A):(B - A) = \frac{a}{a + b}$

$(P - B):(B - A) = \frac{-b}{a + b}$, etc.]

Corollary.



$(P - A):(B - P) = s$

$(P - A):(B - A) = \frac{s}{s + 1}$

$(P - A):(B - A) = t$

$(P - A):(B - P) = t : 1 - t$

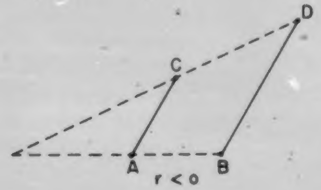
This graphic procedure of computing a ratio involving three points in terms of another such ratio is extended on pages 331 and 332 to apply to cases in which more than three points are involved or two or more ratios are given.

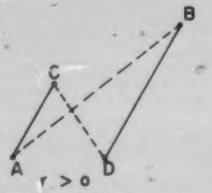
In answers to exercises, we shall usually refer to the appropriate theorems; but, students should probably be allowed to use the graphical method to obtain the desired results.

Answers for Part D

1. By Theorem 8-2, the point in question — say, P — is such that $P = A + (B - A)\cdot /(1 - r)$, where $r = (D - B):(C - A)$. So, by (iii) of Part E on page 326, P divides the interval from A to B in the ratio $t:(1 - t)$, when $t = /(1 - r)$. On subtracting and simplifying, the ratio turns out to be $-/r$, and this is $(C - A):(B - D)$.

2.





3. This result follows from Exercise 3 of Part C on page 322 and the fact that

$$(Q - A):(C - A) = (R - B):(D - B)$$

$$\Longleftrightarrow (Q - A):(C - Q) = (R - B):(D - R).$$

The latter follows most readily from (ii) and (iii) of Part E on page 325.

4. If A, B, C, D, and P are five points such that $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{P\}$ then $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$ if and only if P divides the interval from C to D and the interval from A to B in the same ratio.

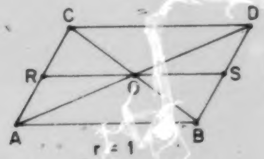
Part E brings together results already deduced by using Theorem 8-2, including those which were first noted in the commentary for page 320 and the results obtained in the exercises of Part D. These results will be of frequent use, and are summarized again in Part B on page 333.

Exercise 4(a) adds to earlier results [see, for example, the figure on TC 320(2)] the fact that, in trapezoid ABDC, the midpoints of the parallel sides, the intersection of the diagonals, and the intersection of the lines containing the nonparallel sides, are collinear. Of course, this can also be interpreted as saying something about △PBD [or △PAC, or △OBD]. It also has consequences concerning quadrilaterals, such as PAOC, in which one diagonal bisects the other.

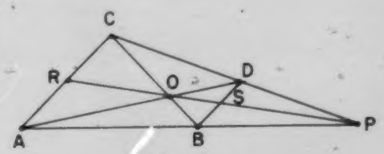
The variety of the interpretations just referred-to may indicate why a mathematician might be satisfied with Theorem 8-2, say, and be happy to forgo talk of triangles, trapezoids, etc.!

Answers for Part E

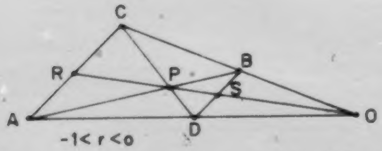
1.



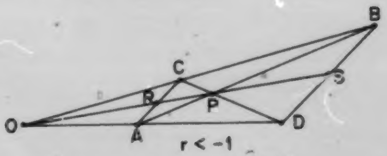
$r = 1$



$0 < r < 1$



$-1 < r < 0$



$r < -1$

[The figures should remind students that, while Theorem 8-2 "explicitly" gives information about the point P at which $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect, it can also be used to locate the point O of intersection of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$.]

2. By Theorem 8-2 [with 'B' and 'D' interchanged and '−r' for 'r'] it follows that, since $(B - D):(C - A) \neq -1$, $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ intersect at a point O such that

$$O = A + (D - A)\cdot/(1 + r) = C + (B - C)\cdot/(1 + r).$$

As in Exercise 1 of Part D [again, interchange 'B' and 'D' and replace 'r' by '−r'], O divides the intervals in question in $1 : r$. [If, as in the figure, $r \neq 1$, then $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at a point P which divides the intervals from A to B and from C to D in $-1 : r$.]

Answers for Part E [cont.]

3. From Exercise 3 of Part C on page 322 or Exercise 3 of Part D, above, with 'B' and 'D' interchanged, $O \in \overleftrightarrow{RS}$ if and only if $(R - A):(C - A) = (S - D):(B - D)$ and, also, if and only if $(R - A):(C - R) = (S - D):(B - S)$.

4. (a) Yes. By the second of the answers for Exercise 3, $O \in \overleftrightarrow{RS}$ if and only if $(R - A):(C - R) = (S - D):(B - S)$. Interchanging 'B' and 'D', $P \in \overleftrightarrow{RS}$ if and only if $(R - A):(C - R) = (S - B):(D - S)$. Hence, O and P both belong to $\overleftrightarrow{RS}$ if and only if $(S - D):(B - S) = (S - B):(D - S)$. This condition is satisfied if and only if the common value of these ratios — each of which is the reciprocal of the other — is 1. [The common value cannot be −1 because $B \neq D$.] In short, R and S must be the midpoints of $\overline{AC}$ and $\overline{BD}$, respectively.

   (b) Yes. Again, R and S must be the midpoints of $\overline{AC}$ and $\overline{BD}$.

5. (a) For $O \in \overleftrightarrow{RS}$, $(R - A):(C - A) = (S - D):(B - D)$. Since the four translations whose ratios are involved here have the same direction it follows that $(R - A):(S - D) = (C - A):(B - D) = -/r$.

   (b) Since $\overline{AR} \parallel \overline{DS}$, $(O - R):(S - O) = (O - A):(D - O) = /r$. [Corollary to Theorem 8-3 and Exercise 2, above.]

Answers for Part E [cont.]

6. (a) $(O - A):(D - A) = /(1 + r)$. So, for $V \in \overleftrightarrow{AB}$, $\overleftrightarrow{VO} \parallel \overleftrightarrow{BD}$ if and only if $(V - A):(B - A) = /(1 + r)$. So, V must be the point $A + (B - A)\cdot/(1 + r)$.

   (b) As in part (a), $W = C + (D - C)\cdot/(1 + r)$.

   (c) $O - V = (O - A) + (A - B)\cdot/(1 + r) = (D - A)\cdot/(1 + r)$
   $+ (A - B)\cdot/(1 + r) = (D - B)\cdot/(1 + r)$

   $W - O = (C - O) + (D - C)\cdot/(1 + r) = (C - B)\cdot/(1 + r)$
   $+ (D - C)\cdot/(1 + r) = (D - B)\cdot/(1 + r)$

   Hence, $(O - V):(W - O) = 1:1$.

   (d) By (c), $W - V = (W - O) + (O - V) = (D - B)(2/(1 + r))$. Since $(D - B):(C - A) = r$ it follows that $W - V = (C - A)(2r/(1 + r))$. Hence,

$$(V - W):(B - D) + (V - W):(A - C) = \frac{2}{1 + r} + \frac{2r}{1 + r} = 2.$$

5. Suppose that $O \in \overrightarrow{RS}$. Compute:
   (a) $(R - A) : (S - D)$ (b) $(O - R) : (S - O)$
6. (a) Find [if possible] a point $V \in \overleftrightarrow{AB}$ such that $\overleftrightarrow{VO} \parallel \overleftrightarrow{BD}$. [*Hint*: There are several ways to go about this, but since you know $(O - A) : (C - A)$ the simplest is to use Exercise 4 of Part D.]
   (b) Find a point $W \in \overleftrightarrow{CD}$ such that $\overleftrightarrow{VO} \parallel \overleftrightarrow{DB}$.
   (c) In what ratio does O divide the interval from V to W?
   (d) What is $(V - W) : (B - D) + (V - W) : (A - C)$?
7. Suppose that $|r| \neq 1$ and that [as shown in the figure] P, R, S, and O are collinear. Express each of the following ratios in terms of 'r'.
   (a) $(P - R) : (S - P)$ (b) $(O - R) : (S - O)$
   (c) $(R - S) : (P - R)$ (d) $(R - S) : (O - R)$
   (e) $(S - P) : (R - S)$ (f) $(S - O) : (R - S)$
   (g) $(O - R) : (P - R)$ (h) $(S - P) : (S - O)$
8. In Exercise 7, what can you say about the ratios in which R and S divide the interval from P to O? In which P and O divide the interval from R to S?

*

You should have found Exercise 7 rather easy if you recalled the answers for parts (a) and (b) and the results you learned in Part D, and used some simple algebra of ratios. This procedure is, however, rather roundabout in case, for example, you happen to need, say, only the answer to part (g). There is an easy graphical method for finding the answers to such problems. [Perhaps you already use a simple form of it to help you recall the results of Part D.] To solve Exercise 7 by this method, we indicate the collinear points P, R, S, and O, and the known ratios $(P - R) : (S - P)$ and $(O - R) : (S - O)$, in a diagram [the role of 't' will appear in a moment]:

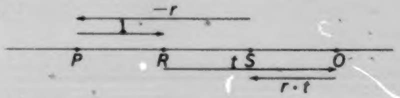


Fig. 8–7

Using the notations above the line we can compute ratios involving the points P, R, and S. For example, $(R - S) : (P - R) = \dfrac{-(r - 1)}{-1} = r - 1$. Similarly, using the notations below the line we can compute ratios which involve R, S, and O. If we wish to compute a ratio which involves both P and O then we must choose t in such a way as to make the notations consistent with one another. For this, we need $t + rt = r - 1$ and, so, $t = \dfrac{r - 1}{r + 1}$. We can now see that, say,

$$(O - R) : (P - R) = \frac{t}{-1} = \frac{1 - r}{1 + r}.$$

[This result is as close as we can come at present to saying that, given two bases, $\overline{AC}$ and $\overline{BD}$, of a trapezoid, the measure m of the segment through the intersection of the diagonals and parallel to the bases is the harmonic mean of the measures b_1 and b_2 of the bases:

$$m = 2/\left(\frac{1}{b_1} + \frac{1}{b_2}\right)$$

Of course, considerably more has been proved. For example, locate V and W on the figures in answer to Exercise 1 for which $r < 0$, and interpret the result of the present exercise.]

*

It may be helpful to summarize some of the results obtained:

Each interval contained in a line through O and whose end points belong to $\overline{AC}$ and $\overline{BD}$, respectively, is divided by O in $1 : r$, and the ratios in which the end points of such an interval divide the intervals from A to C and from D to B, respectively, are the same. [Exercise 5, parts (b) and (a).]

[Replacing 'O' by 'P', 'r' by '−r' and interchanging 'B' and 'D', in the preceding statement gives another correct result.]

P, R, O, and S are collinear if and only if R is the midpoint of $\overline{AC}$ and S is the midpoint of $\overline{BD}$. [Exercise 4(a).]

O is the midpoint of $\overline{VW}$, and

$$(O - V):(D - B) = /(1 + r) \text{ and}$$

$$(O - V):(C - A) = r/(1 + r). \quad \text{[Exercise 6]}$$

These results hold for any value of 'r' other than 1 and −1. [Those not involving P hold even if $r = 1$.]

*

7. [The assumption that $|r| \neq 1$ is, of course, to ensure the existence of P as well as that of O.]
   (a) $(P - R):(S - P) = (P - A):(B - P) = -/r$
   (b) $(O - R):(S - O) = /r$
   (c) $(R - S):(P - R) = (A - B):(P - A) = -//(1 - r) = r - 1$
   (d) $(R - S):(O - R) = \dfrac{/r + 1}{-/r} = -(1 + r)$ [Use (b).]
   (e) $(S - P):(R - S) = -r \cdot /(r - 1) = r/(1 - r)$ [Use (a) and (c).]
   (f) $(S - O):(R - S) = -r/(1 + r)$ [Like (e).]
   (g) $(O - R):(P - R) = (1 - r)/(1 + r)$ [Use (c) and (d).]
   (h) $(S - P):(S - O) = (r + 1)/(r - 1)$ [Use (e) and (f).]
8. In each case, the two ratios are opposites.

*

The method of computing ratios from others which is illustrated on pages 331 and 332 is a considerable time-saver. As is pointed out, the order in which the points are indicated in Figure 8-7 is immaterial.

[If this looks like magic, note that, above the line, we are really saying that $S - R = (R - P)(r - 1)$. Below the line we are saying that $S - R = (R - P)(t + rt)$. In order to be correct both times we must have $t + rt = r - 1$.] Note that it doesn't matter in what order we list the points.

Part F

1. Use the graphical technique to check your answers for Exercise 7 of Part E and to find $(P - O) : (R - S)$ and $(P - S) : (R - O)$.
2. Suppose that A, B, C, and D are four collinear points such that $(A - B) : (C - B) = a$ and $(B - D) : (D - C) = b$. Compute at least six ratios each of which involves three or four of the given points.
3. Suppose that J, K, L, and M are four points of a line l such that $(K - J) : (L - J) = 2$ and $(M - L) : (K - L) = -4$. Draw an appropriate picture and compute these ratios.
   (a) $(J - M) : (L - J)$ (b) $(K - J) : (J - M)$
   (c) $(M - J) : (K - M)$ (d) $(K - L) : (K - M)$
   (e) $(K - L) : (M - J)$ (f) $(M - L) : (K - J)$
   (g) $(M - J) : (M - L)$ (h) $(J - L) : (K - L)$

8.03 Triangles

Definition 8–2
(a) $PQR = \overline{PQ} \cup \overline{QR} \cup \overline{RP}$
(b) PQR is a triangle $\Longleftrightarrow \{P, Q, R\}$ is noncollinear

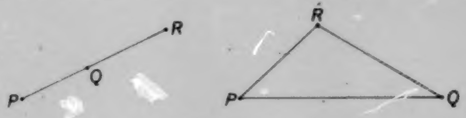


Fig. 8–8

We shall usually write '$\triangle PQR$' when we wish to refer to the set PQR and, at the same time, imply that $\{P, Q, R\}$ is noncollinear. The points P, Q, and R are the *vertices* of $\triangle PQR$, and the intervals $\overline{QR}$, $\overline{RP}$, and $\overline{PQ}$ are its *sides*. P is the vertex of $\triangle PQR$ opposite side $\overline{QR}$, and $\overline{QR}$ is the side of $\triangle PQR$ opposite the vertex P. Etc. Note that, for any points P, Q, and R, $PQR = PRQ$. So, in particular, $\triangle PQR = \triangle PRQ$. Give four other ways of referring to the same triangle.

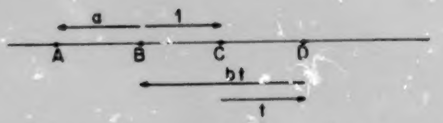
Exercises

Part A

1. (a) Draw a $\triangle KZR$.
   (b) List the vertices and the sides of your triangle, pairing up each vertex with the opposite side.

Answers for Part F

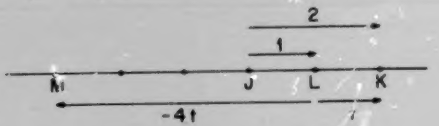
1. $(P - O):(R - S) = -2r/(1 + r)(1 - r)$; $(P - S):(R - O) = r(r + 1)/(r - 1)$
2. [The answers will vary according to which ratios students choose to compute. Computations may be based on a figure like:



where, since $t + bt = -1$, $t = -1/(b + 1)$. As an example:

$$(B - A):(D - C) = -a : t = a(b + 1)$$]

3.



with $t = 1$.

(a) 3 (b) 2/3 (c) 3/5 (d) 1/5
(e) −1/3 (f) −2 (g) 3/4 (h) −1

The letters 'P', 'Q', and 'R' in any order, preceded by a '△', gives a symbol referring to △PQR.

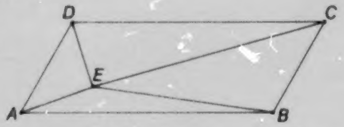
The exercises of Part A provide practice with the language of triangles and can easily be treated in class. Parts B and C can be used as a homework assignment. This is a rather large assignment, however, containing many important applications of ratios. We recommend that in addition to assigning Parts B and C as homework you discuss the exercises in class the next day.

Answers for Part A

1. (a) [Any triangle.] (b) K, $\overline{ZR}$; Z, $\overline{RK}$; R, $\overline{KZ}$

2. $\overline{KB}$ [or: $\overline{BK}$]; K
3. [Since no three of the five points are collinear, any three are vertices of a triangle. For grading purposes it is probably better to know that there are 10 triangles, rather than to have a check-list. It is common to have students who do not "see", for example, that B, E, and D are vertices of a triangle because $\overline{BD}$ is not drawn in the figure.]

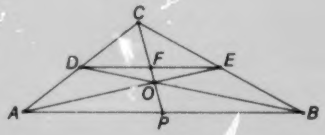
2. Without drawing $\triangle BLK$, tell which side of this triangle is opposite L and which vertex is opposite the side $\overline{BL}$.
3. Name all the triangles whose vertices are among the points A, B, C, D, E in the figure.



4. (a) Draw a $\triangle PQR$ and mark the midpoints, M and N, of $\overline{PR}$ and $\overline{RQ}$, respectively.
   (b) Show that $\overline{MN} \parallel \overline{PQ}$.
   (c) Compute $(N - M) : (Q - P)$.

Part B

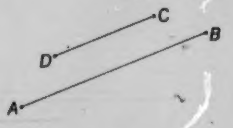
In $\triangle ABC$ suppose that $D \in \overline{CA}$ and $E \in \overline{BC}$.



1. Show that $\overline{DE} \parallel \overline{AB}$ if and only if D and E divide the sides from C to A and from C to B, respectively, in the same ratio.
2. Suppose that $\overline{DE} \parallel \overline{AB}$.
   (a) Show that $(E - D) : (B - A) = r$ if and only if $(D - C) : (A - C) = r$. From this and the hypothesis that $D \in \overline{CA}$, what can you conclude about $(E - D) : (B - A)$?
   (b) Show that $\overleftrightarrow{AE}$ and $\overleftrightarrow{BD}$ intersect at a point O, and compute $(O - D) : (B - O)$ and $(O - E) : (A - O)$. Must O belong to the intervals $\overline{AE}$ and $\overline{BD}$? Explain.
   (c) Show that $\overleftrightarrow{CO}$ intersects $\overline{DE}$ and $\overline{AB}$ at their respective midpoints, F and P.
   (d) Compute $(O - F) : (P - O)$ and $(F - C) : (P - C)$.
   (e) Compute $(O - P) : (C - O)$.

Part C

We have had a good deal to do with ratios of translations. It will turn out shortly that, for some purposes, it is more convenient to speak of *ratios of intervals*. Given parallel intervals $\overline{AB}$ and $\overline{CD}$ we shall say that the ratio of $\overline{CD}$ to to $\overline{AB}$ [for short: $\overline{CD} : \overline{AB}$] is the *absolute value* of the ratio of $D - C$ to $B - A$:

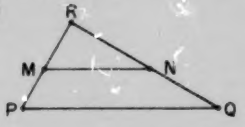


$$\overline{CD} : \overline{AB} = |(D - C) : (B - A)|$$

1. Show that $\overline{CD} : \overline{AB} = \overline{DC} : \overline{AB} = \overline{DC} : \overline{BA} = \overline{CD} : \overline{BA}$ and that $\overline{AB} : \overline{CD}$ is the reciprocal of $\overline{CD} : \overline{AB}$.

Answers for Part A [cont.]

4. (a)



(b) Since $(R - Q):(R - N) = 2 = (R - P):(R - M)$ it follows by the corollary to Theorem 8-3 that $\overline{MN} \parallel \overline{PQ}$.

(c) If $(N - M):(Q - P) = r$ then, by Theorem 8-2, $R = P + (M - P) \cdot/(1 - r)$. Since different values of 'r' yields different values of '/(1 - r)', the converse also holds. So, since $(R - P):(M - P) = 2$, it follows that $1 - r = 1/2$ and that $r = 1/2$. Hence, $(N - M):(Q - P) = 1/2$. [Although it is good practice to learn to use the general theorems developed in this chapter, it is often easier to solve exercises independently of these theorems. In the case of the present exercise it is convenient to let $\vec{p} = P - R$ and $\vec{q} = Q - R$. Then $N - M = (N - R) - (M - R) = \vec{q}\cdot/2 - \vec{p}\cdot/2 = (\vec{q} - \vec{p})\cdot/2 = (Q - P)\cdot/2$. This result yields both (b) and (c).]

Answers for Part B

[As remarked in the commentary for page 330, preceding the discussion of Part E, much of the present Part B is a review of the results obtained in Part E. Exercise 1 is an exception to this. In any case, all follows from Theorems 8-2 and 8-3. The results of Exercise 2 are summarized in Theorems 8-9 and 8-10 on page 338.]

1. This is an immediate consequence of the corollary to Theorem 8-3 [as restated in Exercise 4 on page 330] together with the fact that C divides the intervals from D to A and from E to B in the same ratio if and only if the ratio in which D divides the interval from C to A is the same as the ratio in which E divides the interval from C to B. This last follows from the fact that either of $(C - D):(A - C)$ and $(D - C):(A - D)$ is computable from the other and, in the same way, the corresponding one of $(C - E):(B - C)$ and $(E - C):(B - E)$ is computable from the other. So, starting with the same value for $(C - D):(A - C)$ and $(C - E):(B - C)$ one will obtain by computation the same value for $(D - C):(A - D)$ and $(E - C):(B - E)$, and vice versa.

2. (a) [To show that $(E - D):(B - A) = r$ if and only if $(D - C):(A - C) = r$ amounts just to showing that $(E - D):(B - A) = (D - C):(A - C)$. And, to show the latter, it is sufficient to show that if $(E - D):(B - A) = r$ then $(D - C):(A - C) = r$.] By Theorem 8-2, since $\overleftrightarrow{AD} \cap \overleftrightarrow{BE}$ consists of a single point, $(E - D):(B - A) \neq 1$. Moreover, since $\overleftrightarrow{AD} \cap \overleftrightarrow{BE} = \{C\}$ it follows that if $(E - D):(B - A) = r$ then $C = A + (D - A)\cdot/(1 - r)$. [Note that both parts of the conclusion of Theorem 8-2 have been called on in the preceding argument.] It follows that $(C - A):(D - A) = /(1 - r)$, $(D - A):(A - C) = r - 1$, and, so, $(D - C):(A - C) = (r - 1) + 1 = r$. Hence, if $(E - D):(B - A) = r$ then $(D - C):(A - C) = r$. Since $(E - D):(B - A)$ is some number, it follows that $(E - D):(B - A) = (D - C):(A - C)$. Consequently, $(E - D):(B - A) = r$ if and only if $(D - C):(A - C) = r$.

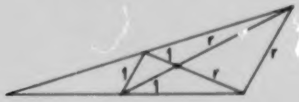
   Assuming that $D \in \overline{CA}$ it follows that $0 < (D - C):(A - C) < 1$. So, under this hypothesis, $0 < (E - D):(B - A) < 1$.

Answers for Part B [cont.]

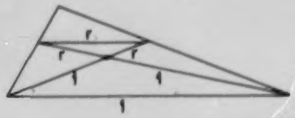
(b) [Students should recognize this as a repetition of Exercise 2 of Part E on page 330. If one identifies the points D, A, B, and E of the present exercise with the points A, B, D, and C of Part E then the points O of the two exercises correspond, but the number r of Part E is the reciprocal of the number r of part (a) of the present exercise. Translating the result of Exercise 2 of Part E into the present notation, we see that O divides the interval from D to B and the interval from E to A in r : 1. So, the value of each of the ratios in question is r — that is, each is (E - D):(B - A).

Since the latter ratio is between 0 and 1, so are the former. It follows from this that O belongs both to $\overline{BD}$ and to $\overline{AE}$.

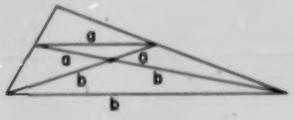
Students may answer by referring to a figure like:



which summarizes the relevant information from Part E. The corresponding figure for the present exercise is:



More "generally":



For a verbal statement of the property used, see the figure of the summarizing statements following the answer for Exercise 6 of Part E.]

(c) [This has been shown in Exercise 4(a) of Part E on page 330.]

(d) (O - F):(P - O) = (E - D):(B - A)

(F - C):(P - C) = (D - C):(A - C) = (E - D):(B - A)

(e) (O - P):(C - O) = (1 - r):(2r), where r = (E - D):(B - A). [Use part (d) and the technique explained on page 331.]

It is sometimes convenient to speak of the ratio of two intervals [or segments]. [See, for example, Theorem 8-6 on page 335 and Theorem 8-7 on page 335.] In doing so, one disregards the notion of sense — which does not apply to intervals — and, so often loses information which is available in statements about ratios of translations or about ratios of sensed distances [see Part D on page 363].

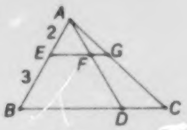
Answers for Part C

1. Since (D - C):(B - A) = -[-(D - C):(B - A)] = -[(C - D):(B - A)], and since numbers which are opposites have the same absolute value, it follows that $|(D - C):(B - A)| = |(C - D):(B - A)|$. So, by definition, $\overline{CD}:\overline{AB} = \overline{DC}:\overline{AB}$. Etc.

   Since (B - A):(D - C) is the reciprocal of (D - C):(B - A), and since the absolute values of reciprocals are, themselves reciprocals, it follows that $|(B - A):(D - C)|$ is the reciprocal of $|(D - C):(B - A)|$. So, by definition, $\overline{AB}:\overline{CD}$ is the reciprocal of $\overline{CD}:\overline{AB}$.

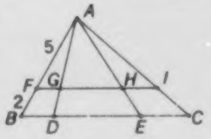
2. In each of the following, you are given certain information about a figure. Also, certain ratios of intervals are indicated in the given figures. Use this information to help you to compute the indicated ratios of intervals. [For help, see Exercise 3, page 322.]

(a)



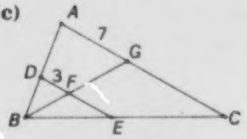
Given: $\overline{EF} \parallel \overline{BC}$
Compute: $\overline{EF} : \overline{BD}$; $\overline{DC} : \overline{FG}$; $\overline{EG} : \overline{BC}$; $\overline{AG} : \overline{AC}$

(b)



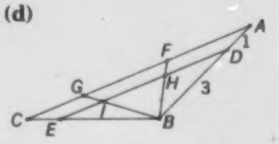
Given: $\overline{FI} \parallel \overline{BC}$
Compute: $\overline{FG} : \overline{BD}$; $\overline{HI} : \overline{EC}$; $\overline{AH} : \overline{AE}$; $\overline{BC} : \overline{FI}$

(c)



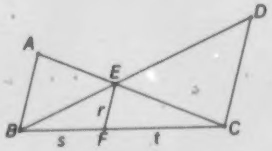
Given: $\overline{DE} \parallel \overline{AC}$
Compute: $\overline{BF} : \overline{FG}$; $\overline{BE} : \overline{EC}$; $\overline{EF} : \overline{CG}$; $\overline{CA} : \overline{ED}$

(d)



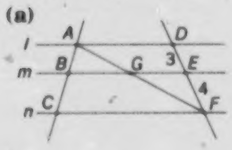
Given: $\overline{DE} \parallel \overline{AC}$
Compute: $\overline{DH} : \overline{AF}$; $\overline{IE} : \overline{GC}$; $\overline{BI} : \overline{BG}$; $\overline{AC} : \overline{DE}$

3. Suppose that $\overline{AB} \parallel \overline{EF} \parallel \overline{DC}$, and that the ratios among the intervals $\overline{EF}$, $\overline{BF}$, and $\overline{FC}$ are as indicated in the figure at the right.
   (a) Compute these ratios:
   $\overline{EF} : \overline{AB}$; $\overline{EF} : \overline{DC}$;
   $\overline{CE} : \overline{EA}$; $\overline{BE} : \overline{BD}$
   (b) Show that
   $\overline{EF} : \overline{AB} + \overline{EF} : \overline{CD} = 1$.



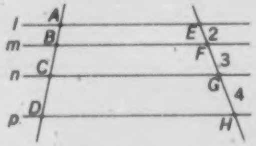
4. In each of the following, you are given that $l \parallel m \parallel n$. Also, certain ratios of intervals are indicated in the given figure. Use this information to compute the indicated ratios of intervals. [For help, see Exercise 5, page 325.]

(a)



Compute: $\overline{AB} : \overline{BC}$; $\overline{AG} : \overline{AF}$;
$\overline{AD} : \overline{GE}$; $\overline{BG} : \overline{CF}$

(b)



Given: $p \parallel l$
Compute: $\overline{AB} : \overline{CD}$; $\overline{BC} : \overline{AB}$;
$\overline{AC} : \overline{CD}$; $\overline{AB} : \overline{AD}$

Answers for Part C [cont.]

2. (a) $\overline{EF} : \overline{BD} = 2/5$; $\overline{DC} : \overline{FG} = 5/2$; $\overline{EG} : \overline{BC} = 2/5$;
$\overline{AG} : \overline{AC} = 2/5$
   (b) $\overline{FG} : \overline{BD} = 5/7$; $\overline{HI} : \overline{EC} = 5/7$; $\overline{AH} : \overline{AE} = 5/7$;
$\overline{BC} : \overline{FI} = 7/3$
   (c) $\overline{BF} : \overline{FG} = 3/4$; $\overline{BE} : \overline{EC} = 3/4$; $\overline{EF} : \overline{CG} = 3/7$;
$\overline{CA} : \overline{ED} = 7/3$
   (d) $\overline{DH} : \overline{AF} = 3/4$; $\overline{IE} : \overline{GC} = 3/4$; $\overline{BI} : \overline{BG} = 3/4$;
$\overline{AC} : \overline{DE} = 4/3$

3. (a) $\overline{EF} : \overline{AB} = t : (s + t)$; $\overline{EF} : \overline{DC} = s : (s + t)$
$\overline{CE} : \overline{EA} = t : s$; $\overline{BE} : \overline{BD} = s : (s + t)$
   (b) $\overline{EF} : \overline{AB} + \overline{EF} : \overline{CD} = \frac{t}{s + t} + \frac{s}{s + t} = 1$

5. It is customary to call each of two intersecting lines a *transversal* of the other. An interval of a line *l* is said to be *intercepted* by any two transversals of *l* which, together, contain the endpoints of the interval. Two or more lines which have a common point are said to be *concurrent*. The following theorem restates earlier results in these new terms. Prove it by referring to the appropriate exercises and showing how these apply.

*

Theorem 8-6

(a) The ratio of two intervals which are intercepted on one of two parallel lines by concurrent transversals of both these lines is the same as that of the corresponding intervals which are intercepted by these transversals on the other.

(b) The ratio of two intervals which are intercepted by parallel lines on one transversal of these lines is the same as that of the corresponding intervals which are intercepted by these lines on any other transversal.

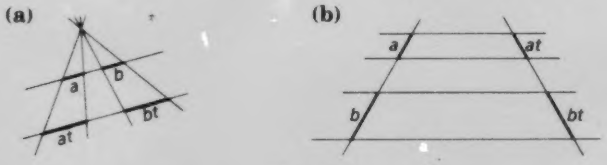


Fig. 8–9

8.04 Ratios in a Triangle

The exercises of Part B suggest several theorems about triangles, and the notion introduced in Part C helps in stating some of them. For example:

Theorem 8-7 The interval whose endpoints are the midpoints of two sides of a triangle is parallel to the third side and its ratio to the third side is 1/2.

Corollary A line through the midpoint of one side of a triangle is parallel to a second side if and only if it contains the midpoint of the third side.

Before stating our next theorem we need a definition.

Answers for Part C [cont.]

4. (a) $\overline{AB}:\overline{BC} = 3/4$; $\overline{AG}:\overline{AF} = 3/7$; $\overline{AD}:\overline{GE} = 7/4$; $\overline{BG}:\overline{CF} = 3/7$

   (b) $\overline{AB}:\overline{CD} = 1/2$; $\overline{BC}:\overline{AB} = 3/2$; $\overline{AC}:\overline{CD} = 5/4$; $\overline{AB}:\overline{AD} = 2/9$

5. Theorem 8-6(a) can be proved by using Exercise 3 of Part D on page 330. Suppose [in the notation of that exercise] the two parallel lines of the theorem are $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ and consider the concurrent transversals $\overleftrightarrow{PA}$, $\overleftrightarrow{PC}$ and $\overleftrightarrow{PQ}$ which intersect $\overleftrightarrow{AC}$ at A, C, and Q and intersect $\overleftrightarrow{BD}$ at B, D, and R, respectively. According to the exercise, $(Q - A):(C - Q) = (R - B):(D - R)$. So, $\overline{AQ}:\overline{QC} = \overline{BR}:\overline{RD}$. This establishes Theorem 8-6(a) in case the intervals are intercepted by three transversals. To establish the general case, as pictured in Figure 8-9(a) it is sufficient to apply the special case twice. Suppose that four concurrent transversals intersect one of two parallel lines at A_1, A_2, A_3, and A_4 and intersect the other at B_1, B_2, B_3, and B_4, respectively, and that we wish to show that $\overline{A_1A_2}:\overline{A_3A_4} = \overline{B_1B_2}:\overline{B_3B_4}$. By the special case, $\overline{A_1A_2}:\overline{A_2A_3} = \overline{B_1B_2}:\overline{B_2B_3}$ and, also, $\overline{A_2A_3}:\overline{A_3A_4} = \overline{B_2B_3}:\overline{B_3B_4}$. The desired result now follows from the fact that $(\overline{A_1A_2}:\overline{A_2A_3})(\overline{A_2A_3}:\overline{A_3A_4}) = \overline{A_1A_2}:\overline{A_3A_4}$ and similarly for the 'B's.

   Theorem 8-6(b) has already been established in Exercise 6 of Part D on page 325.

*

Note that Theorem 8-6(a) has a corollary:

> The ratio of two intervals intercepted on two parallel lines by two concurrent transversals is the same as that of the intervals intercepted on these lines by any two transversals which are concurrent at the same point as the given transversals.

The corollary is also illustrated by Figure 8-9(a), where the common ratio is t. [In the theorem itself, the common ratio is a/b (or b/a).] The proof is easy. Referring to the proof given above for Theorem 8-6(a), since $\overline{A_1A_2}:\overline{A_3A_4} = \overline{B_1B_2}:\overline{B_3B_4}$, and the four intervals are parallel, it follows that $\overline{A_1A_2}:\overline{B_1B_2} = \overline{A_3A_4}:\overline{B_3B_4}$.

There is a similar corollary for Theorem 8-6(b); but, here, we must require that the two transversals be parallel. The two corollaries can be combined into one if we speak of the parallel lines of Theorem 8-6(b) as the transversals:

> The ratio of two intervals intercepted on two parallel lines by two concurrent [or parallel] transversals is the same as that of the intervals intercepted on these lines by any two transversals which are concurrent at the same point as [or are parallel to] the given transversals.

You may wish to give your students a restatement of the parts of Theorem 8-6 using the word 'proportional':

> Concurrent transversals of parallel lines intercept proportional intervals.
>
> Parallel lines intercept proportional intervals on all transversals.

Definition 8–3 The *median* of a triangle from a given vertex is the interval whose endpoints are the given vertex and the midpoint of the opposite side.

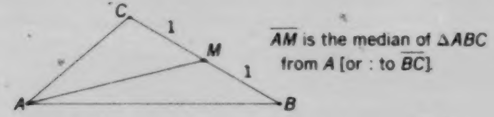

C
1
M
1
A
B
$\overline{AM}$ is the median of ΔABC from A [or : to $\overline{BC}$].

Fig. 8–10

How many medians does a triangle have? Is it possible to have a triangle which has some two of its medians collinear? Is it possible to have a triangle which has some two of its medians parallel and noncollinear? Explain your answers.

Theorem 8–8 The three medians of a triangle intersect at a point which divides each of them, from vertex to midpoint of opposite side, in 2 : 1.

[*Hint*: From Exercises 2(b) and 2(e) of Part B, page 333, what are $(O - A) : (E - O)$, $(O - B) : (D - O)$, and $(O - C) : (P - O)$ when $r = 1/2$?]

Theorems 8–7 and 8–8 take account of only a very special case of the results you obtained in the exercises of Part B on page 333. To state these results in a more easily remembered form it is helpful to generalize the notion of midpoint. The midpoint of the interval $\overline{AB}$ is the point $A + (B - A) \cdot \frac{1}{2}$. [It is also the point $B + (A - B) \cdot \frac{1}{2}$.] We shall refer to the point $A + (B - A)r$ as:

the r-point, from A, of $\overline{AB}$

So, for example, the midpoint of $\overline{AB}$ is the $\frac{1}{2}$-point, from either A or B, of $\overline{AB}$. Note that

the r-point, from A, of $\overline{AB}$ is the $(1 - r)$-point, from B, of $\overline{AB}$.

Note also that, by the Corollary to Theorem 8–4, that [for $0 \neq r \neq 1$]

the r-point, from A, of $\overline{AB}$ divides the interval from A to B in $r : 1 - r$,

and that [for $-1 \neq s \neq 0$]

the point which divides the interval from A to B in $s : 1$ is the $\frac{s}{s+1}$-point, from A, of $\overline{AB}$.

Two proofs for Theorem 8-7 are given in the answer for Exercise 4 of Part A on page 333. The theorem is also a special case [$r = 1/2$] of Theorem 8-9(a) on page 337. This theorem is a consequence of Exercises 1 and 2(a) of Part B on page 333.

The if-part of the corollary is obviously a restatement of part of the theorem. The only if-part is a consequence of the if-part: For, there is just one line through the midpoint of the given side and parallel to the second side and, by the if-part, the line through the midpoints of the first and third side is this line. [So, this line contains the midpoint of the third side.]

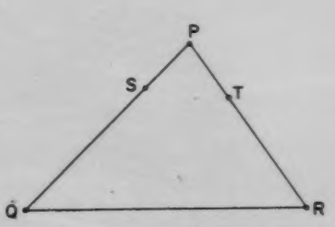
A triangle has three medians, one from each vertex. [The median from one vertex — say, A — cannot be, also, the median from another vertex — say, B — for, if it were, the midpoints of $\overline{BC}$ and of $\overline{AC}$ would belong to $\overleftrightarrow{AB}$, and {A, B, C} would be collinear.] No two medians of a triangle can be collinear. [Same reasons as given for distinctness of medians.] That no two medians of a triangle can be parallel follows from Exercise 2(b) of Part B, on page 333.

As indicated in the hint, Theorem 8-8 follows at once from Exercises 2(b) and 2(e) of Part B. Several other proofs of this theorem are given later in this chapter.

*

Sample Quiz

In the picture at the right, $\overline{PS}$ is 1/4 as long as $\overline{PQ}$ and T divides the segment from P to R in the ratio 1/3.



True or false?

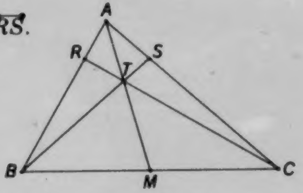
1. S divides the segment from P to Q in the ratio 1/4.
2. $\overline{PT}$ is 1/4 as long as $\overline{PR}$.
3. $\overleftrightarrow{ST}$ is parallel to $\overleftrightarrow{QR}$.
4. $\overline{ST}$ is 1/3 as long as $\overline{QR}$.
5. The ratio $\overline{ST}:\overline{QR}$ is 1/4.
6. The midpoints of $\overline{ST}$ and $\overline{QR}$ and the point P are collinear.
7. The segments $\overline{SR}$ and $\overline{QT}$ intersect.

Key to Sample Quiz

1. False. 2. True. 3. True. 4. False.
5. True. 6. True. 7. True.

Part A

1. In each of the following, you are given that P and Q are certain r-points of the sides $\overline{AB}$ and $\overline{AC}$, respectively, of $\triangle ABC$. Carefully draw a picture for each case and answer the following:
   - (i) What is $(Q - P) : (C - B)$?
   - (ii) Given that R is the point of intersection of $\overline{PC}$ and $\overline{QB}$, compute $(R - P) : (C - R)$ and $(R - Q) : (B - Q)$.
   - (iii) Given that S is the point of intersection of $\overleftrightarrow{AR}$ and $\overline{BC}$, compute $(S - B) : (C - S)$.
   - **(a)** Given that P is the $\frac{1}{3}$-point, from A, of $\overline{AB}$ and that Q is the $\frac{1}{3}$-point, from A, of $\overline{AC}$.
   - **(b)** Given that P and Q are the $\frac{2}{3}$-points, from A, of $\overline{AB}$ and $\overline{AC}$, respectively.
   - **(c)** Given that P and Q are the $\frac{3}{4}$-points, from A, of $\overline{AB}$ and $\overline{AC}$, respectively.
   - **(d)** Given that P and Q are the midpoints of sides $\overline{AB}$ and $\overline{AC}$, respectively.
2. Suppose that R and S are points on the sides $\overline{DE}$ and $\overline{DF}$, respectively, of $\triangle DEF$.
   - **(a)** Given that P is the $\frac{1}{3}$-point, from A, of $\overline{AB}$ and that Q is the $\frac{1}{3}$-point, from A, of $\overline{AC}$.
   - **(b)** Given that R and S are the $\frac{2}{3}$-points, from D, of $\overline{DE}$ and $\overline{DF}$, respectively, what can you say about $\overleftrightarrow{RS}$ and $\overleftrightarrow{EF}$? Try to show that your answer is correct.
3. Suppose that $\overline{CM}$ and $\overline{BN}$ are the medians from C and B, respectively, of $\triangle ABC$. Let $R = C + (M - C)2$ and $S = B + (N - B)2$.
   - **(a)** Draw an appropriate picture and show that $\overleftrightarrow{SR} \parallel \overleftrightarrow{BC}$
   - **(b)** What is $\overline{BC} : \overline{RS}$? $\overline{MN} : \overline{RS}$?
   - **(c)** Show that A is the midpoint of $\overline{RS}$.
4. In the figure at the right, $\overline{AM}$ is the median from A in $\triangle ABC$, R and S are the $\frac{1}{4}$-points, from A, of $\overline{AB}$ and $\overline{AC}$, respectively, and $\overline{RC}$ and $\overline{BS}$ intersect in the point T. Compute these ratios.



   - **(a)** $(T - R) : (C - T)$
   - **(b)** $(T - B) : (S - T)$
   - **(c)** $(T - A) : (M - T)$ [*Hint*: Let U be the point of intersection of $\overline{RS}$ and $\overline{AM}$. Make use of the ratios $(U - A) : (M - U)$ and $(T - U) : (M - T)$. Or, see Exercise 2(e) of Part B, page 333.]

The following theorems are suggested by our work with ratios in triangles in Part A and in earlier work.

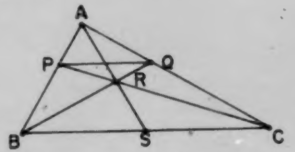
Theorem 8-9

(a) The interval whose endpoints are the r-points of two sides of a triangle, from their common endpoint, is parallel to the third side, and its ratio to the third side is r.

Part A should be a class activity due to the introduction of some new language. Parts B and C constitute one reasonable homework assignment. Part D is a second homework assignment. Part E presents some interesting in-class activities. Part F is a third possible homework assignment but, because of its length, should be carefully discussed the following day. The exercises of Part F contain many interesting and important geometric relationships.

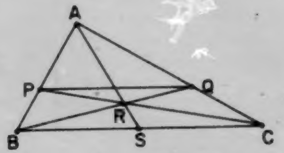
Answers for Part A

1. (a)



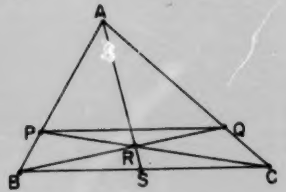
(i) $(Q - P) : (C - B) = 1/3$
(ii) $(R - P) : (C - R) = 1/3$; $(R - Q) : (B - Q) = 1/4$
(iii) $(S - B) : (C - S) = 1$

(b)



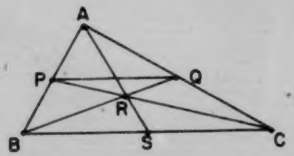
(i) $2/3$
(ii) $2/3$; $2/5$
(iii) 1

(c)



(i) $3/4$
(ii) $3/4$; $3/7$
(iii) 1

(d)



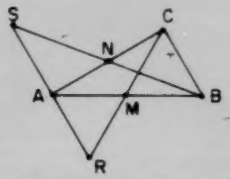
(i) $1/2$
(ii) $1/2$; $1/3$
(iii) 1

Answers for Part A [cont.]

2. (a) 3/4; 3/4

   (b) $\overleftrightarrow{RS} \parallel \overleftrightarrow{EF}$. This follows from Exercise 1 of Part B on page 333.

3. (a)



$\overleftrightarrow{SR} \parallel \overleftrightarrow{SA}$; and $\overleftrightarrow{SA} \parallel \overleftrightarrow{CB}$ because $(N - B):(N - S) = (N - C):(N - A)$

[Corollary to Theorem 4-3.]

[There are many other proofs that $\overleftrightarrow{SR} \parallel \overleftrightarrow{CB}$. All, however, go back to Theorem 4-3.]

   (b) $\overline{BC} : \overline{RS} = 1/2$; $\overline{MN} : \overline{RS} = 1/4$

   (c) In $\triangle ACR$, $(A - R):(N - M) = 2$; and, in $\triangle ABS$, $(S - A):(N - M) = 2$. So, $S - A = (N - M)2 = A - R$ and A is the midpoint of $\overline{RS}$.

4. (a) 1/4

   (b) 4

   (c) 2/3 [From the given information and the hint, U is the $\frac{1}{4}$-point, from A, of $\overline{AM}$ and T is the $\frac{1}{5}$-point, from U, of $\overline{UM}$. So, T is the $\frac{2}{5}$-point, from A, of $\overline{AM}$ (for, $\frac{2}{5} = \frac{1}{4} + \frac{1}{5}\cdot\frac{3}{4}$). Thus, T divides the segment from A to M in the ratio 2/3 (for, $(\frac{2}{5})/(\frac{3}{5}) = 2/3$).]

(b) A line through the r-point of one side of a triangle, from one of its vertices, is parallel to the side opposite that vertex if and only if it contains the r-point, from that vertex, of the third side.

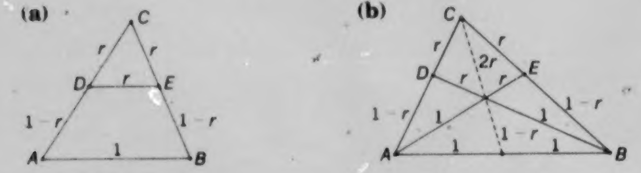


Fig. 8-11

Theorem 8-10

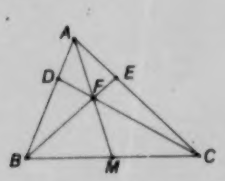
(a) Intervals from two vertices of a triangle to r-points of the opposite sides [from their common vertex] intersect at the $\frac{2r}{r+1}$-point of the median from that vertex. The point of intersection divides each of the two intervals, from vertex to opposite side in $1 : r$ and divides the median, from vertex to side, in $2r : 1 - r$.

(b) Lines through two vertices of a triangle which intersect at the s-point of the median from the third vertex intersect the opposite sides at their $\frac{s}{2-s}$-points from this vertex.

Part B

1. Prove these theorems. [*Hint*: You may, of course, use the results you obtained in Part B on page 333.]
   - **(a)** Theorem 8-9(a)
   - **(b)** Theorem 8-9(b)
   - **(c)** Theorem 8-10(a)
   - **(d)** Theorem 8-10(b)
2. In each of the following, $\overline{AM}$ is the median from A in $\triangle ABC$, D and E are given r-points, from A, of $\overline{AB}$ and $\overline{AC}$, respectively, and F is the point common to $\overline{AM}$, $\overline{BE}$, and $\overline{CD}$. Make use of Theorem 8-10 to help answer the given questions.

   (a)



Given: D and E are $\frac{1}{3}$-points from A.
Complete:

(i) F is the ____-point, from A, of $\overline{AM}$.

(ii) $(F - B) : (E - F) =$ ____ and $(F - C) : (D - F) =$ ____.

(iii) F divides $\overline{AM}$, from vertex to side, in the ratio ____.

Theorem 8-9(a) is obviously a generalization of Theorem 8-7, and Theorem 8-9(b) is the corresponding generalization of the corollary to Theorem 8-7. The proof of Theorem 8-9 is analogous to the proof of Theorem 8-7 and its corollary. [See TC 335(2).] Theorem 8-10 is related in a similar way to Theorem 8-8. [Theorem 8-10 is a special case of a much more general theorem — see Exercises 5 and 6 on pages 365 and 366.]

Figures 8-11(a) and 8-11(b) illustrate Theorem 8-9(a) and 8-10(a), respectively, in the case in which $0 < r < 1$. Students should illustrate the cases in which $r < 0$ and $r > 1$.

In all strictness, these theorems require some mild restrictions. In the case of Theorem 8-9, r must be restricted to be nonzero. In Theorem 8-10, r can be neither 0, 1, nor -1 and s cannot be 2. Students should draw figures to illustrate these excluded cases.

Answers for Part B

1. (a) This is an immediate consequence of Exercises 1 and 2(a) of Part B on page 333. Alternatively, using the notation of Figure 8-11(a), $D = C + (A - C)r$ and $E = C + (B - C)r$. So, $E - D = (B - A)r$. Hence, $\overline{DE} \parallel \overline{AB}$ and $\overline{DE} : \overline{AB} = r$.

   (b) Since there is one and only one line through D parallel to $\overline{AB}$, and, by (a), the line through D and E is parallel to $\overline{AB}$, it follows that the line through D parallel to $\overline{AB}$ is the line through D which contains E. Hence, a line through D is parallel to $\overline{AB}$ if and only if it contains E.

   (c) This is an immediate consequence of Exercises 1 and 2 of Part B on page 333. Using the notation of this exercise, D and E are the r-points in question and, by Exercise 1, the assumption made for Exercise 2 is satisfied. By Exercise 2(b), the intervals in question intersect at a point O which divides each of them, from side to vertex, in the ratio r and, so, divides each, from vertex to side in $1 : r$. By Exercise 2(c), O belongs to the median from C; and, by Exercise 2(e), O divides the median, from P to C, in $1 - r : 2r$. So, O divides the median from C to P in $2r : 1 - r$, and the point which so divides the median is its $(2r)/(r + 1)$-point [see Part A on page 337].

   (d) in the notation of the answer just given for part (c), one need only note that, subject to appropriate restrictions [see TC 333(1)],

$$s = \frac{2r}{r + 1} \Longleftrightarrow r = \frac{s}{2 - s}.$$

*

Theorem 8-10(a) can be proved directly, without reference to earlier exercises. Using the notation of Part B on page 333 [which is that of Figure 8-11(b)] we have that $D = C + (A - C)r$ and $E = C + (B - C)r$ and are seeking a point O such that [for some u and v] $O = B + (D - B)u = A + (E - A)v$. We wish, then, to solve:

$$B + (D - B)u = A + (E - A)v$$

or, equivalently:

$$B + [(C - B) + (A - C)r]u = A + [(C - A) + (B - C)r]v$$

$$B - A = (C - A)(v + ru) + (B - C)(rv + u)$$

Recalling that $B - A = (C - A) + (B - C)$ and that $(C - A, B - C)$ is linearly independent, this amounts to solving:

$$\begin{cases} ru + v = 1 \\ u + rv = 1 \end{cases}$$

which is satisfied if and only if $u = v = 1/(r + 1)$ [assuming that $r \neq -1$]. It follows that the desired point O exists [uniquely] and is the $1/(r + 1)$-point of both $\overline{AE}$ and $\overline{BD}$, from A and B, respectively. It is readily computed [Part A] that an $1/(r + 1)$-point of an interval divides the interval in $1 : r$.

It remains to be shown that O belongs to the median from C and to compute its location on this median. This can be done by comparing $O - C$ and $P - O$ where P is given as the midpoint of $\overline{AB}$. Alternatively, one may avoid prejudgement of the issue by showing, by the technique used above, that $\overleftrightarrow{CO}$ and $\overleftrightarrow{AB}$ intersect at a point P which turns out to be the midpoint of $\overline{AB}$.

Students should be required to practice arguments like the preceding [from basic principles], as well as arguments which, like that previously given in proof of Theorem 8-10(a), are based on general theorems.

*

Answers for Part B [cont.]

2. (a) (i) $1/2$ (ii) $3;\ 3$ (iii) 1

(b) Given: D and E are $\frac{1}{2}$-points from A.
Complete: (i) F is the ____-point, from A, of $\overline{AM}$.
(ii) $(F - B) : (E - F) =$ ____ and
$(F - C) : (D - F) =$ ____.
(iii) F divides $\overline{AM}$, from vertex to side, in the ratio ____.

(c) Given: F is the $\frac{2}{3}$-point, from A, of $\overline{AM}$.
Complete: (i) Both D and E are ____-points from A.
(ii) $(F - C) : (D - F) =$ ____
(iii) $(F - A) : (M - A) =$ ____

3. Draw a picture of triangle PQR. Locate A on $\overline{PQ}$ such that A is the $\frac{1}{5}$-point, from P. In the following, add to your picture and answer the questions.
   (a) Locate B on $\overline{QR}$ such that $\overline{AB} \parallel \overline{PR}$. What is $(B - Q) : (R - B)$?
   (b) Locate C on $\overline{PR}$ such that $\overline{BC} \parallel \overline{PQ}$. What is $(C - R) : (P - C)$?
   (c) Locate D on $\overline{PQ}$ such that $\overline{CD} \parallel \overline{RQ}$. What is $(D - Q) : (P - D)$?
   (d) Locate E on $\overline{QR}$ such that $\overline{DE} \parallel \overline{PR}$. What is $(E - Q) : (R - E)$?
   (e) Locate F on $\overline{PR}$ such that $\overline{EF} \parallel \overline{PQ}$. What is $(F - P) : (R - F)$?
   (f) What can you say about $\overline{AF}$ and $\overline{QR}$? About $\overline{AF}$ and $\overline{DC}$?
   (g) Show that $(F - A) + (C - D) = R - Q$.

4. Prove:

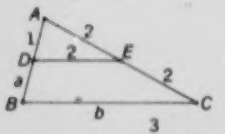
> **Theorem 8-11.** If, in $\triangle ABC$ and $\triangle A'B'C'$, $\overline{AB} \parallel \overline{A'B'}$, $\overline{BC} \parallel \overline{B'C'}$, and $\overline{CA} \parallel \overline{C'A'}$ then
> (a) $(B' - A') : (B - A) = (C' - B') : (C - B)$
> $= (A' - C') : (A - C)$, and
> (b) for $A \neq A'$, $B \neq B'$, and $C \neq C'$, the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are parallel or concurrent.

[*Hint*: Part (a) is just a restatement of a theorem in Chapter 6; for part (b), use Theorems 8-2 and 8-11(a).]

Part C

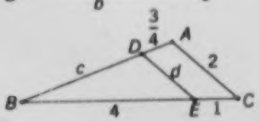
In each of the following, you are given certain information about a figure. Also, certain ratios of segments are indicated in the given figures. Use this information to determine the values asked for.

1.



Given: $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$
Find: a; b

2.



Given: $\overleftrightarrow{DE} \parallel \overleftrightarrow{AC}$
Find: c; d

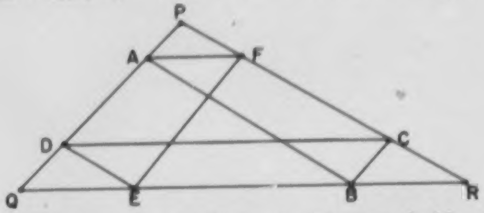
Answers for Part B [cont.]

(b) (i) 2/3 (ii) 2; 2 (iii) 2

(c) (i) 1/2 (ii) 2; 2 (iii) 2/3

[You may consider it worthwhile to make up other exercises, like those of Exercise 2, using less familiar values of 'r'. Remember that negative values and values greater than 1 are permissible.]

3. [This is exploration for Theorem 8-12 on page 340.]
   (a) 4 (b) 1/4 (c) 1/4 (d) 1/4 (e) 1/4
   (f) $\overline{AF} \parallel \overline{QR}$ [also, $\overline{AF} : \overline{QR} = 1/5$];
   $\overline{AF} \parallel \overline{DC}$ [also, $\overline{AF} : \overline{DC} = 1/4$]
   (g) $F - A = (R - Q)(1/5)$, $C - D = (R - Q)(4/5)$; $(1/5) + (4/5) = 1$.

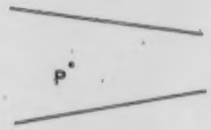


4. For part (a), the theorem in question is Theorem 6-12. Let $\vec{a} = C - B$, $\vec{b} = A - C$, $\vec{c} = B - A$, $\vec{a}' = C' - B'$, $\vec{b}' = A' - C'$, and $\vec{c}' = B' - A'$. Then $(\vec{a}, \vec{b})$ is linearly independent, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, and $\vec{a}' + \vec{b}' + \vec{c}' = \vec{0}$. Since $\vec{a}' = \vec{a}(\vec{a}' : \vec{a})$, etc. it follows by Theorem 6-12 that $\vec{a}' : \vec{a} = \vec{b}' : \vec{b} = \vec{c}' : \vec{c}$.

   For (b), let $r = \vec{a}' : \vec{a}$. By Theorem 8-2, $\overleftrightarrow{CC'} \parallel \overleftrightarrow{BB'}$ if $r = 1$ and, if $r \neq 1$, $\overleftrightarrow{CC'}$ and $\overleftrightarrow{BB'}$ intersect at the $/(1 - r)$-point, from C', of $\overline{C'C}$. Since, by Theorem 8-11(a), $\vec{b}' : \vec{b} = r$, it follows that, if $r \neq 1$, $\overleftrightarrow{CC'} \parallel \overleftrightarrow{AA'}$ and, if $r \neq 1$, $\overleftrightarrow{CC'}$ and $\overleftrightarrow{AA'}$ intersect at the $/(1 - r)$-point, from C', of $\overline{CC'}$. So, if $r = 1$ the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are parallel and, if $r \neq 1$, these lines are concurrent.

*

Theorem 8-11(b) is part of the converse of Desargue's Theorem [TC 36(1) for page 36]. [The other part — which some of your more interested students may wish to prove — says that if the intersections of lines containing corresponding sides of the triangles are collinear then the lines containing corresponding vertices are, again, either parallel or concurrent.] Theorem 8-11(b) gives a way of solving the following "construction problem". Given two lines and a point, as shown, draw the line through P which is concurrent with the given lines, assuming that it is impossible to find the point of concurrency by extending the pictures of the given lines. [The use of parallel rulers, or other instruments for drawing parallel lines is allowed.]

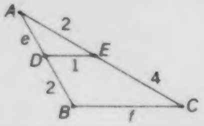


Answers for Part C

1. 1; 4

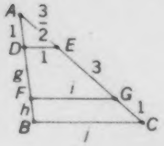
2. 3; 8/5

3.



Given: $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$
Find: e; f

4.



Given: $\overleftrightarrow{DE} \parallel \overleftrightarrow{FG} \parallel \overleftrightarrow{BC}$
Find: g; h; i; j

Part D

The results in Exercise 3 of Part B suggest the following interesting theorem. If you will review the manner in which the critical points were obtained, you will see why it is reasonable to call this "the twice-around theorem".

> **Theorem 8-12** [The Twice-Around Theorem]
> If, in $\triangle ABC$, G and D are in $\overline{BC}$, E and H are in $\overline{CA}$, I and F are in $\overline{AB}$, and $\overline{DE} \parallel \overline{BA} \parallel \overline{GH}$, $\overline{EF} \parallel \overline{CB} \parallel \overline{HI}$, $\overline{FG} \parallel \overline{AC}$, then $\overline{ID} \parallel \overline{AC}$.

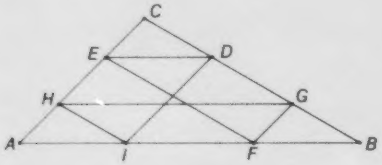


Fig. 8–12

1. Prove the twice-around theorem. [*Hint*: Suppose that D and G are the r-point and the s-point, from C, on $\overline{BC}$. Which points are E and H, from C? Which points are F and I, from B? Which points are G and D, from B?]
2. Show that, in the twice-around theorem, $(D - E) + (G - H) = B - A$.
3. Draw a figure for the case of the twice-around theorem in which $D = G$.
4. The lines $\overleftrightarrow{EF}$ and $\overleftrightarrow{ID}$ intersect at a point O. In what ratio does O divide the interval from E to F? That from D to I? [Answer in terms of the number r referred to in the hint for Exercise 1.]
5. The line $\overleftrightarrow{CO}$ intersects $\overline{DE}$ at a point P and $\overline{AB}$ at a point Q. In what ratio does O divide the interval from P to Q? How might you describe the interval $\overline{CQ}$? Compute other ratios involving C, O, P, and Q.

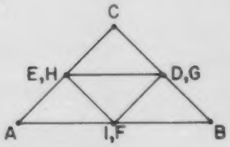
---

3. 1; 3

4. 2; 2/3; 3; 11/3

Answers for Part D

1. Assuming that D is an r-point from C, E is an r-point from C and F is an r-point from B. Assuming that G is an s-point from C, H is an s-point from C and I is an s-point from B. Since G is an s-point from C, G is a (1 - s)-point from B. So, F is a (1 - s)-point from B. Hence, 1 - s = r. Since D is an r-point from C, D is a (1 - r)-point from B. Since I is an s-point from B and s = 1 - r it follows that $\overline{ID} \parallel \overline{AC}$.
2. Since E and D are r-points from C, and H and G are s-points from C, it follows that D - E = (B - A)r and G - H = (B - A)s. But, r + s = 1.
3.



[Figure 8-12 illustrates the case in which $0 < r < 1/3$. The figure for Exercise 3 illustrates the case in which $r = 1/2$. Students should draw other figures: $r = 1/3$, $1/3 < r < 1/2$, and $1/2 < r < 1$.]

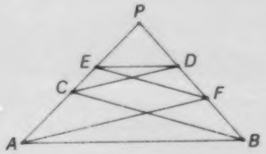
4. (O - E):(F - O) = (D - C):(G - D) = r/(1 - 2r);
   (O - D):(I - O) = r:(1 - 2r)
5. (C - P):(Q - O) = (O - E):(F - O) = r/(1 - 2r); the median of $\triangle$ABC from C

6. Draw a figure like that for the twice-around theorem, but choose D in $-\overrightarrow{CB}$. Does the conclusion of the theorem still hold for such a choice of D?

Part E

1. (a) Draw two lines, l and m, intersecting at a point P. Mark a point A on l and draw an interval from A to a point B on m. Draw a second interval from B to a point C of l and a third interval from C to a point D of m. Continue this procedure, but draw the fourth interval parallel to $\overline{AB}$, the fifth parallel to $\overline{BC}$, and the sixth parallel to $\overline{CD}$.
   (b) Repeat part (a) at least twice, trying to choose the first three points in such a way as to obtain figures which look different.
   (c) Choose a point $C \in l$ and a point $F \in m$. From C, draw two intervals to points of m. From F, draw intervals, parallel to those from C, to points of l. What do you notice?
2. Part of what you have discovered in Exercise 1 is stated in

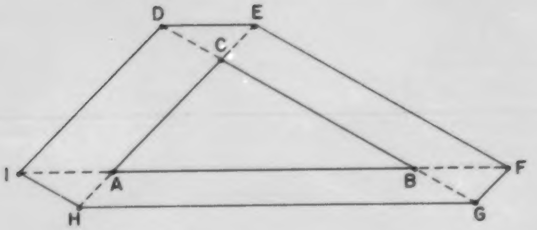
> **Theorem 8-13**
> If, in $\triangle ABP$, D and F are in $\overline{BP}$, C and E are in $\overline{PA}$, $\overline{EF} \parallel \overline{BC}$, and $\overline{CD} \parallel \overline{AF}$, then $\overline{DE} \parallel \overline{AB}$.



Prove this theorem. [*Hint*: Suppose that F is the r-point, from P, in $\overline{BP}$ and that C is the s-point, from P, in $\overline{AP}$. Which points are D and E, from P? The technique used in Part F on page 332 should help.]

3. Once the points C and F have been chosen, the points D and E are determined by the parallelism requirements. Your work in Exercise 1 should have shown you that $\overline{DE}$ will be parallel to $\overline{AB}$ if D is any point, other than P, in $\overleftrightarrow{BP}$ and E is any point, other than P, in $\overline{PA}$. Does your proof of Theorem 8-13 show this to be the case?
4. The extension of Theorem 8-13 which is referred to in Exercise 3 is called *Pappus' Theorem*. State Pappus' Theorem, using the notation of Exercise 1(a).
5. There is a second case of Pappus' Theorem in which the lines l and m are assumed to be parallel. Investigate this case.
6. A surveying party of three—Todd, Taylor, and Reid—are running a line due east across country which is mostly flat. Todd operates a transit, and Taylor and Reid act as rodmen. Unfortunately, their line runs into the west side of an isolated, unclimbable mountain. Somehow, they must locate a point to the east of the mountain

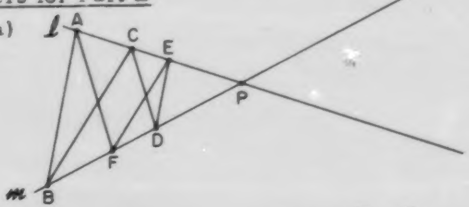
6.



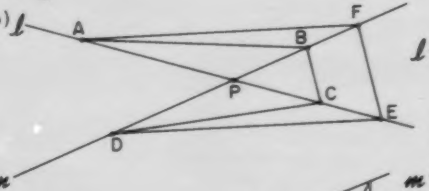
The conclusion holds in this case, of course — for, the proof in answer to Exercise 1 requires no restriction on the value of 'r'.

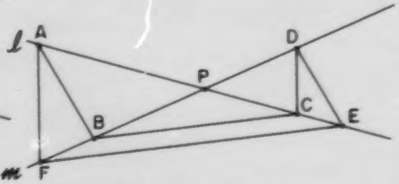
Answers for Part E

1. (a)



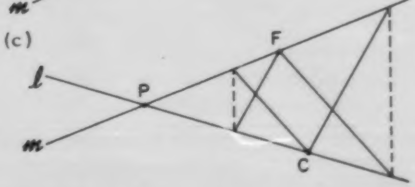
(b)



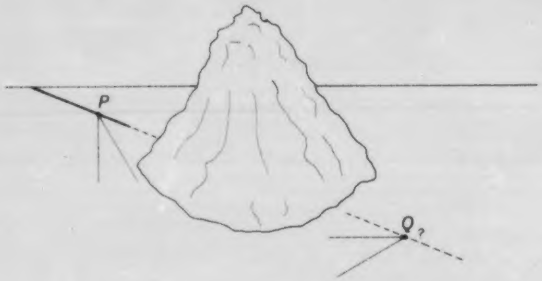


The intervals indicated by dashed lines are parallel.

(c)



2. Proof of Theorem 8-13 Suppose that F is an r-point, from P, in $\overline{PB}$, and that C is an s-point, from P, in $\overline{PA}$. Then, D divides the segment from P to F in s : 1 - s, and E divides the segment from P to C in r : 1 - r. So, for some a, (E - P) : (C - E) = ra : (1 - r)a, where s = ra + (1 - r)a = a. Also, for some b, (D - P) : (F - D) = sb : (1 - s)b, where r = sb + (1 - s)b = b. So, (E - P) : (A - E) = rs : 1 - rs and (D - P) : (B - D) = rs : 1 - rs, which means that E divides the segment from P to A in the same ratio that D divides the segment from P to B. Thus, $\overline{DE} \parallel \overline{AB}$.
3. Yes, for the proof shows that neither D nor E is P; D is an s-point of $\overline{PF}$ and E is an r-point of $\overline{PC}$.

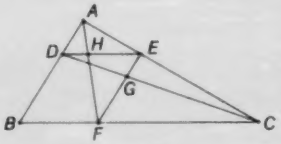


from which to extend their line. Fortunately, they have a spare transit which Taylor can operate, and Taylor knows Pappus' Theorem. He points out to the others that the land south of the mountain is flat, and shows them how to locate the required point.

Given a point P west of the mountain, show how to locate a point Q due east from P and east of the mountain. [*Hint*: Using a compass and transit, Todd [or Taylor] can direct Reid to a point at any bearing from a point at which a transit is set up – provided the mountain is not in the way. Using both transits, Reid can be directed to the point at which lines through the positions of the two transits intersect.]

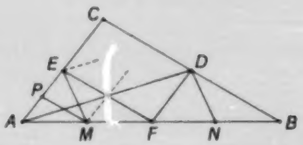
Part F

1. Suppose that $\overline{DE} \parallel \overline{BC}$ and $\overline{EF} \parallel \overline{AB}$. Also, $(D - A) : (B - A) = \frac{1}{3}$. Determine the following ratios.



(a) $(D - A) : (B - D)$
(b) $(E - A) : (C - A)$
(c) $(E - D) : (C - B)$
(d) $(E - D) : (B - F)$
(e) $(G - C) : (D - G)$
(f) $(F - E) : (B - A)$
(g) $(A - H) : (A - F)$
(h) $(G - E) : (F - G)$
(i) $(E - D) : (H - D)$
(j) $(D - A) : (G - E)$

2. Suppose that D, E, and F are the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, and that M, N, and P are the midpoints of $\overline{AF}$, $\overline{FB}$, and $\overline{EA}$.

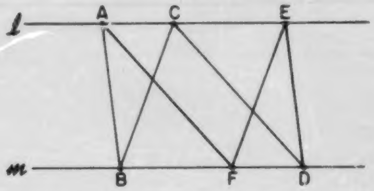


(a) Show that $\overline{MP} \parallel \overline{EF}$.
(b) Describe the location of the point of intersection of $\overleftrightarrow{E[D - A]}$ and $\overleftrightarrow{M[D - F]}$.
(c) In what ratio does the point of intersection of $\overline{AD}$ and $\overline{EF}$ divide these segments?
(d) Show that $\overline{ME} \parallel \overline{ND}$.

Answers for Part E [cont.]

4. If $\ell \cap m = \{P\}$ and A, C, and E are three points other than P on ℓ and B, D, and F are three points other than P on m, and $\overline{DE} \parallel \overline{AB}$ and $\overline{EF} \parallel \overline{BC}$, then $\overline{FA} \parallel \overline{CD}$.

5.



The theorem is like that of Exercise 4 except that ℓ and m are two parallel lines. By Theorem 8-2 it follows that $D - E = B - A$ and $E - F = C - B$. Hence [by addition] $D - F = C - A$ and, so, $D - C = F - A$. Consequently, $\overline{FA} \parallel \overline{CD}$.

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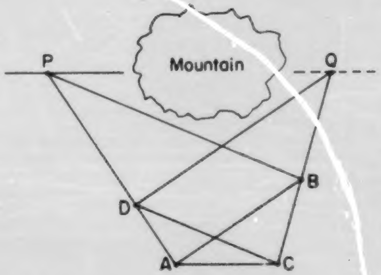
6.



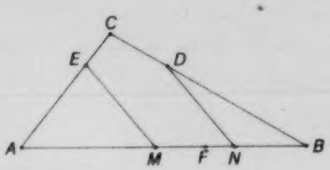
Todd and Taylor first take up positions at A and B. [Todd has a transit; Taylor carries a rod and the spare transit.] Reid remains, with his rod, at P. Todd, at A, can see both P and B, and records their bearings from A. Taylor at B can see P, whose bearing he records, and, also, can see past the eastern slope of the mountain. Once the bearings of P from A and B have been determined, Reid leaves P and is directed by Todd, at A, along a line running east from A. Reid drives a stake at a point C of this line from which he can see through B along a line running east of the mountain. Taylor, at B, records the bearing of C from B. Taylor now moves to C and directs Reid along a line through C parallel to $\overleftrightarrow{BP}$. Todd signals to Reid when the latter reaches a point D on $\overleftrightarrow{AB}$. Reid drives a stake to mark D. Todd moves to D and directs Reid along a line through D parallel to $\overleftrightarrow{AB}$. Taylor, at C, signals Reid when the latter reaches the point Q of $\overleftrightarrow{CB}$. Reid drives a stake at this point.

Answers for Part F

1. (a) 1/2 (b) 1/3 (c) 1/3 (d) −1 (e) 2
(f) 2/3 (g) 1/3 (h) 1/2 (i) 3 (j) 3/2

2. (a) $\overline{MP} \parallel \overline{EF}$ by Theorem 8-7.
(b) The lines intersect at the midpoint of $\overline{CD}$.
(c) 1
(d) By Theorem 8-7, each of $\overline{ME}$ and $\overline{ND}$ is parallel to $\overline{CF}$. Hence $\overline{ME} \parallel \overline{ND}$.

3. Suppose that D, E, and F are points of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, that $M \in \overline{AF}$, and that $N \in \overline{FB}$. Suppose, also, that $(M - A) : (F - M) = (E - A) : (C - E)$ and that $(N - F) : (B - N) = (D - C) : (B - D)$.

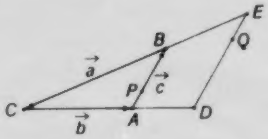


(a) Show that $\overline{ME} \parallel \overline{ND}$.

(b) How must $(M - A) : (F - M)$ and $(N - F) : (B - N)$ be related in order that $\overline{DE} \parallel \overline{AB}$?

(c) Suppose that $\overline{DE} \parallel \overline{AB}$ and that $(M - A) : (F - M) = s$. Compute $(D - E) : (B - A)$.

4. In $\triangle ABC$, $C - B = \vec{a}$, $A - C = \vec{b}$, and $B - A = \vec{c}$. Also, $C - E = \vec{a}e$, $D - C = \vec{b}d$, and $E - D = \vec{c}c$.



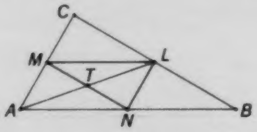
(a) Show that $c = d = e$.

(b) Suppose that $P - A = (B - A)p$ and $Q - D = (E - D)q$. Show that $\{C, P, Q\}$ is collinear if and only if $p = q$. [*Hint*: Express '$(P - C)a + (Q - C)b$' in terms of '$\vec{b}$' and '$\vec{c}$'.]

5. An interval is said to be *bisected* by a given set if [and only if] the intersection of the given set and the interval consists of the midpoint of the interval. [Two intervals *bisect each other* if their intersection consists of a single point which is the midpoint of each of them.]

Show that if L, M, and N are the midpoints of the sides of a triangle, L being on the side opposite the vertex A, then $\overline{AL}$ and $\overline{MN}$ bisect each other.

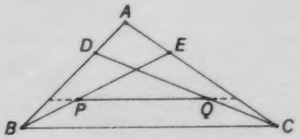
6. The *centroid* of a triangle is the point of intersection of its medians. In $\triangle ABC$, let L, M, and N be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively.



(a) Show that $\triangle ABC$ and $\triangle LMN$ have the same centroid.

(b) Let T be the common midpoint of $\overline{AL}$ and $\overline{MN}$. In what ratio does the centroid of $\triangle ABC$ divide the interval from T to L? The interval from T to A?

7. In $\triangle ABC$, suppose that D and E divide the intervals from A to B and from A to C, respectively, in the same ratio, and that P and Q divide the intervals from B to E and from C to D, respectively, in the same ratio.



(a) Show that $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$. [*Hint*: Consider $\triangle BCD$ and $\triangle DEC$.]

Answers for Part F [cont.]

3. (a) By Theorem 8-9(a), each of $\overline{ME}$ and $\overline{ND}$ is parallel to $\overline{CF}$. Hence, $\overline{ME} \parallel \overline{ND}$.

(b) The ratios must be reciprocals.

(c) Since $(M - A):(F - M) = s$, $(E - A):(C - E) = s$. So, $(E - C):(A - C) = 1/(s + 1)$. Hence, $(D - E):(B - A) = 1/(s + 1)$.

4. (a) This is an immediate consequence of Theorem 8-11(a) [or, of Theorem 6-12].

(b) [This has been established in each of several earlier exercises; but the hint suggests a solution, the details of which we give here.] Using the hint, we note that $P - C = \vec{b} + \vec{c}p$ and $Q - C = \vec{b}d + \vec{c}(cq)$. So, $(P - C)a + (Q - C)b = \vec{b}(a + bd) + \vec{c}(ap + bcq)$. Since $(\vec{b}, \vec{c})$ is linearly independent $\{C, P, Q\}$ is collinear if and only if the equations:

$$(*) \qquad \begin{cases} a + bd = 0 \\ ap + bcq = 0 \end{cases}$$

are satisfied by numbers a and b which are not both zero. Recalling that $c = d$, it can be shown in several ways that (*) has a solution other than $(0, 0)$. [The simplest is to note that this is the case if and only if (*) has more than one solution, and thus occurs if and only if the determinant $1 \cdot (cq) - p \cdot d = 0$.]

5. That $\overline{AL}$ bisects $\overline{MN}$ follows from Exercise 4(b), with $p = q = 1/2$; it also follows from Theorem 8-6(a), and from any of several earlier exercises. That $\overline{MN}$ bisects $\overline{AL}$ follows from Theorem 8-6(b). [Consider $\overline{A[B - C]}$ as a third transversal.]

6. (a) The median of $\triangle LMN$ from L is a subset of the median of $\triangle ABC$ from A and contains the 2/3-point, from A, of the latter.

(b) $1:3$; $-1:4$

7. (a) Let $(D - A):(B - D) = r = (E - A):(C - E)$ and $(P - B):(E - P) = s = (Q - C):(D - Q)$. Let R and T be the points which divide the intervals from B to D and from C to E in the ratio s. It follows that R and T divide the intervals from B to A and from C to A in the same ratio. [This ratio turns out to be $s : 1 + r(s + 1)$; but this is not important.] It follows that $\overleftrightarrow{PT}$, $\overleftrightarrow{RQ}$, and $\overleftrightarrow{RT}$ are all parallel to $\overleftrightarrow{BC}$. Since there is only one line through R parallel to $\overleftrightarrow{BC}$, $\overleftrightarrow{RT} = \overleftrightarrow{RQ}$. Similarly, $\overleftrightarrow{RT} = \overleftrightarrow{PT}$. Hence, $\overleftrightarrow{RT} = \overleftrightarrow{PQ}$ and, so, $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$.

(b) Let S be the point of intersection of $\overline{CD}$ and $\overline{BE}$. By Theorem 8-10(a), S belongs to the median from A of $\triangle ABC$, and divides the intervals from B to E and from C to D in the same ratio. It follows that P and Q divide the intervals from B to S and from C to S in the same ratio. Hence, $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$ [this is another solution for part (a)] and, by Theorem 8-6(a), the median of $\triangle RST$ from S bisects $\overline{PQ}$.

7. (c) $(T - R):(E - D) = \dfrac{a + (b/2)}{a}$ and $(P - R):(E - D) = \dfrac{1}{2}$. So, $(Q - P):(E - D) = \dfrac{a + (b/2)}{a} - 1 = \dfrac{b}{2a}$.

(b) Show that A, the midpoint of $\overline{PQ}$, and the midpoint of $\overline{BC}$ are collinear.

(c) Suppose that D divides the interval from A to B in $a : b$ and that P is the midpoint of $\overline{BE}$. What is the ratio of $Q - P$ to $E - D$? [*Hint*: Let R and T be the points in which $\overleftrightarrow{PQ}$ intersects $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, respectively. What is the ratio of $T - R$ to $E - D$? Of $P - R$ to $E - D$?]

8. Suppose that $\overline{BM}$ and $\overline{CN}$ are medians of $\triangle ABC$, that M is the midpoint of $\overline{BQ}$ and that N is the midpoint of $\overline{CR}$. Show that
   (a) $Q = B + (M - B)2$,
   (b) $R - Q = (B - C)2$, and
   (c) A is the midpoint of $\overline{QR}$.

9. Suppose that, in $\triangle ABC$, M and N are the a-points, from A, of $\overline{AC}$ and $\overline{AB}$, that $Q = B + (M - B)d$, and that $R = C + (N - C)d$. Suppose, also, that $a \neq 0$ and that $d \neq 1/(a + 1)$.
   (a) Show that $\overline{MN} \parallel \overline{BC}$.
   (b) Find the ratio of $M - N$ to $C - B$.
   (c) Show that $\overline{QR} \parallel \overline{BC}$.
   (d) Find $(Q - R) : (C - B)$.
   (e) Show that A is the midpoint of $\overline{QR}$ if and only if $d = 1/(1 - a)$ $[a \neq 1]$.
   (f) What is $(Q - R) : (C - B)$ if A is the midpoint of $\overline{QR}$?
   (g) Apply some of the preceding results to check your answers for Exercise 8.

10. Consider two lines, $\overleftrightarrow{OA}$ and $\overleftrightarrow{OA'}$. Let $B = O + (A - O)2$, $C = O + (A - O)\frac{7}{2}$, $B' = O + (A' - O)\frac{7}{4}$, and $C' = O + (A' - O)4$.
   (a) Find points P, Q, and R such that $\overleftrightarrow{BC'} \cap \overleftrightarrow{B'C} = \{P\}$, $\overleftrightarrow{C'A} \cap \overleftrightarrow{CA'} = \{Q\}$, and $\overleftrightarrow{AB'} \cap \overleftrightarrow{A'B} = \{R\}$.
   (b) Show that $\{P, Q, R\}$ is collinear.

8.05 Two Ways of Setting Up Problems

Up to now in this chapter we have dealt mostly with ratios of translations expressed in the form '$(B - A) : (C - D)$'. As you have seen, there are quite general results—for examples, Theorem 8–2, Theorem 8–3, and the results collected in Part E on page 330 and in Part B on page 338—which can be expressed in this form and which lead to general theorems like Theorems 8–9 and 8–10. Often it is more convenient, however, to make a fresh start on a problem instead of tailoring a general theorem to fit. In this section we shall illustrate two helpful ways of using arrow-notation. One of them you are already somewhat acquainted with.

Answers for Part F [cont.]

8. (a) Since M is the midpoint of $\overline{BQ}$, $Q - M = M - B$. Since $Q - M = (B - M) + (Q - B)$ it follows that $Q - B = (M - B)2$ and, so, $Q = B + (M - B)2$.

   (b) As in (a), $R = C + (N - C)2$. So, $R - Q = [C + (N - C)2] - [B + (M - B)2] = (C - B) + [(N - C) - (M - B)]2 = (C - B) + [(N - M) + (B - C)]2 = (C - B) + (B - C)3 = (B - C)2$.

   (c) $A - R = (A - C) + (C - N)2 = (A - N) + (C - N) = (N - B) + (C - N) = C - B$;
   $Q - A = (B - A) + (M - B)2 = (M - A) + (M - B) = (C - M) + (M - B) = C - B$.
   Hence, $Q - A = A - R$ and, so, A is the midpoint of $\overline{QR}$.

9. (a), (b) By Theorem 8-9(a), $\overline{MN} \parallel \overline{BC}$ and $(M - N):(C - B) = a$.

   (c), (d) $Q = B + (M - B)d$, $R = C + (N - C)d$, $Q - R = (B - C) + [(M - B) - (N - C)]d = (B - C) + [(M - N) + (C - B)]d = (C - B)[-1 + (a + 1)d]$. So, since $d \neq 1/(a + 1)$, $\overline{QR} \parallel \overline{BC}$ and $(Q - R):(C - B) = (a + 1)d - 1$.

   (e) $A - R = (A - C) + (C - N)d = (A - C) + [(C - A) + (A - B)a]d = (C - A)(d - 1) + (A - B)ad$; similarly, $Q - A = (A - B)(d - 1) + (C - A)ad$. Since $\{A, B, C\}$ is noncollinear it follows that $A - R = Q - A$ if and only if $ad = d - 1$ — that is, if and only if $d = 1/(1 - a)$.

   (f) $(2a)/(1 - a)$

10. This exercise involves a great deal of computation. The work can be simplified somewhat by using position vectors of points with respect to O. If, for each X and $\vec{x}$, $\vec{x} = X - O$ then the given data reduces to $\vec{b} = \vec{a}2$, $\vec{c} = \vec{a}\frac{7}{2}$, $\vec{b}' = \vec{a}'\frac{7}{4}$, and $\vec{c}' = \vec{a}'4$. We seek $\vec{p}$, $\vec{q}$, and $\vec{r}$ where $\vec{p} = \vec{b} + (\vec{c}' - \vec{b})p_1 = \vec{b}' + (\vec{c} - \vec{b}')p_2$, $\vec{q} = \vec{c} + (\vec{a}' - \vec{c})q_1 = \vec{c}' + (\vec{a} - \vec{c}')q_2$, and $\vec{r} = \vec{a} + (\vec{b}' - \vec{a})r_1 = \vec{a}' + (\vec{b} - \vec{a}')r_2$. Substituting the given data and collecting terms we find that $\vec{p} = \vec{a}(2 - 2p_1) + \vec{a}'(4p_1) = \vec{a}(\frac{7}{2}p_2) + \vec{a}'(\frac{7}{4} - \frac{7}{4}p_2)$, $\vec{q} = \vec{a}(\frac{7}{2} - \frac{7}{2}q_1) + \vec{a}'q_1 = \vec{a}q_2 + \vec{a}'(4 - 4q_2)$, $\vec{r} = \vec{a}(1 - r_1) + \vec{a}'(\frac{7}{4}r_1) = \vec{a}(2r_2) + \vec{a}'(1 - r_2)$. Noting that $(\vec{a}, \vec{a}')$ is linearly independent we find that $p_1 = \frac{1}{4}$, $p_2 = \frac{3}{7}$, $q_1 = \frac{10}{13}$, $q_2 = \frac{21}{26}$, $r_1 = \frac{2}{5}$, and $r_2 = \frac{3}{10}$. So, $\vec{p} = \vec{a}\frac{3}{2} + \vec{a}'$, $\vec{q} = \vec{a}\frac{21}{26} + \vec{a}'\frac{10}{13}$, and $\vec{r} = \vec{a}\frac{3}{5} + \vec{a}'\frac{7}{10}$. It follows that $\vec{q} - \vec{p} = \vec{a} \cdot -\frac{9}{13} + \vec{a}' \cdot -\frac{3}{13}$ and $\vec{r} - \vec{p} = \vec{a} \cdot -\frac{9}{10} + \vec{a}' \cdot -\frac{3}{10}$. Hence, $(\vec{q} - \vec{p}, \vec{r} - \vec{p})$ is linearly dependent and, consequently $\{P, Q, R\}$ is collinear.

As an illustration, let's reconsider the theorem concerning the concurrency of the medians of a triangle. Consider a triangle – say, $\triangle ABC$. For convenience, let $\vec{a} = C - B$, $\vec{b} = A - C$, and $\vec{c} = B - A$. Note that we have chosen $\vec{a}$, $\vec{b}$, and $\vec{c}$ in such a way that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Note, also, that since ABC is a triangle, $(\vec{a}, \vec{b})$ is linearly independent. [So, for that matter, is each of $(\vec{b}, \vec{c})$ and $(\vec{c}, \vec{a})$.]

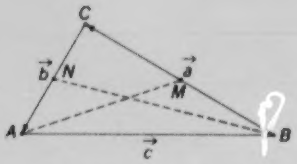


Fig. 8–13

Suppose, now, that M and N are the midpoints of $\overline{BC}$ and $\overline{CA}$ respectively. We wish to know whether – and, if so, where – $\overleftrightarrow{AM}$ and $\overleftrightarrow{BN}$ intersect. This we know we can find out by considering the solutions of the equation:

$$A + (M - A)m = B + (N - B)n$$

Since $M = C - \vec{a}/2$ and $N = C + \vec{b}/2$ this equation is equivalent to:

$$A + [(C - \vec{a}/2) - A]m = B + [(C + \vec{b}/2) - B]n$$

or, equivalently:

$$A - [(A - C) + \vec{a}/2]m = B + [(C - B) + \vec{b}/2]n$$

Since $C - B = \vec{a}$, $A - C = \vec{b}$, and $B - A = \vec{c}$, this can be rewritten as:

$$(a + \vec{b}/2)n + (\vec{b} + \vec{a}/2)m + \vec{c} = \vec{0}$$

and simplified to:

$$\vec{a}(n + m/2) + \vec{b}(n/2 + m) + \vec{c}1 = \vec{0}$$

From what has previously been said about $\vec{a}$, $\vec{b}$, and $\vec{c}$, this equation is equivalent to:

$$n + \frac{m}{2} = \frac{n}{2} + m = 1$$

and so, by a little real number algebra to:

$$m = 2/3 = n$$

Thus $\overline{AM}$ and $\overline{BN}$ intersect at a point which divides each, from vertex to opposite side, in 2:1. Since, obviously, the same must hold of $\overline{BN}$ and the median $\overline{CP}$ from C it follows that all three medians intersect at a point which divides each of them, from vertex to opposite side, in the ratio 2:1.

Clearly, a similar procedure can be used to prove Theorem 8–10 by taking $M = C - \vec{a}r$ and $N = C + \vec{b}r$. In fact, the less easily stated results concerning the intersection of $\overleftrightarrow{AM}$ and $\overleftrightarrow{BN}$ when $M = C - \vec{a}u$ and $N = C + \vec{b}v$ $[u \neq v]$ are nearly as simple to establish. [And we have proved no general theorem previously from which these results could be derived.]

In applying the method just illustrated one introduces arrow-notation for translations which are determined solely by the points in which one happens to be interested. This method is particularly suited to the solution of problems concerning triangles. For other problems it is often more convenient to introduce what are called *position vectors* of the points in question. Doing so amounts to choosing, arbitrarily, a point – say, O – and associating with each point X its position vector $X - O$ with respect to O. It is convenient to let, for example, $\vec{a} = A - O$, $\vec{b} = B - O$, etc. One advantage of this choice of notation is that, for any points A and B and their respective position vectors $\vec{a}$ and $\vec{b}$,

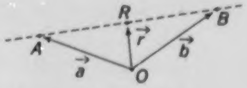


Fig. 8–14

$$B - A = (B - O) - (A - O) = \vec{b} - \vec{a}.$$

Using this result it is easy to see that

$$\begin{aligned} R = A + (B - A)r &\Longleftrightarrow \vec{r} = \vec{a} + (\vec{b} - \vec{a})r \\ &\Longleftrightarrow \vec{r} = \vec{a}(1 - r) + \vec{b}r. \end{aligned}$$

Because of this the equation:

(1) $$\vec{r} = \vec{a}(1 - r) + \vec{b}r$$

is sometimes referred to as *the vector equation* of $\overleftrightarrow{AB}$. This means that a point whose position vector is $\vec{r}$ belongs to the line in question if and only if there exists a number x such that $\vec{r} = \vec{a}(1 - x) + \vec{b}x$. Of course, *for given $\vec{a}$ and $\vec{b}$*, what line it is that is "in question" depends on the choice of the point O. But, *for given points*, A and B, (1) remains the equation of $\overleftrightarrow{AB}$, whatever O may be, as long as $\vec{a} = A - O$ and $\vec{b} = B - O$.

The result which led to consideration of (1) yields our first basic theorem concerning position vectors:

> **Theorem 8-14** If $\vec{a}$, $\vec{b}$, and $\vec{r}$ are position vectors of A, B, and R [with respect to any point O] then, for $A \neq B$ and $0 \neq r \neq 1$, R is the point which divides the interval from A to B in $r{:}1 - r$ if and only if $\vec{r} = \vec{a}(1 - r) + \vec{b}r$.

For example, the midpoint of $\overline{AB}$ is the point whose position vector is $(\vec{a} + \vec{b}) \cdot \frac{1}{2}$. What is the position vector of the point which divides the interval from A to B in 2:1?

If we note that

$$\vec{r} = \vec{a}(1 - r) + \vec{b}r \Longleftrightarrow \vec{a}(1 - r) + \vec{b}r + \vec{r}\cdot -1 = \vec{0}$$

and that $(1 - r) + r + -1 = 0$, the preceding theorem suggests a second basic theorem:

> **Theorem 8-15** $\vec{a}$, $\vec{b}$, and $\vec{c}$ are position vectors of collinear points if and only if there exist numbers x, y, and z, not all 0, such that $\vec{a}x + \vec{b}y + \vec{c}z = \vec{0}$ and $x + y + z = 0$.

In one of the later exercises you will be asked to prove this theorem.

Exercises

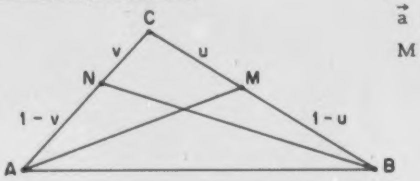
Part A

In ΔABC, let M be the u-point, from C, of $\overline{BC}$ and let N be the v-point, from C, of $\overline{CA}$. Let $\vec{a} = C - B$, $\vec{b} = A - C$, and $\vec{c} = B - A$.

1. Express $M - A$ and $N - B$ in terms of '$\vec{a}$', '$\vec{b}$', 'u', and 'v'.
2. Show that $\overleftrightarrow{AM} \parallel \overleftrightarrow{BN}$ if and only if $uv = 1$.
3. Show that $\overleftrightarrow{AM}$ and $\overleftrightarrow{BN}$ intersect at a single point if and only if $uv \neq 1$ and compute the ratios in which the point of intersection divides the intervals from A to M and from B to N.
4. Check your result in Exercise 3 when $u = v$ against the result of an earlier theorem.

Due to the introduction of position vectors in the preceding text, we recommend that Parts A and B be treated as a class activity. Be sure to emphasize the similarities between the "point-difference" notation for translations and the new "position vector" notation. Parts C and D make a rather long homework assignment. Perhaps you could allow students to work in teams for the derivations. Part E is rather involved and is treated most easily under the direction of the teacher. Parts F, G, and H present some important relationships in geometric figures and together make another reasonable homework assignment.

Answers for Part A



$\vec{a} = C - B$, $\vec{b} = A - C$, $\vec{c} = B - A$,
$M = C - \vec{a}u$, $N = C + \vec{b}v$

1. $M - A = (C - \vec{a}u) - A = -\vec{b} - \vec{a}u$; $N - B = (C + \vec{b}v) - B = \vec{a} + \vec{b}v$
2. $\overleftrightarrow{AM} \parallel \overleftrightarrow{BN}$ if and only if $(M - A, N - B)$ is linearly dependent. By Exercise 1, this is the case if and only there are numbers — say, a and b — not both zero, such that

   $$(*) \qquad (\vec{a}u + \vec{b})a + (\vec{a} + \vec{b}v)b = \vec{0}.$$

   Since $(\vec{a}, \vec{b})$ is linearly independent, numbers a and b satisfy (*) if and only if

   $$au + b = 0 \quad \text{and} \quad a + bv = 0.$$

   If a and b satisfy these equations and are not both 0 then neither is 0, and $u = -(b/a)$, $v = -(a/b)$, and $uv = 1$. On the other hand, if $uv = 1$ then the equations are equivalent and any pair of nonzero numbers which satisfies one equation satisfies both. For example, let $a = 1$ and $b = -u$. So, $\overleftrightarrow{AM} \parallel \overleftrightarrow{BM}$ if and only if $uv = 1$.
3. $O \in \overleftrightarrow{AM} \cap \overleftrightarrow{BN}$ if and only if there are numbers — say, m and n — such that $O = A + (M - A)m = B + (N - B)n$. Using the results of Exercise 1, we need to show that:

   $$A - (\vec{a}u + \vec{b})m = B + (\vec{a} + \vec{b}v)n$$

   has a unique solution if and only if $uv \neq 1$. The equation is equivalent to:

   $$(B - A) + (\vec{a}u + \vec{b})m + (\vec{a} + \vec{b}v)n = \vec{0}$$
   $$\vec{a}(um + n) + \vec{b}(m + vn) + \vec{c} = \vec{0}$$

   which, since $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $(\vec{a}, \vec{b})$ is linearly independent, is equivalent to:

   $$um + n = 1$$
   $$m + vn = 1$$

Answers for Part A [cont.]

In case uv = 1, multiplication with 'u' on both sides of the second equation shows that the equations are inconsistent [if $u \neq 1$] or equivalent [if $u = 1$]. This agrees with Exercise 2; if uv = 1 then $\overleftrightarrow{AM}$ and $\overleftrightarrow{BN}$ are two parallel lines or are the same line. If $uv \neq 1$ the equations are easily solved for 'm' and 'n':

$$m = \frac{1 - v}{1 - uv}, \quad n = \frac{1 - u}{1 - uv}$$

So, if $uv \neq 1$, $\overleftrightarrow{AM}$ and $\overleftrightarrow{BN}$ intersect at a point O which is the $\frac{v - 1}{uv - 1}$-point, from A, of $\overline{AM}$ and the $\frac{u - 1}{uv - 1}$-point, from B, of $\overline{BN}$. It follows that O divides the intervals from A to M and from B to N, respectively, in the ratios $\frac{1 - v}{(1 - u)v}$ and $\frac{1 - u}{(1 - v)u}$.

4. In case u = v, the point O of Exercise 3 divides the intervals in the ratio 1/u. This is in agreement with Theorem 8-10(a).

*

The results in Part A can be developed further to obtain a proof of Theorem 8-20. Suppose that the u-point M divides the interval from B to C in the ratio r and that the v-point N divides the interval from C to A in the ratio s. Then $r = (1 - u)/u$, $s = v/(1 - v)$, $u = 1/(1 + r)$, and $v = s/(1 + s)$. Computing $1 - uv$ in terms of r and s shows that $\overleftrightarrow{AM} \parallel \overleftrightarrow{BN}$ if and only if $1 + r + rs = 0$ and that, otherwise, these lines intersect at O, where

$$O = A + (M - A)m = B + (N - B)n$$

with $m = \frac{1 + r}{1 + r + rs}$ and $n = \frac{r(1 + s)}{1 + r + rs}$.

Consider, now, a third point P which divides the interval from A to B in the ratio t. It follows from the preceding results that $\overleftrightarrow{BN} \parallel \overleftrightarrow{CP}$ if and only if $1 + s + st = 0$ and that, otherwise, these lines intersect at O′, where

$$O' = B + (N - B)n' = C + (P - C)p$$

with $n' = \frac{1 + s}{1 + s + st}$ and $p = \frac{s(1 + t)}{1 + s + st}$.

Now suppose that $\overleftrightarrow{AM} \nparallel \overleftrightarrow{BN} \nparallel \overleftrightarrow{CP}$. It then follows that the three lines are concurrent if and only if O′ = O — that is, if and only if n′ = n. But n′ = n if and only if

$$\frac{1 + s}{1 + s + st} = \frac{r(1 + s)}{1 + r + rs}$$

$$1 + r + rs = r(1 + s + st)$$

$$1 = rst.$$

This result has been obtained under the assumption that $\overleftrightarrow{BN}$ is not parallel to either $\overleftrightarrow{AM}$ or $\overleftrightarrow{CP}$. But [by cyclically permuting the notation] it is clear that the same conclusion holds if there is one of the three lines which is not parallel to either of the others — that is, it holds unless the three lines are parallel. So, if $\overleftrightarrow{AM}$, $\overleftrightarrow{BN}$, and $\overleftrightarrow{CP}$ are not parallel then they are concurrent if and only if rst = 1. In particular,

if rst = 1 then $\overleftrightarrow{AM}$, $\overleftrightarrow{BN}$, and $\overleftrightarrow{CP}$
are either parallel or concurrent.

On the other hand, suppose that $\overleftrightarrow{AM} \parallel \overleftrightarrow{BN} \parallel \overleftrightarrow{CP}$. Then, as previously shown, $1 + r + rs = 0$ and $1 + s + st = 0$. From the second of these it follows that $r + rs + rst = 0$; and this, in combination with the first, shows that rst = 1. Suppose, finally, that $\overleftrightarrow{AM}$, $\overleftrightarrow{BN}$, and $\overleftrightarrow{CN}$ are concurrent. Since M and N are points of division, $M \neq B$ and $N \neq A$. So, $\overleftrightarrow{AM} \neq \overleftrightarrow{BN}$ and, since $\overleftrightarrow{AM} \cap \overleftrightarrow{BN} \neq \emptyset$, $\overleftrightarrow{AM} \nparallel \overleftrightarrow{BN}$. Similarly $\overleftrightarrow{BN} \nparallel \overleftrightarrow{CP}$. But, it has already been proved in this case that, $\overleftrightarrow{AM}$, $\overleftrightarrow{BN}$, and $\overleftrightarrow{CP}$ being concurrent, rst = 1.

Consequently, we have arrived at the following theorem:

$\overleftrightarrow{AM}$, $\overleftrightarrow{BN}$, and $\overleftrightarrow{CP}$ are parallel or
concurrent if and only if rst = 1.

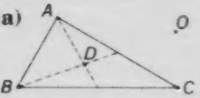
This is the result of Exercise 4(a) on page 365. Exercise 4(b) is easily obtained from results which are already known, and Theorem 8-20 can be obtained as outlined in Exercise 5.

Part B

In each of the following you are given a figure showing points A, B, C, D, and O. In each case, copy the figure and draw arrows indicating the position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ of A, B, C, and D with respect to O.

1. Express each of the described translations in terms of '$\vec{a}$', '$\vec{b}$', '$\vec{c}$' and '$\vec{d}$'.

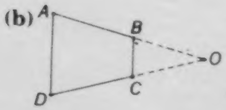
(a)



(i) $C - A$ **(ii)** $C - B$ **(iii)** $A - B$

(iv) $(C - B) + (B - D)$ **(v)** $D - C$

(b)



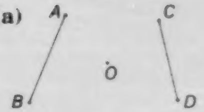
(i) $A - B$ **(ii)** $D - A$ **(iii)** $C - D$

(iv) $C - B$ **(v)** $D - B$ **(vi)** $C - A$

(c) Like part (b), but with $O = D$. Compare your answers with those of part (b).

2. Express the position vector of each of the described points in terms of '$\vec{a}$', '$\vec{b}$', '$\vec{c}$', and '$\vec{d}$'.

(a)



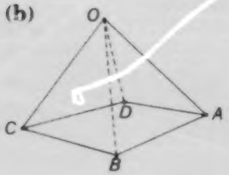
(i) the midpoint, M, of $\overline{AB}$

(ii) the midpoint, N, of $\overline{CD}$

(iii) the midpoint of $\overline{MN}$

(iv) the midpoint of the interval whose end points are the midpoints, P and Q, of $\overline{AC}$ and $\overline{BD}$

(b)



(i) the midpoint, P, of $\overline{AB}$

(ii) the point Q which divides the interval from A to D in 1:4 [Note, in (ii), that, in applying Theorem 8–14, the value of 'r' is *not* 1/4.]

(iii) the point which divides the interval from P to Q in 2:3

3. The following questions refer to the figures in Exercises 1 and 2. Answer them by expressing the translations referred to in terms of '$\vec{a}$', '$\vec{b}$', '$\vec{c}$', and '$\vec{d}$'.

(a) Suppose that, in Exercise 1(a), M and N are the midpoints of $\overline{AC}$ and $\overline{BC}$, respectively. By comparing $M - N$ and $A - B$, reach a conclusion concerning $\overline{MN}$ and $\overline{AB}$.

(b) Suppose that, in Exercise 1(b), $\overline{AD} \parallel \overline{BC}$, M, R, and S are the midpoints of $\overline{AC}$, $\overline{AB}$, and $\overline{CD}$. By comparing $S - R$ and $M - R$, reach a conclusion concerning M, R, and S.

Answers for Part B

1. [There seems no need for giving, here, either the figures or the formulas which are requested. The latter are obtained merely by replacing capital letters by the corresponding arrow-letters. For example, the answer for (a) (i) is '$\vec{c} - \vec{a}$'.]

2. (a) (i) $(\vec{a} + \vec{b})/2$ (ii) $(\vec{c} + \vec{d})/2$

(iii) $(\vec{a} + \vec{b} + \vec{c} + \vec{d})/4$ (iv) $(\vec{a} + \vec{c} + \vec{b} + \vec{d})/4$

(b) (i) $(\vec{a} + \vec{b})/2$ (ii) $(\vec{a}4 + \vec{b})/5$

(iii) $(\vec{a} \cdot 31 + \vec{b} \cdot 19)/50$

3. (a) $M - N = (\vec{a} + \vec{c})/2 - (\vec{b} + \vec{c})/2 = (\vec{a} - \vec{b})/2 = (A - B)/2$. So, $\overline{MN} \parallel \overline{AB}$ and $\overline{MN} : \overline{AB} = 1/2$.

(b) $S - M = (\vec{c} + \vec{d})/2 - (\vec{a} + \vec{c})/2 = (\vec{d} - \vec{a})/2$;

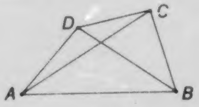
$M - R = (\vec{a} + \vec{c})/2 - (\vec{a} + \vec{b})/2 = (\vec{c} - \vec{b})/2$

Since, by hypothesis $(\vec{d} - \vec{a}, \vec{c} - \vec{b})$ is linearly dependent, so is $(S - M, M - R)$. Hence, $\{M, R, S\}$ is collinear.

(c) In Exercise 2(a), compare $M - P$ and $Q - N$. Reach a conclusion about $\overline{MP}$ and $\overline{QN}$. About $\overline{MQ}$ and $\overline{PN}$.

(d) Suppose that, in Exercise 2(b), K, L, M, and N are the midpoints of $\overline{OA}$, $\overline{AB}$, $\overline{BC}$, and $\overline{OC}$, respectively. Discover, by using position vectors, as much as you can about the intervals $\overline{KL}$, $\overline{LM}$, $\overline{MN}$, $\overline{NK}$, $\overline{KM}$, and $\overline{LN}$. [*Hint*: You should be able to make a significant discovery about each of three pairs of these intervals.]

Part C



Suppose that A, B, C, and D are four points, no three of which are collinear.

1. Use what you know about lines to show that $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ have at most one point in common, and that none of the given points belongs to both of these lines.
2. Supposing that $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ have a common point, R, it follows from Exercise 1 that this point divides each of the intervals, from A to C and from B to D, in some ratio. Suppose that the ratio is the same for both intervals – say, $r : 1 - r$. Show that $\overline{CD} \parallel \overline{AB}$ and that $C - D \neq A - B$. [*Hint*: Introduce position vectors for the given points, equate two expressions for the position vector of R, and deduce an equation concerning $C - D$ and $B - A$.]
3. Suppose that $\overline{CD} \parallel \overline{AB}$ and $C - D \neq A - B$. Show that the lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at a point which divides the interval from A to C and the interval from B to D in the same ratio. [*Hint*: If $(C - D)a = (B - A)b$, with a and b not both 0 and $a + b \neq 0$, then you can find a number r such that $r : 1 - r = a : b$ and retrace the steps you made in solving Exercise 2.]
4. Show that the *intervals* $\overline{AC}$ and $\overline{BD}$ intersect at a point which divides both [from A and from B, respectively] in the same ratio if and only if $\overline{AB} \parallel \overline{CD}$ and $(B - A) : (C - D) > 0$.

Part D

Mark points O, A, and B, with $A \neq B$, and indicate the position vectors, $\vec{a}$ and $\vec{b}$, of A and B with respect to O.

1. On your figure, locate six points, C_1 through C_6, whose position vectors are given by:

$$\vec{c}_1 = \vec{a}\tfrac{1}{2} + \vec{b}\tfrac{1}{2}, \qquad \vec{c}_2 = \vec{a}\tfrac{1}{4} + \vec{b}\tfrac{3}{4}, \qquad \vec{a}3 + \vec{b} - \vec{c}_3 4 = \vec{0}$$
$$\vec{c}_4 = \vec{a}2 + \vec{b} \cdot -1, \qquad \vec{c}_5 = \vec{a} \cdot -2 + \vec{b}3, \qquad \vec{c}_6 = \vec{a} + \vec{b}\tfrac{1}{2}$$

2. If, in your figure, you made a different choice for O, would you obtain different points as $C_1, \ldots, C_6$?
3. Which of the six points belong to $\overline{AB}$? To $-\overrightarrow{BA}$? To $-\overrightarrow{AB}$? Justify your answers by referring to Theorem 8–14 and Theorem 8–5.

3. (c) $M - P = (\vec{a} + \vec{b})/2 - (\vec{a} + \vec{c})/2 = (\vec{b} - \vec{c})/2$;

$Q - N = (\vec{b} + \vec{d})/2 - (\vec{c} + \vec{d})/2 = (\vec{b} - \vec{c})/2$.

So, $M - P = Q - N$ and, hence, $M - Q = P - N$. It follows that $\overline{MP} \parallel \overline{QN}$, $\overline{MQ} \parallel \overline{PN}$, and $\overline{MP} : \overline{QN} = 1 = \overline{MQ} : \overline{PN}$.

(d) $L - K = (\vec{a} + \vec{b})/2 - \vec{a}/2 = \vec{b}/2$; $M - N = (\vec{b} + \vec{c})/2 - \vec{c}/2 = \vec{b}/2$. So, $\overline{KL}$ and $\overline{MN}$ are parallel and their ratio is 1.

$L - M = (\vec{a} + \vec{b})/2 - (\vec{b} + \vec{c})/2 = (\vec{a} - \vec{c})/2$; $K - N = \vec{a}/2 - \vec{c}/2 = (\vec{a} - \vec{c})/2$. So, $\overline{LM}$ and $\overline{MN}$ are parallel and their ratio is 1.

The midpoint of $\overline{LN}$ has position vector $(\vec{a} + \vec{b} + \vec{c})/4$, and this is also the position vector of the midpoint of $\overline{KM}$. So, $\overline{KM}$ and $\overline{LN}$ bisect each other.

Answers for Part C

1. Since no three of the four points A, B, C, and D are collinear, the four points are certainly not all on one line. So, $\overleftrightarrow{AC} \neq \overleftrightarrow{BD}$ and, hence, $\overleftrightarrow{AC} \cap \overleftrightarrow{BD}$ contains at most one point. Also, since no three are collinear, $A \notin \overleftrightarrow{BD}$, $C \notin \overleftrightarrow{BD}$, $B \notin \overleftrightarrow{AC}$, and $D \notin \overleftrightarrow{AC}$. Hence, none of the four points belongs to both $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$.
2. Suppose that $\overleftrightarrow{AC} \cap \overleftrightarrow{BD} = \{R\}$ and that R divides the intervals from A to C and from B to D in the same ratio — say, $r : 1 - r$. Using position vectors, $\vec{r} = \vec{a}(1 - r) + \vec{c}r = \vec{b}(1 - r) + \vec{d}r$. It follows that $(\vec{c} - \vec{d})r = (\vec{b} - \vec{a})(1 - r)$ and, hence, that $(C - D)r = (B - A)(1 - r)$. Since neither r nor $1 - r$ is zero it follows that $[C - D] = [B - A]$. Hence, $\overline{CD} \parallel \overline{AB}$.
3. Since $\overline{CD} \parallel \overline{AB}$, there are numbers — say, a and b — not both zero, such that $(C - D)a = (B - A)b$. Since $C - D \neq A - B$, $a + b \neq 0$. So, $(C - D)\frac{a}{a + b} = (B - A)\frac{b}{a + b}$. Let $r = a/(a + b)$. It follows that $b/(a + b) = 1 - r$. So, $(C - D)r = (B - A)(1 - r)$ and, introducing position vectors, $(\vec{c} - \vec{d})r = (\vec{b} - \vec{a})(1 - r)$. Hence, $\vec{a}(1 - r) + \vec{c}r = \vec{b}(1 - r) + \vec{d}r$. But, $\vec{a}(1 - r) + \vec{c}r$ is the position vector of the point R_1 which divides the interval from A to C in $r : 1 - r$, and $\vec{b}(1 - r) + \vec{d}r$ is the position vector of the point R_2 which divides the interval from B to D in the same ratio. Since, as we have seen, $R_1 = R_2$, it follows that $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at a point which divides both intervals in the same ratio.
4. After Exercise 3, what remains to be shown is that $0 < r < 1$ if and only if $(B - A) : (C - D) > 0$. But, since $(C - D)r = (B - A)(1 - r)$, $(B - A) : (C - D) = r/(1 - r)$ and, as previously shown, $r/(1 - r) > 0$ if and only if $0 < r < 1$.

Answers for Part D

1. [Various.]
2. No. [A different choice for O would result in different values for '$\vec{a}$', '$\vec{b}$', etc., but the points would be unaffected. For example, C_1 will be the midpoint of $\overline{AB}$, however O is chosen.]
3. By Theorem 8-14, C_1 divides the interval from A to B in the ratio 1; C_2 divides the same interval in the ratio 3; C_3, in $1/3$; C_4, in $-1/2$, and C_5, in $-3/2$. So, by Theorem 8-5, C_1, C_2, and C_3 belong to $\overline{AB}$, $C_4 \in -\overrightarrow{AB}$, and $C_5 \in -\overrightarrow{BA}$. Since $1 + \frac{1}{2} \neq 1$, C_6 is not a point which divides the interval from A to B in any ratio. Also, it is neither A nor B. So, $C_6 \notin \overline{AB}$.

4. As we have seen,

$$C \in \overleftrightarrow{AB} \text{ if and only if } \exists_x\, \vec{c} = \vec{a}(1 - x) + \vec{b}x.$$

Prove, using Theorem 8-14, that, for $A \neq B$,
(a) $C \in \overline{AB} \longleftrightarrow \vec{c} = \vec{a}(1 - c) + \vec{b}c$, where $0 < c < 1$;
(b) $C \in \overrightarrow{AB} \longleftrightarrow \vec{c} = \vec{a}(1 - c) + \vec{b}c$, where $c > 0$;
(c) $C \in -\overrightarrow{AB} \longleftrightarrow \vec{c} = \vec{a}(1 - c) + \vec{b}c$, where $c < 0$;
(d) $C \in \overrightarrow{BA} \longleftrightarrow \vec{c} = \vec{a}(1 - c) + \vec{b}c$, where $c \leq 1$.

5. For each of the translations $\vec{c}_1$ through $\vec{c}_6$ of Exercise 1, determine whether there exist numbers x, y, and z, not all zero, such that

$$x + y + z = 0 \quad \text{and} \quad \vec{a}x + \vec{b}y + \vec{c}_iz = \vec{0}. \; [i = 1, 2, 3, 4, 5, \text{ or } 6]$$

6. What can you say of $\{A, B, C_i\}$ if there are numbers x, y, and z as described in Exercise 5? What can you say if there are no such numbers?

Part E

1. Prove Theorem 8-15 by first showing that
(a) if $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$ and $a + b + c = 0$, and not all of a, b, and c are zero, then $(\vec{a} - \vec{c})a + (\vec{b} - \vec{c})b = \vec{0}$ and not both a and b are 0,
and
(b) if $(\vec{a} - \vec{c})a + (\vec{b} - \vec{c})b = \vec{0}$ and not both a and b are 0, then there exists a number z such that $\vec{a}a + \vec{b}b + \vec{c}z = \vec{0}$, $a + b + z = 0$, and a, b, and z are not all zero.

[To complete the proof, note that it follows that

there exist numbers x and y, not both 0,
such that $(\vec{a} - \vec{c})x + (\vec{b} - \vec{c})y = \vec{0}$
if and only if
there exist numbers x, y, and z, not all 0,
such that $\vec{a}x + \vec{b}y + \vec{c}z = \vec{0}$ and $x + y + z = 0$.]

2. From Exercise 1(a) it follows that if $a + b + c = 0$, $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$, and a, b, and c are not all 0, then $(\vec{a} - \vec{c}, \vec{b} - \vec{c})$ is linearly dependent. This sentence is of the form:

$$(*) \qquad [p \text{ and } q \text{ and not } r] \longrightarrow s$$

(a) Verify the statement just made by giving the sentences which should replace 'p', 'q', 'r', and 's'.
(b) Explain why a sentence of the form (*) is equivalent to corresponding sentences of the forms:
(i) $p \longrightarrow [(q \text{ and not } r) \longrightarrow s]$
(ii) $p \longrightarrow [\text{not } s \longrightarrow \text{not } (q \text{ and not } r)]$
(iii) $(p \text{ and not } s) \longrightarrow \text{not } (q \text{ and not } r)$
(iv) $(p \text{ and not } s) \longrightarrow [q \longrightarrow r]$

Answers for Part D [cont.]

4. (a) $C \in \overline{AB}$ if and only if $C \in \overleftrightarrow{AB}$ and divides the interval from A to B in a positive ratio; $\vec{c} = \vec{a}(1 - c) + \vec{b}c$ if and only if C is the point of $\overleftrightarrow{AB}$ which divides the interval from A to B in $c/(1 - c)$; $c/(1 - c) > 0$ if and only if $0 < c < 1$.

[(b) - (d) are similar.]

5. Such numbers exist for C_1, C_2, C_3, C_4, and C_5, but not for C_6.

6. $C_i \in \overleftrightarrow{AB}$; $C_i \notin \overleftrightarrow{AB}$

Answers for Part E

1. (a) Suppose that $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$ and $a + b + c = 0$. Then, since $c = -a + -b$, $(\vec{a} - \vec{c})a + (\vec{b} - \vec{c})b = \vec{0}$. Also, if $a = 0 = b$ then $c = -a + -b = 0$. So, if not all of a, b, and c are zero then not both a and b are 0.

[So, if there exist numbers x, y, and z not all 0, such that $\vec{a}x + \vec{b}y + \vec{c}z = \vec{0}$ and $x + y + z = 0$ then there exist numbers x and y, not both 0, such that $(\vec{a} - \vec{c})x + (\vec{b} - \vec{c})y = \vec{0}$ — that is, then the points whose position vectors are $\vec{a}$, $\vec{b}$, and $\vec{c}$ are collinear.]

(b) Suppose that $(\vec{a} - \vec{c})a + (\vec{b} - \vec{c})b = \vec{0}$, where not both a and b are 0. Let $c = -a + -b$. It follows that $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$, $a + b + c = 0$, and, since not both a and b are zero, not all of a, b, and c are zero. So, there exists a z such that $\vec{a}a + \vec{b}b + \vec{c}z = \vec{0}$, $a + b + z = 0$, and not all of a, b, and z are zero.

[So, if there exist numbers x and y — then there exist numbers x, y, and z . . . .]

2. (a) p: $a + b + c = 0$; q: $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$; r: $a = b = c = 0$; s: $(\vec{a} - \vec{c}, \vec{b} - \vec{c})$ is linearly dependent

(b) (*) is equivalent to (i) by importation and exportation.

(i) is equivalent to (ii) because a sentence is equivalent to its contrapositive, and because of the replacement rule for biconditional sentences.

(ii) is equivalent to (iii) by importation and exportation.

(iii) is equivalent to (iv) because sentences of the forms 'not (q and not r)' and '$q \Longrightarrow r$' are equivalent [pages 255 and 269] and because of the replacement rule.

(c) Show that Exercise 1(a) implies:
If $(\vec{a} - \vec{c}, \vec{b} - \vec{c})$ is linearly independent and $a + b + c = 0$ then, if $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$, $a = 0$ and $b = 0$ and $c = 0$.

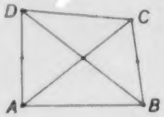
(d) Prove the following corollary of Theorem 8–15:

> Corollary If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are position vectors of noncollinear points, and a, b, and c are numbers such that $a + b + c = 0$, then $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$ if and only if $a = 0$, $b = 0$, and $c = 0$.

[*Hint*: In part (c) you have essentially proved the part of the corollary which you would obtain by replacing 'if and only if' by 'only if'. The other part is very easy.]

Part F

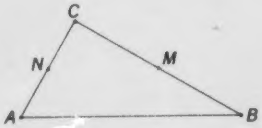
Suppose that [as in Part C], A, B, C, and D are four points, no three of which are collinear.



1. Introduce position vectors for these points and compute the position vectors of the midpoints of $\overline{BD}$, $\overline{AD}$, and $\overline{BC}$.
2. Suppose that the three midpoints of Exercise 1 are collinear.
   - **(a)** Make a conjecture concerning $\overline{AB}$ and $\overline{CD}$.
   - **(b)** Use the only if-part of Theorem 8–9(b) to establish a result concerning $\overline{AB}$ and $\overline{CD}$.
3. Suppose that $\overline{AB} \parallel \overline{CD}$. By retracing the steps you took in Exercise 2, show that the midpoints of Exercise 1 are collinear.

Part G

The corollary to Theorem 8–15 can be used to give another proof that two medians of a triangle intersect at a point which divides each in the ratio 2 : 1. Suppose that, in $\triangle ABC$, M and N are the midpoints of $\overline{BC}$ and $\overline{CA}$, respectively.



1. Introduce position vectors of A, B, and C.
   - **(a)** Compute the position vectors of M and N.
   - **(b)** Compute the position vectors of the points which divide the interval from A to M and the interval from B to N in the ratios $m : 1 - m$ and $n : 1 - n$, respectively.
2. **(a)** Obtain an equation of the form '$\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$' [where '$a$', '$b$', and '$c$' are expressions in '$m$' and '$n$'] which has a solution if and only if $\overleftrightarrow{AM}$ and $\overleftrightarrow{BN}$ intersect.
   - **(b)** Show that the corresponding equation of the form '$a + b + c = 0$' is satisfied.

Answers for Part E [cont.]

(c) Exercise 1(a) is of the form (*), with the replacements for 'p', 'q', 'r', and 's' given in part (a). This is equivalent to the sentences of the form:

$$(\text{not } s \text{ and } p) \Longrightarrow [q \Longrightarrow r]$$

with the same replacements. In particular, the former implies the latter.

(d) To prove the corollary, all that remains is to note that if $a = b = c = 0$ then $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$.

Answers for Part F

1. midpoint of $\overline{BD}$: $(\vec{b} + \vec{d})/2$; midpoint of $\overline{AD}$: $(\vec{a} + \vec{d})/2$; midpoint of $\overline{BC}$: $(\vec{b} + \vec{c})/2$
2. (a) $\overline{AB} \parallel \overline{CD}$
   - (b) In $\triangle ABD$, the interval joining the midpoints of $\overline{BD}$ and $\overline{AD}$ is parallel to $\overline{AB}$; in $\triangle BCD$, the interval joining the midpoints of $\overline{BD}$ and $\overline{BC}$ is parallel to $\overline{CD}$. So, if the three midpoints are collinear then the line containing them is parallel to both $\overline{AB}$ and $\overline{CD}$ and, so, $\overline{AB} \parallel \overline{CD}$.
3. Suppose that $\overline{AB} \parallel \overline{CD}$. It follows that the two intervals joining midpoints are parallel and, since they have a common end point, are collinear.

Answers for Part G

1. (a) $\vec{m} = (\vec{b} + \vec{c})/2$, $\vec{n} = (\vec{a} + \vec{c})/2$
   - (b) $\vec{a}(1 - m) + (\vec{b} + \vec{c})(m/2)$, $\vec{b}(1 - n) + (\vec{c} + \vec{a})(n/2)$
2. (a) $\vec{a}(1 - m - \frac{n}{2}) + \vec{b}(\frac{m}{2} - 1 + n) + \vec{c}(\frac{m}{2} - \frac{n}{2}) = \vec{0}$
   - (b) $(1 - m - \frac{n}{2}) + (\frac{m}{2} - 1 + n) + (\frac{m}{2} - \frac{n}{2}) = 0$

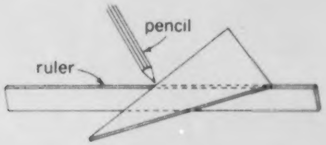
(c) Apply the corollary to find the values of 'm' and 'n' which satisfy the equation of part (a).

3. Interpret the result of Exercise 2.

Part H

1. Given $\triangle ABC$, let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be the position vectors of A, B and C with respect to an arbitrary point O.
   - (a) Show that the position vectors of the intersection of the medians of $\triangle ABC$ is $(\vec{a} + \vec{b} + \vec{c})\frac{1}{3}$.
   - (b) Compare the result in part (a) with the expression for the position vector of the midpoint of $\overleftrightarrow{AB}$.

2. A uniform rod of constant cross-section is a reasonable physical model of the segment $\overleftrightarrow{AB}$. Such a rod will balance if it is supported by a fulcrum placed beneath its center. Similarly, a triangular piece of cardboard is a reasonable physical model – not of a triangle, but – of a "solid triangle".
   - (a) Cut out several triangular pieces of cardboard.
   - (b) Hold a corner of one of your pieces of cardboard on the edge of a ruler and twist the cardboard until it balances. When it does, mark the point in the side opposite the chosen vertex which is over the edge of the ruler.



Repeat this experiment, choosing different corners, and using different pieces of cardboard.

   - (c) Draw the medians on each of your pieces of cardboard. What happens when you place the cardboard so that a median lies along the edge of your ruler?
   - (d) Try to balance one of your pieces of cardboard on the point of a pencil.

8.06 Quadrilaterals

> Definition 8-4
> (a) $PQRS = \overleftrightarrow{PQ} \cup \overleftrightarrow{QR} \cup \overleftrightarrow{RS} \cup \overleftrightarrow{SP}$
> (b) $PQRS$ is a quadrilateral $\longleftrightarrow$ each of $\{P, Q, R\}$, $\{Q, R, S\}$, $\{R, S, P\}$, $\{S, P, Q\}$ is noncollinear

When we write 'quadrilateral *PQRS*' we shall be referring to the set *PQRS* and, at the same time, implying that it is a quadrilateral. The points P, Q, R, and S are the *vertices* of quadrilateral *PQRS*, and the

---

(c) By the corollary, since $\vec{a}$, $\vec{b}$, and $\vec{c}$ are position vectors of noncollinear points and equation (b) holds, equation (a) is satisfied if and only if

$$m + \frac{n}{2} = 1, \quad \frac{m}{2} + n = 1, \quad \text{and } \frac{m}{2} = \frac{n}{2}$$

— that is, if and only if $m = \frac{2}{3} = n$.

3. The result of Exercise 2 shows that the medians intersect at a point which is the $\frac{2}{3}$-point, from A, of $\overline{AM}$ and, also, the $\frac{2}{3}$-point, from B, of $\overline{BN}$.

Answers for Part H

1. (a) From Exercise 1(b), with $m = \frac{2}{3}$, we see that the desired position vector is $(\vec{a} + \vec{b} + \vec{c})/3$.
   (b) [The position vector of the midpoint of $\overleftrightarrow{AB}$ is $(\vec{a} + \vec{b})/2$.]

2. The cardboard triangles should balance when the edge of the ruler lies under a median, and when the pencil point is under the intersection of the medians.

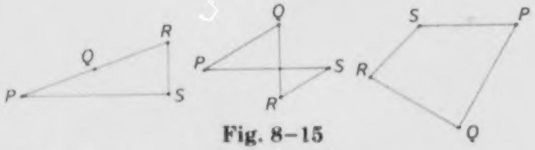


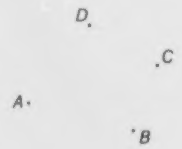
Fig. 8–15

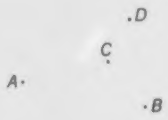
intervals $\overline{QR}$, $\overline{RS}$, $\overline{SP}$, and $\overline{PQ}$ are its sides. $\overline{QR}$ and $\overline{SP}$ are *opposite sides*, as are $\overline{RS}$ and $\overline{PQ}$. The intervals $\overline{PR}$ and $\overline{QS}$ are the *diagonals* of quadrilateral $PQRS$. The endpoints of a diagonal of a quadrilateral are called *opposite vertices* of the quadrilateral.

Some care must be exercised in listing the vertices of a quadrilateral. If P, Q, R, and S are four points, no three of which are collinear, then there is more than one quadrilateral which has these points as vertices. An easy way to see how many such quadrilaterals there are is to pay attention to opposite vertices. Choosing one of the four points – say, P – how many choices do you have for the vertex which is to be opposite P? How many quadrilaterals have P, Q, R, and S as vertices? [Make some sketches. Indicate the diagonals by dotted lines.] Note that quadrilateral $PQRS$ and quadrilateral $RQPS$ have the same pairs of opposite vertices and, so, are the same quadrilateral. [How many names like these two can you find for this quadrilateral?]

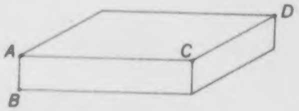
Exercises

1. Consider four points, A, B, C, and D, situated as in the upper figure on the right.
   (a) Draw a picture of quadrilateral $ABCD$ and use dotted lines to represent its diagonals.
   (b) Do any two sides of this quadrilateral intersect?
   (c) Do its diagonals intersect?
   (d) Repeat parts (a) – (c) for quadrilateral $ACBD$.
   (e) Repeat parts (a) – (c) for quadrilateral $ADBC$.





2. Repeat Exercise 1 when A, B, C, and D are situated as in the lower figure.
3. Suppose that A, B, C, and D are four corners of a box, as shown in the figure.
   (a) Repeat Exercise 1(a).
   (b) Are you less sure of any of your answers for Exercises 1 and 2?



Suppose given four points, P, Q, R, and S, no three of which are collinear. There is a quadrilateral having these points as vertices and which has $\overline{PQ}$ as a diagonal, another which has $\overline{PR}$ as a diagonal, and a third which has $\overline{PS}$ as a diagonal. There are six intervals with end points P, Q, R, or S, and the choice of one of these as a diagonal determines which of the remaining five is the other diagonal. The remaining four intervals are left as the sides of the quadrilateral. So, there are exactly three quadrilaterals with the given points as vertices.

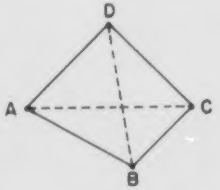
PQRS, QRSP, RSPQ, SPQR, RQPS, QPSR, PSRQ, and: SRQP all name the same quadrilateral. [That there are eight such names is to be expected, since there are 24 permutations of the letters and each such names one of three quadrilaterals.]

* * *

The exercises which follow can be easily treated in class. We recommend that you have a stick model for Exercise 3.

Answers for Exercises

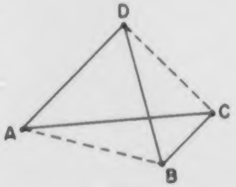
1. (a)



(b) No.

(c) Yes. [At least, they appear to!]

(d)

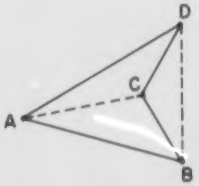


Yes, $\overline{AC}$ and $\overline{BD}$ [seem to] intersect.

No.

(e) [As for part (d).]

2. (a)



(b) No.

(c) No.

One of the quadrilaterals you drew in the preceding exercises appeared to have intersecting diagonals. [Which one?] One appeared to have two opposite sides which intersect. [Which one?] As Exercise 3 points out, appearances may be deceiving. On the basis of these examples we define two properties of quadrilaterals:

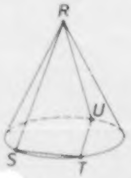
> Definition 8-5
> (a) A quadrilateral is *simple* if and only if no two of its sides intersect.
> (b) A quadrilateral is *convex* if and only if its diagonals intersect.

The quadrilateral of Exercise 1(a) is simple and [apparently] convex. That of Exercises 1(d) and 1(e) is not convex and [apparently] is not simple. The quadrilaterals of Exercises 2 and 3 are simple and not convex. One might expect that, in addition to quadrilaterals of these three kinds [simple and convex, simple but not convex, and neither simple nor convex], there would be quadrilaterals of a fourth kind. What would the fourth kind be? Do you think there are quadrilaterals of this kind?

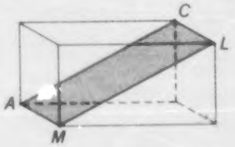
Exercises

Part A

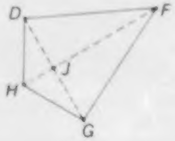
1. Which of the following are pictures of (i) simple quadrilaterals, (ii) convex quadrilaterals, (iii) simple convex quadrilaterals?



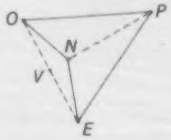
(a) quadrilateral *RSTU*
(b) quadrilateral *RUST*



(c) quadrilateral *AMLC*
(d) quadrilateral *CLAM*

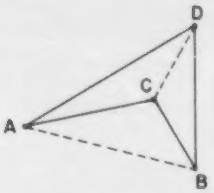


(e) quadrilateral *DFGH*
(f) quadrilateral *DFHG*



(g) quadrilateral *OPEN*
(h) quadrilateral *PEON*

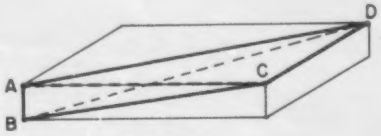
(d)



No.
No.

(e) [As for part (d).]

3. (a)



(b) Students who gave unqualified 'Yes.' answers in Exercise 1 should reconsider. The points A, B, C, and D of Exercise 1 may or may not lie in one plane. If they do, then the 'Yes.'s are correct; if they don't, they aren't.

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Answers for Questions. ABCD of Exercise 1; ACBD of Exercise 1

Any convex quadrilateral is simple, so there are no convex and nonsimple quadrilaterals. [This is proved in Exercise 4(b) of Part A on page 355.]

Answers for Part A

1. (a) simple (and not convex) (b) simple and convex
(c) simple (and not convex) (d) neither simple nor convex
(e) simple and convex (f) simple (and not convex)
(g) neither simple nor convex (h) simple (and not convex)

2. Suppose that A, B, C, and D are four points no three of which are collinear.
   - **(a)** How many quadrilaterals have these points as vertices? Justify your answer.
   - **(b)** How many such quadrilaterals have $\overline{AB}$ as a diagonal?
   - **(c)** How many such quadrilaterals have $\overline{AB}$ as a side?
3. Can you choose the points of Exercise 2 in such a way that
   - **(a)** all of the quadrilaterals with these points as vertices are simple?
   - **(b)** just two of the quadrilaterals are simple? [Justify your answer.]
   - **(c)** all the quadrilaterals are simple and at least one of them is convex?
4. **(a)** Try to draw a convex quadrilateral which is not simple.
   - ◦**(b)** Prove that there are no such quadrilaterals. [*Hint*: This is difficult. It is sufficient to prove that if $\overline{AC} \cap \overline{BD} = \{M\}$ then $\overline{AB} \cap \overline{CD} = \varnothing$. (Why is this sufficient?) One way to proceed is to assume that $M = A + (C - A)m = B + (D - B)n$, where $0 < m < 1$ and $0 < n < 1$, and to try to find numbers p and q such that $A + (B - A)p = C + (D - C)q$. (Explain.) It can be shown that if there are such numbers then $(n - m)(q - p) = 1$. From this (and the assumption concerning m and n) it follows that $|q - p| > 1$. From this it follows that no point of $\overline{AB} \cap \overline{CD}$ belongs both to $\overline{AB}$ and to $\overline{CD}$. There are other ways to proceed, and you may have better success using your own judgment as to how to start.]

Part B

1. **(a)** Draw a quadrilateral $PQRS$ whose diagonals, $\overline{PR}$ and $\overline{QS}$, are parallel.
   - **(b)** Mark the midpoints of the sides of the quadrilateral of part (a) and make a conjecture.
2. Suppose that $\{A, B, C\}$ and $\{B, C, D\}$ are noncollinear. Prove that
   - **(a)** $\overline{AC} \parallel \overline{BD}$ if and only if the midpoints of $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ are collinear, and that
   - **(b)** if $\overline{AC} \parallel \overline{BD}$ then the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ are collinear. [*Hint*: For (a), use Theorem 8–6; for (b), use (a).]
3. Show that the midpoints of the sides of a given quadrilateral are the vertices of a second quadrilateral if and only if the diagonals of the first quadrilateral are not parallel.
4. **(a)** Prove that, in a quadrilateral, the midpoints of two opposite sides and the midpoint of a diagonal are collinear if and only if the other two sides are parallel.
   - **(b)** Can the midpoints of two non-opposite sides and the midpoint of a diagonal of a quadrilateral be collinear?

Answers for Part A [cont.]

2. (a) Three [ABCD, ABDC, and ACBD]; each of the three points B, C, and D is the vertex opposite A in one of the possible quadrilaterals.
   - (b) One.
   - (c) Two.
3. (a) Yes. One way is to choose three noncollinear points and choose a fourth point not in the plane determined by the other three. Another way is to choose coplanar points as in Exercise 2 on page 353.
   - (b) No. If one of the three quadrilaterals is nonsimple then a pair of its opposite sides intersect. Taking these and its diagonals as sides, one obtains another nonsimple quadrilateral. So, it is not possible that just two of the three quadrilaterals be simple. [Note that taking the intersecting sides of the given quadrilateral as diagonals, one obtains a convex quadrilateral. So, for any four points, no three of which are collinear, the three quadrilaterals with these points as vertices are such that either two are nonsimple and the third is convex, or all three are simple and none is convex. The first case is illustrated in Exercise 1 on page 353 if the points A, B, C, and D are coplanar. The second is illustrated in Exercise 2 for coplanar points and, also, by any nonplanar quadrilateral. The only question remaining is whether or not, in the first case the third quadrilateral is simple. This is settled in Exercise 4.]
   - (c) No. If all the quadrilaterals are simple then no two of the six intervals in question can serve as the diagonals of a convex quadrilateral.
4. (a) This is impossible [see answer for part (b)].
   - (b) Suppose that $\overline{AC} \cap \overline{BD} = \{M\}$. If we can deduce that $\overline{AB} \cap \overline{CD} = \varnothing$ then, since $\overline{DB} = \overline{BD}$, it will also follow that $\overline{AD} \cap \overline{CB} = \varnothing$. The result will then, show that if ABCD is a convex quadrilateral, it is also simple. [The result also shows that there is no quadrilateral such that both pairs of opposite sides intersect.] Our assumption amounts to saying that $M = A + (C - A)m = B + (D - B)n$, where $0 < m < 1$ and $0 < n < 1$ and that $(C - A, B - D)$ is linearly independent. To investigate $\overline{AB} \cap \overline{CD}$, we consider the equation:

$$A + (B - A)p = C + (D - C)q \tag{1}$$

Our problem is to show that this equation has no solution (p, q) such that $0 < p < 1$ and $0 < q < 1$. From our assumption it follows that $B - A = (C - A)m + (B - D)n$. Substituting in (1) yields:

$$A + (C - A)mp + (B - D)np = C + (D - C)q$$

which is equivalent to:

$$(C - A)(mp - 1) + (B - D)np = (D - C)q \tag{2}$$

Answers for Part A [cont.]

Leaving this for the moment we note that, by Postulate 3,

$$(C - A) + (D - C) + (B - D) = B - A$$
$$= (C - A)m + (B - D)n$$

and, so, that

$$(3) \qquad (C - A)(m - 1) + (B - D)(n - 1) = D - C.$$

From (2) and (3) [''by subtraction''] we obtain:

$$(C - A)(mp - 1 - mq + q) + (B - D)(np - nq + q) = \vec{0}$$

Since $(C - A, B - D)$ is linearly independent this equation — which, under our assumptions is equivalent to (1) — is equivalent to:

$$(4) \qquad \begin{cases} mp - (m - 1)q = 1 \\ np - (n - 1)q = 0 \end{cases}$$

Now [''by subtraction''], (4) implies:

$$(5) \qquad (m - n)(p - q) = 1$$

But, since $0 < m < 1$ and $0 < n < 1$, $|m - n| < 1$. So, if (p, q) is a solution of (1) then $|p - q| > 1$. Hence, not both p and q can be between 0 and 1. Consequently, if $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ do intersect, their intersection cannot belong both to $\overline{AB}$ and $\overline{CD}$.

*

The preceding conclusion is sufficient to our needs; but, of course, much more can be derived from the preceding argument. The argument concluding with equation (5) makes no use of the assumption that m and n are between 0 and 1. So, (4) and (5) continue to hold under merely the assumption that $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{M\}$, and they can be made to yield much information concerning the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ [or of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$] under this assumption. For example, if $m = n$ — equivalently, if M divides the intervals from A to C and from B to D in the same ratio — it follows from (5) that $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \emptyset$, and one may argue further to show that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. This result has already been established in Exercise 2 of Part C on page 349. Supposing, now, that $m \neq n$, equations (4) are equivalent to:

$$(6) \qquad p = \frac{1 - n}{m - n}, \quad q = \frac{-n}{m - n}$$

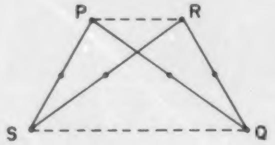
So, if $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are two intersecting lines and $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are not parallel, then $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at a unique point which can be located by using (6) and (1) — assuming, of course, that one knows the location of M with respect to A and C and to B and D. For example, if $M \in \overline{AC} \cap \overline{BD}$ then, using (6), one can show that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at a point which is either on $-\overrightarrow{BA}$ and $-\overrightarrow{CD}$ or on $-\overrightarrow{AB}$ and on $-\overrightarrow{DC}$. [For, suppose that m and n are two numbers between 0 and 1. It follows from (6) that if $m > n$ then $p > 1$ and $q < 0$. So, by (1), in this case the point of intersection belongs to $-\overrightarrow{BA}$ and to $-\overrightarrow{CD}$. The case in which $m < n$ can be treated similarly.]

It is noteworthy that Theorem 8-2 treats of the case in which $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are two parallel lines and forms the basis for our study of trapezoids [including parallelograms]. The preceding considerations deal with the case in which $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are two intersecting lines and forms the basis for a study of quadrilaterals no two of whose sides are parallel. Although many interesting results are obtainable which concern ratios related to such quadrilaterals, the theory is obviously more complex than is that of trapezoids, and we shall do very little with it.

Answers for Part B

[These exercises are of some interest in themselves but are mainly preparatory for Theorem 8-18.]

1. (a)



(b) If the diagonals of a quadrilateral are parallel then the midpoints of its sides are collinear.

2. (a) Let K, L, and M be the respective midpoints of $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$. Note that $\overline{KL} \parallel \overline{AC}$ and that $\overline{LM} \parallel \overline{BD}$. So, $\overline{AC} \parallel \overline{BD}$ if and only if $\overline{KL} \parallel \overline{LM}$. But, $\overline{KL} \parallel \overline{LM}$ if and only if K, L, and M are collinear. Hence, the conclusion.

   (b) Using the notation from (a), assume further that N is the midpoint of $\overline{DA}$ and that $\overline{AC} \parallel \overline{BD}$. By (a), K, L, and M are collinear and, also N, K, and L are collinear if it is the case that {A, B, D} as well as {A, B, C}, is noncollinear. This is the case, for if $A \in \overleftrightarrow{BD}$ then, since $\overline{AC} \parallel \overline{BD}$, $C \in \overleftrightarrow{BD}$, contrary to the assumption that {B, C, D} is noncollinear. Since K, L, and M are collinear and N, K, and L are collinear, and since $K \neq L$ [because {A, B, C} is noncollinear], it follows that K, L, M, and N are collinear.

3. Using the notation of Exercise 2, suppose that ABCD is a quadrilateral. It follows from Theorem 8-7 that $\overline{KL} \parallel \overline{MN} \parallel \overline{AC}$ and $\overline{NK} \parallel \overline{LM} \parallel \overline{BD}$. So, if $\overline{AC} \nparallel \overline{BD}$ then $\overline{KL} \nparallel \overline{LM}$, $\overline{LM} \nparallel \overline{MN}$, $\overline{MN} \nparallel \overline{NK}$, and $\overline{NK} \nparallel \overline{KL}$. In particular, if $\overline{AC} \nparallel \overline{BD}$ then {K, L, M}, {L, M, N}, {M, N, K}, and {N, K, L} are noncollinear and, so, K, L, M, and N are vertices of a quadrilateral. On the other hand, if K, L, M, and N are vertices of a quadrilateral then {K, L, M} is noncollinear and, so, $\overline{KL} \nparallel \overline{LM}$ and $\overline{AC} \nparallel \overline{BD}$.

4. (a) Consider quadrilateral ACDB and apply Exercise 2(a). $\overline{AC} \parallel \overline{BD}$ if and only if the midpoints of the sides $\overline{AB}$ and $\overline{CD}$ and the midpoint of the diagonal $\overline{BC}$ are collinear. Also, $\overline{CA} \parallel \overline{DB}$ if and only if the midpoints of the sides $\overline{CD}$ and $\overline{AB}$ and the midpoint of the diagonal $\overline{DA}$ are collinear. So, the opposite sides $\overline{AC}$ and $\overline{BD}$ are parallel if and only if the midpoints of the sides $\overline{AB}$ and $\overline{CD}$ and the midpoint of one of the diagonals are collinear. [Note that, in an entirely similar way one can show that the midpoints of the diagonals of a quadrilateral and the midpoint of one of its sides are collinear if and only if the two sides adjacent to the one just mentioned are parallel. So, for example, it follows that a quadrilateral whose diagonals bisect each other is a parallelogram.]

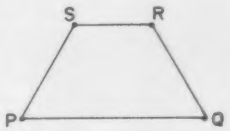
(b) No. For, the remaining two sides would be collinear.

Since, by Definition 8-6(a), opposite sides of a parallelogram are parallel and, because of the noncollinearity of the vertices of a quadrilateral, are noncollinear, they have no common point. So, a parallelogram is a simple quadrilateral. Hence, by Definition 8-6(b), any parallelogram is a trapezoid. Note that there are nonsimple quadrilaterals with two parallel sides. These "bow-ties" are not trapezoids.

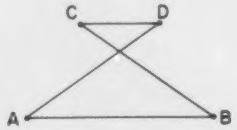
Parts A - D would make a very long homework assignment. We recommend using Parts A and B as in-class exercises. Parts C and D still make a long assignment, however. It would probably be best to identify 8 or 10 of the exercises for all students to do and then to assign a team of students to each of the remaining exercises. Follow this with a careful class discussion of all the exercises.

Answers for Part A

1. (a)



(b)



2. Suppose that {A, B, C} is noncollinear and B - A = C - D. It follows that A ≠ B ≠ C and B - C = A - D. So, $\overline{AB} \parallel \overline{DC}$ and $\overline{CB} \parallel \overline{DA}$. Since $\overline{AB} \parallel \overline{DC}$ and $C \notin \overleftrightarrow{AB}$, $D \notin \overleftrightarrow{AB}$. So, {A, B, D} is noncollinear. Similarly, {B, C, D} is noncollinear and, using this and the fact that $\overline{AB} \parallel \overline{DC}$, {A, C, D} is noncollinear. Hence, if {A, B, C} is noncollinear and B - A = C - D then ABCD is a parallelogram.

   On the other hand, suppose that ABCD is a parallelogram. Then {A, B, C} is noncollinear and $C - B \in [D - A]$ and $B - A \in [C - D]$. It follows that there are nonzero numbers — say, c and a — such that D - A = (C - B)c and C - D = (B - A)a. It follows that C - A = (B - A)a + (C - B)c. Since, also, C - A = (B - A) + (C - B) and since (B - A, C - B) is linearly independent, it follows that a = 1 and c = 1. In particular, C - D = B - A. Hence, if ABCD is a parallelogram then {A, B, C} is noncollinear and B - A = C - D.

3. If A, B, C, and D are vertices of a parallelogram then the parallelogram is either ABCD [D opposite B] or ACBD [D opposite C] or CABD [D opposite A]. Suppose, now, that {A, B, C} is noncollinear. It follows from part (a) that the figures listed above are parallelograms if and only if, respectively, B - A = C - D or C - A = B - D or A - C = B - D — that is, if and only if D = C + (A - B) or D = B + (A - C) or D = B + (C - A). [Students should, of course, draw figures to illustrate this result.]

4. This follows at once from Exercise 2(b) of Part B on page 355. For, if, in that exercise, we interchange 'B' and 'C' all we need note is that, by Definition 8-6(a), if ABCD is a trapezoid with bases $\overline{AB}$ and $\overline{CD}$ then {A, C, B} and {C, B, D} are noncollinear and $\overline{AB} \parallel \overline{CD}$.

8.07 Trapezoids and Parallelograms

Definition 8-6

(a) A quadrilateral is a *trapezoid* if and only if it is simple and has two parallel sides.

(b) A quadrilateral is a *parallelogram* if and only if its opposite sides are parallel.

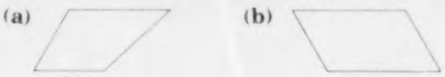


Fig. 8-16

Since a parallelogram is certainly simple [Why?], any parallelogram is a trapezoid. [Note, however, that the word 'trapezoid' is sometimes – but not in this book – used to describe a simple quadrilateral which has *exactly* two parallel sides.] Any pair of parallel sides of a trapezoid is called a pair of *bases* of the trapezoid.

Exercises

Part A

1. **(a)** Draw a picture of a trapezoid $PQRS$ with bases $\overline{PQ}$ and $\overline{RS}$.
   (b) Draw a quadrilateral $ABCD$ with $\overline{AB} \parallel \overline{CD}$ which is not a trapezoid.
2. Show that $ABCD$ is a parallelogram if and only if $\{A, B, C\}$ is noncollinear and $B - A = C - D$. [This is Theorem 8-16.]
3. Show that if $\{A, B, C\}$ is noncollinear then there are three, and only three, parallelograms each of which has A, B, and C as three of its vertices.
4. Show that, in trapezoid $ABCD$ with bases $\overline{AB}$ and $\overline{CD}$, the midpoints of the sides $\overline{BC}$ and $\overline{DA}$ and those of the diagonals $\overline{AC}$ and $\overline{BD}$ are collinear.
5. Prove each of the following.
   (a) If the midpoints of two sides and of a diagonal of a simple quadrilateral are collinear then the quadrilateral is a trapezoid with the remaining sides as bases.
   (b) If the midpoints of two diagonals and of a side of a simple quadrilateral are collinear then the quadrilateral is a trapezoid with the sides which are not opposite to the given side as bases.
6. A quadrilateral is a parallelogram if and only if its diagonals bisect each other. [This is Theorem 8-17.]

Part B

1. **(a)** Suppose that $PQRS$ is a trapezoid with bases $\overline{PQ}$ and $\overline{RS}$. What do you guess to be true of the ratio $(R - S) : (Q - P)$?

Answers for Part A [cont.]

5. (a) This follows at once from Exercise 4(a, b) of Part B on page 355, and Definition 8-6(a).
   (b) Like (a). [See note to solution of Exercise 4(a) at the end of TC 355(4).]

6. Suppose that the diagonals of quadrilateral ABCD have midpoints K and L and that the sides $\overline{AB}$ and $\overline{BC}$ have midpoints M and N. If K = L then {K, L, M} and {N, K, L} are collinear and so $\overline{BC} \parallel \overline{DA}$ and $\overline{AB} \parallel \overline{CD}$. Hence, if K = L then the quadrilateral is a parallelogram. On the other hand, if ABCD is a parallelogram then, since $\overline{BC} \parallel \overline{DA}$, {K, L, M} is collinear and, since $\overline{AB} \parallel \overline{CD}$, {N, K, L} is collinear. If K were not L it would follow that {M, N, K} is collinear. But, by Exercise 5(b), this is not the case. Hence, if ABCD is a parallelogram then K = L.

*

Exercise 6 can, of course, be proved in many ways. Here is one which makes use of the position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ of A, B, C, and D, respectively. The position vectors of the midpoints of $\overline{AC}$ and $\overline{BD}$ are $(\vec{a} + \vec{c})/2$ and $(\vec{b} + \vec{d})/2$. Hence, the midpoint of $\overline{AC}$ is that of $\overline{BD}$ if and only if $\vec{a} + \vec{c} = \vec{b} + \vec{d}$ — that is, if and only if $\vec{c} - \vec{d} = \vec{b} - \vec{a}$ — that is, if and only if B - A = C - D. Assuming, now, that ABCD is a quadrilateral it follows that $\overline{AC}$ and $\overline{BD}$ have the same midpoint if and only if they bisect each other, and, by Theorem 8-16 of Part A, that ABCD is a parallelogram if and only if B - A = C - D. [Note that intervals $\overline{AC}$ and $\overline{BD}$ may have the same midpoint without bisecting each other. This can occur if the intervals are collinear.]

Answers for Part B

1. (a) $(R - S):(Q - P) > 0$
   (b) PQRS would be nonsimple

(b) Suppose that *PQRS* is a quadrilateral in which $\overline{PQ} \parallel \overline{RS}$ and $(R - S) : (Q - P) < 0$. What kind of quadrilateral do you guess *PQRS* to be?

(c) Suppose that $\overline{PQ}$ and $\overline{SR}$ are noncollinear parallel intervals. Show that quadrilateral *PQRS* is simple if and only if $(R - S) : (Q - P) > 0$. [*Hint*: Use Theorem 8-2 to show that if $(R - S) : (Q - P) = r$ then $\overleftrightarrow{PS} \parallel \overleftrightarrow{QR}$ if $r = 1$ and $\overleftrightarrow{PS}$ and $\overleftrightarrow{QR}$ intersect at a point which divides both the interval from *P* to *S* and the interval from *Q* to *R* in $1 : -r$.]

2. Prove:

(a)

Theorem 8-18

(a) *PQRS* is a trapezoid with bases $\overline{PQ}$ and $\overline{RS}$ if and only if $\overline{PQ}$ and $\overline{RS}$ are noncollinear parallel intervals such that $(Q - P) : (R - S) > 0$.

(b) If, in trapezoid *PQRS*, $\overline{PS} \nparallel \overline{QR}$ then $\overleftrightarrow{PS}$ and $\overleftrightarrow{QR}$ intersect at a point which divides both the interval from *P* to *S* and the interval from *Q* to *R* in $-(\overline{PQ} : \overline{RS})$.

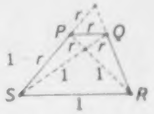
(b)

Theorem 8-19

(a) A trapezoid is convex.

(b) If *PQRS* is a trapezoid with bases $\overline{PQ}$ and $\overline{RS}$ then the intersection of its diagonals divides each of them, from *P* to *R* and from *Q* to *S*, respectively, in $\overline{PQ} : \overline{RS}$.

[*Hint*: Reread the hint for Exercise 1(c), interchanging '*R*' and '*S*'.]



3. (a) Show that the diagonals of a simple quadrilateral are not parallel. [*Hint*: The vertices of a quadrilateral with parallel diagonals are also vertices of a trapezoid. Use Theorem 8-19(a) to show that a quadrilateral with parallel diagonals is not simple.]

(b) Prove:

Theorem 8-20

(a) The midpoints of successive sides of a simple quadrilateral are the successive vertices of a parallelogram.

(b) The intervals joining the midpoints of opposite sides of a simple quadrilateral bisect each other.

Answers for Part B [cont.]

(c) Since, by hypothesis, $\overline{PQ} \cap \overline{RS} = \emptyset$, PQRS is simple if and only if $\overline{PS} \cap \overline{QR} = \emptyset$. Following the hint, $\overleftrightarrow{PS} \cap \overleftrightarrow{QR} = \emptyset$ or, for $r \neq 1$, $\overleftrightarrow{PS} \cap \overleftrightarrow{QR}$ is the $\frac{1}{1 - r}$-point, from P and from Q, respectively, of the interval from P to S and the interval from Q to R. So, it is the point which divides each of these intervals in $1 : -r$. The point belongs to the intervals, themselves, if and only if this ratio is positive — that is, if and only if $r < 0$. Hence, it fails to belong to these intervals if and only if $r > 0$.

2. (a) Suppose that PQRS is a trapezoid with bases $\overline{PQ}$ and $\overline{RS}$. It follows, by definition that $\overline{PQ} \parallel \overline{RS}$ and that $\{P, Q, R\}$ is noncollinear. So, $\overline{PQ}$ and $\overline{RS}$ are two noncollinear parallel intervals. Since, by definition, PQRS is simple it follows from Exercise 1(c) that $(Q - P):(R - S) > 0$.

On the other hand, suppose that $\overline{PQ}$ and $\overline{RS}$ are noncollinear parallel intervals such that $(Q - P):(R - S) > 0$. It follows that PQRS is a quadrilateral with $\overline{PQ}$ and $\overline{RS}$ as parallel sides and, by Exercise 1(c), that PQRS is simple. So, by definition, PQRS is a trapezoid.

Suppose, now, that PQRS is a trapezoid such that $\overline{PS} \nparallel \overline{QR}$. It follows that $\overline{PQ}$ and $\overline{SR}$ are noncollinear parallel intervals and, by Theorem 8-2, $\overline{PS}$ and $\overline{QR}$ intersect in the $/(1 - r)$-points, from P and Q, respectively of the intervals from P to S and from Q to R, where $r = (R - S):(Q - P)$ and, by Theorem 8-18(a), is positive. This point divides the intervals in question in the negative ratio $1 : -r$ — that is, $(Q - P):(S - R)$. Since this ratio is negative it is $-(\overline{PQ} : \overline{RS})$.

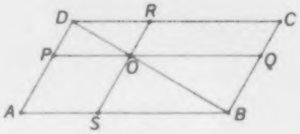
The figure illustrates Theorems 8-18 and 8-19 in the case in which $0 < (Q - P):(R - S) < 1$. Ratios read off the figure, with '$(Q - P):(R - S)$' for 'r' [with proper regard for sense] will hold in all cases.

(b) Suppose that PQRS is a trapezoid with bases $\overline{PQ}$ and $\overline{RS}$. It follows that $\overline{PQ}$ and $\overline{RS}$ are noncollinear parallel intervals and $(S - R):(Q - P) < 0$. By Theorem 8-2, $\overleftrightarrow{PR}$ and $\overleftrightarrow{QS}$ intersect at $P + (R - P)/(1 - r)$ [which is, also, $Q + (S - Q)/(1 - r)$] where $r = (S - R):(Q - P)$. Since $r < 0$, $0 < /(1 - r) < 1$, from which it follows that the point of intersection belongs to both $\overline{PR}$ and $\overline{QS}$. Since these intervals are the diagonals of quadrilateral PQRS it follows that PQRS is convex.

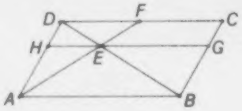
The ratio in which the point of intersection divides the diagonals is, in the notation of the preceding paragraph, $1 : -r$. This ratio is then $(Q - P):(R - S)$. Since it is a positive number it is, also, $\overline{PQ} : \overline{RS}$.

Part C

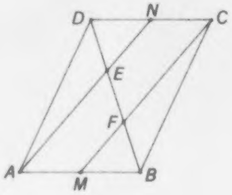
1. In parallelogram $ABCD$, show that the point O of $\overleftrightarrow{BD}$ divides the intervals from P to Q and from R to S in the same ratio. [Assume that $\overline{PQ} \parallel \overline{AB}$ and $\overline{RS} \parallel \overline{AD}$.]



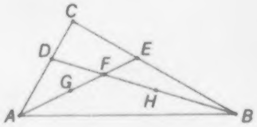
2. In parallelogram $ABCD$, show that E divides the intervals from A to F, B to D, and G to H in $\overline{AB} : \overline{DF}$. Show, also, that $\overline{HE} : \overline{DF} = \overline{EG} : \overline{AB}$.



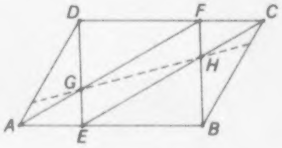
3. In parallelogram $ABCD$, M and N are the midpoints of $\overline{AB}$ and $\overline{CD}$. Show that E and F are the trisection points of $\overline{BD}$. [The trisection points of an interval are the points which divide the interval, from one endpoint or the other, in 1 : 2.] [Incidentally, what theorem assures you that $\overline{AN}$ and $\overline{BD}$ intersect?]



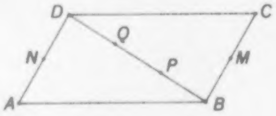
4. In $\triangle ABC$, suppose that D divides the interval from C to A, E divides the interval from C to B, G divides the interval from F to A, and H divides the interval from F to B, all in the same ratio. Show that $GHED$ is a parallelogram.



5. In parallelogram $ABCD$, E and F divide the sides from A to B and C to D, respectively, in the same ratio.
   - (a) Show that $EHFG$ is a parallelogram.
   - (b) What can you say about $\overline{EF}$ and about $\overline{GH}$ if the given ratio is 1?



6. In parallelogram $ABCD$, M and N are the midpoints of $\overline{BC}$ and $\overline{DA}$, respectively, and P and Q are two points which divide the intervals from B to



TC 357 (2)

3. (a) Suppose that ABCD is a quadrilateral whose diagonals, $\overline{AC}$ and $\overline{BD}$ are parallel. Since ABCD is a quadrilateral, $\overline{AC}$ and $\overline{BD}$ are noncollinear parallel intervals. By Theorem 8-18(a), one of the two quadrilaterals ACBD and ACDB is a trapezoid and, so, either $\overline{AB} \cap \overline{CD} \neq \emptyset$ or $\overline{AD} \cap \overline{BC} \neq \emptyset$. In either case, quadrilateral ABCD is not simple. Hence, the diagonals of a simple quadrilateral are not parallel.

   (b) Since the diagonals of a simple quadrilateral are not parallel it follows from Exercise 3 of Part B on page 355 that the midpoints of the sides of a simple quadrilateral are vertices of a quadrilateral. From the solution of Exercise 3 — using Theorem 8-7 — it follows that opposite sides of the quadrilateral whose successive vertices are the midpoints of successive sides of the given quadrilateral are parallel.

   Theorem 8-20(b) follows at once from Theorem 8-20(a) and Theorem 8-17 on page 356.

TC 358

Answers for Part C

1. Since $\overleftrightarrow{PQ}$ and $\overleftrightarrow{DB}$ are transversals of the parallel lines $\overleftrightarrow{AD}$, $\overleftrightarrow{RS}$, and $\overleftrightarrow{BC}$ it follows by Theorem 8-6(b) that $\overline{PO} : \overline{OQ} = \overline{DO} : \overline{OB}$. Similarly, $\overline{DO} : \overline{OB} = \overline{RO} : \overline{OS}$. So, $\overline{PO} : \overline{OQ} = \overline{RO} : \overline{OS}$. [Actually, this does not quite solve the exercise, since the problem is to show that $(O - P) : (Q - O) = (O - R) : (S - O)$, and in using Theorem 8-6 we have lost track of senses. For a complete solution, use Exercise 5 on page 325.]
2. By Theorem 8-19(a), in trapezoid ABFD, E divides the intervals from B to D and from G to H in $\overline{AB} : \overline{DF}$, $\overline{HE} : \overline{EG} = \overline{DE} : \overline{EB} = \overline{EF} : \overline{AE}$ by Theorem 8-6(b), and $\overline{EF} : \overline{AE} = \overline{DF} : \overline{AB}$ by Theorem 8-9(a). So, $\overline{HE} : \overline{EG} = \overline{DF} : \overline{AB}$ and, consequently, $\overline{HE} : \overline{DF} = \overline{EG} : \overline{AB}$.
3. $D - A = C - B$ and $N - D = (C - D)/2 = (B - A)/2 = B - M$. So, $N - A = (D - A) + (N - D) = (B - M) + (C - B) = C - M$. Hence, $\overline{AN} \parallel \overline{MC}$. So, in $\triangle BAE$, F is the midpoint of $\overline{BE}$. Hence, $E - D = F - E = B - F$ and, consequently, E and F are the trisection points. $[E - D = (B - E)/2,\ F - B = (D - F)/2]$ [$\overline{AN}$ and $\overline{BD}$ intersect because they are diagonals of the trapezoid ABND.]
4. By Theorem 8-9(a), in $\triangle ACB$ and $\triangle AFB$, $(E - D) : (B - A) = (H - G) : (B - A)$. Hence, $E - D = H - G$. So, unless $\{D, E, H\}$ is collinear, DEHG is a parallelogram. [Exercise 3 of Part A] But, if $H \in \overleftrightarrow{DE}$ then $\overleftrightarrow{DB} = \overleftrightarrow{DH} = \overleftrightarrow{DE}$ and $E = B$, contrary to the assumption that E divides the interval from C to B.
5. (a) As in Exercise 3, $F - A = C - E$. But, in $\triangle FAB$ and $\triangle CED$, $(H - E) : (F - A) = (B - E) : (B - A) = (D - F) : (D - C) = (G - F) : (E - C) = (F - G) : (C - E)$. Hence, $H - E = F - G$ and, unless $\{E, H, F\}$ is collinear, EHFG is a parallelogram. Since $\overline{DC}$ and $\overline{AB}$ are noncollinear parallel intervals and $F \in \overline{DC}$ while $E \in \overline{AB}$ and $H \in \overline{EC}$, $F \notin \overleftrightarrow{EH}$.

   (b) $\overline{EF} \parallel \overline{AD}$ and $\overline{GH} \parallel \overline{AB}$. The first follows by Exercise 3(i) of Part D on page 330. The second follows in the same manner once one notes that [as shown in part (a)] $\overline{DE} \parallel \overline{FB}$ and that, in the present case, G and H are the midpoints of $\overline{DE}$ and $\overline{FB}$.

D and from D to B, respectively, in the same ratio. Show that $MPNQ$ is a parallelogram.

7. Suppose that $\overline{AB} \parallel \overline{CD}$ and that $\overline{AC}$ bisects $\overline{BD}$. Show that $ABCD$ is a parallelogram.

8. Suppose that $ABCD$ is a parallelogram and l is a line containing points A', B', C', and D' such that any two of $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$, and $\overleftrightarrow{DD'}$ which are lines are parallel. Show that $D' - C' = A' - B'$. [*Hint*: Let $P = A' + (B - B')$ and $Q = D' + (C - C')$. In case $P \notin \overleftrightarrow{AB}$ and $Q \notin \overleftrightarrow{CD}$ you can show that Theorem 8–11 applies. Suppose, then, that $P \in \overleftrightarrow{AB}$ and consider two cases, $P \neq B$ and $P = B$.]

9. In $\triangle ABC$, let M be the midpoint of $\overline{AB}$, N the midpoint of $\overline{BC}$, P a point of $\overline{AC}$ and R and S the midpoints of $\overline{AP}$ and $\overline{PC}$, respectively. Show that $MNSR$ is a parallelogram.

10. Suppose that the diagonals of quadrilateral $ABCD$ intersect at O. Let P_1, P_2, P_3, and P_4 be the centroids of $\triangle ABO$, $\triangle BCO$, $\triangle CDO$, and $\triangle DAO$, respectively. Make a conjecture about $P_1P_2P_3P_4$ and show that it is correct.

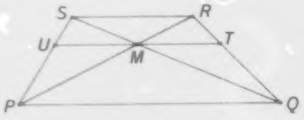
11. In $\triangle ABC$, let P divide the interval from A to B in $t : 1 - t$ and Q divide the interval from A to C in $s : 1 - s$. Suppose that $APQR$ is a parallelogram. Show that
    (a) $\overleftrightarrow{AR} \cap \overleftrightarrow{BC} \neq \varnothing \Longleftrightarrow s \neq t$, and that
    (b) if $\overleftrightarrow{AR} \cap \overleftrightarrow{BC} = \{D\}$ then $D = B + (C - B)\dfrac{s}{s - t}$.

12. In $\triangle ABC$, the medians $\overline{BM}$ and $\overline{AN}$ intersect at P, R is the midpoint of $\overline{AP}$, and Q is the midpoint of $\overline{BP}$. Prove that $MNQR$ is a parallelogram.

Part D

1. (a) Show that the midpoints of a pair of bases of a trapezoid and the intersection of its diagonals are collinear.
   (b) Show that the ratio in which the interval between the midpoints of two bases of a trapezoid is divided by the intersection of the diagonals is the same as the ratio of the bases.

2. In trapezoid $PQRS$, $\overline{TU}$ is parallel to the bases, $\overline{PQ}$ and $\overline{RS}$, and contains the point M at which the diagonals intersect. Show that
   (a) M is the midpoint of $\overline{TU}$,
   (b) U divides the side $\overline{PS}$ in the ratio of the bases, and
   (c) $(T - U) : (R - S) + (T - U) : (Q - P) = 2$.



3. Show that a quadrilateral whose diagonals intersect at a point which divides them in the same ratio is a trapezoid. What is the ratio of the bases of this trapezoid?

4. Show that the lines containing two nonparallel sides of a trapezoid intersect at a point which is collinear with the midpoints of the bases and the intersection of the diagonals of the trapezoid.

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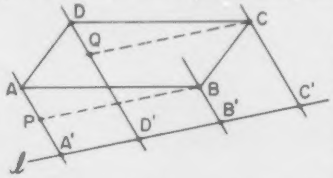
Answers for Part C [cont.]

6. Let R be the midpoint of $\overline{BD}$. Then $R - N = (B - A)/2 = (C - D)/2 = M - R$ and, so, R is the midpoint of $\overline{MN}$. Also, $R - D = B - R$ and, so, $(R - Q) + (Q - D) = (B - P) + (P - R)$. But, by hypothesis, $(Q - D):(B - D) = (P - B):(D - B)$ and, so, $Q - D = B - P$. Hence, $R - Q = P - R$ and, so, R is the midpoint of $\overline{PQ}$. It follows by Theorem 8-17 of Part A on page 356 that MPNQ is a parallelogram if it is a quadrilateral. It is easy to show that no three of M, P, N, and Q are collinear.

7. Since $\overline{AB} \parallel \overline{CD}$ and $\overline{AC}$ and $\overline{BD}$ intersect at a single point — say, M — ABCD is a quadrilateral and $\overline{AB}$ and $\overline{CD}$ are noncollinear parallel intervals. By Theorem 8-6(b), since M is the midpoint of $\overline{BD}$, it is also the midpoint of $\overline{AC}$. So, by Theorem 8-17, ABCD is a parallelogram.

8.



Let $P = A' + (B - B')$ and $Q = D' + (C - C')$. Then $Q - C = (D' - C) + (C - C') = D' - C'$ and $P - B = (A' - B) + (B - B') = A' - B'$. So, our problem is to show that $Q - C = P - B$. From what we have already done it follows that if $P \neq B$ and $Q \neq C$ then $A' \neq B'$ and $D' \neq C'$. Furthermore, in this case $\overleftrightarrow{PB} \parallel \overleftrightarrow{A'B'} = \ell = \overleftrightarrow{D'C'} \parallel \overleftrightarrow{QC}$. Also, $P - A = (A' - A) + (B - B')$ and $Q - D = (D' - D) + (C - C')$. From this it follows that if $P \neq A$ and $Q \neq D$ then the direction of $\overleftrightarrow{AP}$ and $\overleftrightarrow{DQ}$ is the common direction of those of $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$ and $\overleftrightarrow{DD'}$ which are lines. Finally, $\overleftrightarrow{BA} \parallel \overleftrightarrow{CD}$. Since $A - B = D - C$ it now follows by Theorem 8-11 that if $P \notin \overleftrightarrow{AB}$ and $Q \notin \overleftrightarrow{DC}$ then $Q - C = P - B$.

Suppose, now that $P \neq B$ and $Q \neq C$. It follows as before that $\overleftrightarrow{PB} \parallel \overleftrightarrow{QC}$ and, so, if either $P \in \overleftrightarrow{AB}$ or $Q \in \overleftrightarrow{DC}$ then it is the case that $P \in \overleftrightarrow{AB}$ and $Q \in \overleftrightarrow{DC}$. Assuming that $P \in \overleftrightarrow{AB}$ [and $P \neq B$] it follows, as before, that $\overleftrightarrow{AB} = \overleftrightarrow{PB} \parallel \ell$. Suppose that $A \notin \ell$. It follows that $B \notin \ell$ and so that $A \neq A'$ and $B \neq B'$. In this case, $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'}$ and $ABB'A'$ is a parallelogram. Hence, $A' - B' = A - B$. Suppose, on the other hand, that $A \in \ell$, in which case $\overleftrightarrow{AB} = \ell$. If $A \neq A'$ then either $C = C'$ or $\overleftrightarrow{CC'} \parallel \overleftrightarrow{AA'} = \ell$. In either case, since $C' \in \ell$, $C \in \ell$. Since, however, $\{A, B, C\}$ is noncollinear, $C \notin \overleftrightarrow{AB} = \ell$. So, $A = A'$. Similarly $B = B'$, and, again, $A' - B' = A - B$. So, in any case, if $P \in \overleftrightarrow{AB}$ and $P \neq B$ then $A' - B' = A - B$. Similarly, if $Q \in \overleftrightarrow{DC}$ and $Q \neq C$ then $D' - C' = D - C$. Since $D - C = A - B$ it follows that, for $P \neq B$ and $Q \neq C$, $D' - C' = A' - B'$ if both $P \in \overleftrightarrow{AB}$ and $Q \in \overleftrightarrow{DC}$. However, as we have seen, this condition is satisfied if either $P \in \overleftrightarrow{AB}$ or $Q \in \overleftrightarrow{DC}$.

The only case which remains to be considered is that in which $P = B$ or $Q = C$. Note, first that if both $P = B$ and $Q = C$ then $A' = B'$ and $D' = C'$ and, so $D' - C' = \vec{0} = A' - B'$. All that remains, then, is to show that if either $P = B$ or $Q = C$ then both $P = B$ and $Q = C$. Suppose, then, that $P = B$. As noted, it follows [since $P - A' = B - B'$] that $A' = B'$. It follows, since

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Answers for Part C [cont.]

$A \neq B$, that either $A \neq A'$ or $B \neq A'$. Suppose that $A \neq A'$. If $B = B'$ then $B = B' = A' \in \overleftrightarrow{AA'}$ and, since $A \in \overleftrightarrow{AA'}$ and $B \neq A$, $\overleftrightarrow{AA'} = \overleftrightarrow{AB}$. If, on the other hand, $B \neq B'$ then $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'}$ and, since $A' = B'$, $\overleftrightarrow{AA'} = \overleftrightarrow{BB'} = \overleftrightarrow{AB}$. Hence, if $A \neq A'$ then $\overleftrightarrow{AA'} = \overleftrightarrow{AB}$. Similarly, if $B \neq B'$ then $\overleftrightarrow{BB'} = \overleftrightarrow{AB}$. It follows that, in any case, if $P = B$ then either $C = C'$ or $\overleftrightarrow{CC'}$ is the line through C parallel to AB. In either case, C' is the intersection of $\overleftrightarrow{CD}$ with ℓ. Similarly, D' is the intersection of $\overleftrightarrow{CD}$ with ℓ. Hence, if $P = B$ then $D' = C'$ and, since $Q = D' + (C - C')$, $Q = C$. Similarly, if $Q = C$ then $P = B$. Hence, if either $P = B$ or $Q = C$ then both $P = B$ and $Q = C$. As pointed out earlier, this completes the proof.

The treatment of the special cases in which $P \in \overleftrightarrow{AB}$ or $Q \in \overleftrightarrow{DC}$ are, as the preceding shows, quite messy. Students should, however, be able to handle the principle case — that in which $P \notin \overleftrightarrow{AB}$ and $Q \notin \overleftrightarrow{DC}$ — and should be encouraged to do so.

It is worth pointing out that, by Theorem 8-6(b), the result of Exercise 8 continues to hold if, instead of requiring that ABCD be a parallelogram, one merely requires that $D - C = A - B$. For, if $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$ are noncollinear, then ABCD is a parallelogram and Exercise 8 applies, while if $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$ are collinear then Theorem 8-6(b) applies [except in the case in which $A = B$ and $C = D$ — but, this case is trivial].

There is another way to solve Exercise 8 which at first may appear to be simpler than the solution given here. Unfortunately it runs into many difficulties. The idea of this solution is to let B'', say, be the intersection of $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CD}$ and let A'' be the intersection of $\overleftrightarrow{AA'}$ and $\overleftrightarrow{CD}$. Then, since $ABB''A''$ is a parallelogram, $A'' - B'' = A - B = D - C$ and, by Theorem 8-6(b), $(A' - B'):(D' - C') = (A'' - B''):(D - C) = 1$. The difficulty is in making sure of the points A'' and B''. In plane geometry we would be able to argue that a line — say, $\overleftrightarrow{BB'}$ — which intersects one of two parallel lines must also intersect the other. But, to use this fact we would, here, have to show that $\overleftrightarrow{BB'}$ is in the plane of ABCD. This is not too easy to do, even if we know about planes. In addition to this fundamental difficulty, we would still have to deal with the special cases in which $B = B'$ or $A = A'$. Taken all in all, there seems no very simple way of establishing the result of Exercise 8. The simple arguments all require stronger assumptions.

9. Since $S - P = (C - P)/2$ and $P - R = (P - A)/2$ it follows that $S - R = (C - A)/2 = N - M$. Since, for example, $M \notin \overline{CA}$, $\{R, S, M\}$ is noncollinear. So, by Theorem 8-16 of Part A, MNSR is a parallelogram.

10. $P_1P_2P_3P_4$ is a parallelogram [unless it turns out that these points may be collinear!]. Using position vectors with respect to O,

$$\vec{p}_1 = (\vec{a} + \vec{b})/3,\ \vec{p}_2 = (\vec{b} + \vec{c})/3,$$
$$\vec{p}_3 = (\vec{c} + \vec{d})/3, \text{ and } \vec{p}_4 = (\vec{d} + \vec{a})/3.$$

So, $P_1 - P_4 = \vec{p}_1 - \vec{p}_4 = (\vec{b} - \vec{d})/3$ and $P_2 - P_3 = \vec{p}_2 - \vec{p}_3 = (\vec{b} - \vec{d})/3$. Hence, $P_1 - P_4 = P_2 - P_3$. It follows that, unless $\{P_1, P_2, P_3\}$, say, is collinear, $P_1P_2P_3P_4$ is a parallelogram. We shall use Theorem 8-15 to check on the collinearity of $\{P_1, P_2, P_3\}$.

According to this theorem, if $\{P_1, P_2, P_3\}$ is collinear then there are numbers — say, a, b, and c — which are not all zero, such that $\vec{p}_1c + \vec{p}_2a + \vec{p}_3b = \vec{0}$ and $a + b + c = 0$. So, for such numbers,

$$(\vec{a} + \vec{b})c + (\vec{b} + \vec{c})a + (\vec{c} + \vec{d})b = \vec{0}$$
$$\vec{a}c + \vec{b}(c + a) + \vec{c}(a + b) + \vec{d}b = \vec{0}$$
$$\vec{a}c + \vec{b} \cdot -b + \vec{c} \cdot -c + \vec{d}b = \vec{0}$$
$$(\vec{a} - \vec{c})c + (\vec{d} - \vec{b})b = \vec{0}$$
$$(A - C)c + (D - B)b = \vec{0}$$

Now, the last equation implies that if not both b and c are 0, the diagonals $\overline{AC}$ and $\overline{DB}$ are parallel. Since both contain O, they would then be collinear. In this case, ABCD would not be a quadrilateral. So, $b = c = 0$ and, since $a + b + c = 0$, $a = 0$. So, by Theorem 8-15, $\{P_1, P_2, P_3\}$ is noncollinear.

11. (a) $P = A + (B - A)t$, $Q = A + (C - A)s$, $R = A + (Q - P) = A + (C - A)s + (A - B)t$. By Theorem 8-1, $\overleftrightarrow{AR} \cap \overleftrightarrow{BC} \neq \emptyset$ if and only if $B - A \in [R - A, B - C]$. So, from this [or directly, if the theorem is forgotten] $\overleftrightarrow{AR} \cap \overleftrightarrow{BC} \neq \emptyset$ if and only if there are numbers — say, p and q — such that

$$B - A = [(C - A)s + (A - B)t]p + (B - C)q$$
$$(B - A)(1 - ps + pt) = (C - B)(ps - q) \quad [C - A = (B - A) + (C - B)]$$

Since $\{A, B, C\}$ is noncollinear, p and q satisfy this equation if and only if

$$p(s - t) = 1 \text{ and } ps = q.$$

For there to exist such numbers it is evidently necessary that $s \neq t$. On the other hand, if $s \neq t$ then $p = 1/(s - t)$ and $q = s/(s - t)$.

So, $\overleftrightarrow{AR} \cap \overleftrightarrow{BC} \neq \emptyset$ if and only if $s \neq t$.

(b) If $\overleftrightarrow{AR} \cap \overleftrightarrow{BC} = \{D\}$ then, $D = B + (C - B)a = A + (R - A)b$ and

$$(B - A) = (R - A)b + (B - C)a.$$

From part (a) it is evident that $b = p = 1/(s - t)$ and $a = q = s/(s - t)$. So, $D = B + (C - B)\frac{s}{s - t}$.

*

The preceding exercise can be modified by assuming either that ARPQ is a parallelogram or that APRQ is a parallelogram. The exercise parts need no change in the former case. In the latter, merely replace 't' by '−t', beginning with the expression for R.

12. Since P is the 2/3-point on either of the medians, it is the midpoint of $\overline{RN}$ and $\overline{MQ}$. Since P is the only point common to $\overline{AN}$ and $\overline{MB}$, M, N, Q, and R are vertices of a quadrilateral. So, MNQR is a parallelogram.

The exercises of Part D are, for the most part, restatements of results obtained in Part E on pages 330 and 331. Some students may refer to these, or to results of other exercises in answer to those of Part D. We shall, however, give alternative answers.

Answers for Part D

1. (a) Suppose that $PQRS$ is a trapezoid with $R - S = (Q - P)r$. By Theorem 8-18(a), $r > 0$, and by Theorem 8-19(a) the intersection of the diagonals is $P + (R - P)/(1 + r)$. The midpoints of $\overline{PQ}$ and $\overline{RS}$ are $P + (Q - P)/2$ and $P + (R - P) + (S - R)/2$, respectively. The latter is $P + (R - P) + (P - Q)(r/2)$. To show that the three points are collinear it is sufficient to show that

$$\left((Q - P)\frac{1}{2} - (R - P)\frac{1}{1 + r},\ (R - P)(1 - \frac{1}{1 + r}) + (P - Q)\frac{r}{2}\right)$$

is linearly dependent. This is obviously the case — the second term is the product of the first by $-r$.

(b) Since the intersection of the diagonals divides each of them in the ratio of the bases, it also divides any interval through it, whose end points are on the bases, in the same ratio. More directly, if M is the midpoint of $\overline{PQ}$, N is the midpoint of $\overline{RS}$, and O is the intersection of the diagonals, then $O - N = (S - N) + (O - S)$ and $M - O = (Q - O) + (M - Q)$. Since $(S - N):(M - Q) = (S - R):(P - Q) = r$ and $(O - S):(Q - O) = r$ it follows that $(O - N):(M - O) = r$.

2. (a) Since $\overline{UT}$ is parallel to $\overline{PQ}$ and $\overline{SR}$, U and T divide the intervals from S to P and from R to Q in the same ratio. So, $(P - U):(P - S) = (Q - T):(Q - R)$. Also, $(T - M):(R - S) = (Q - T):(Q - R)$ and $(M - U):(R - S) = (P - U):(P - S)$. It follows that $(T - M):(R - S) = (M - U):(R - S)$ and, so, that $T - M = M - U$. Hence, M is the midpoint of $\overline{TU}$.

(b) U divides the interval from S to P in the same ratio as M divides the interval from S to Q [Theorem 8-6(b)]. Since the latter ratio is the ratio of the bases, so is the former.

(c)
$$\begin{aligned}(T - U):(R - S) &= (M - U):(R - S) + (T - M):(R - S)\\ &= (Q - T):(Q - R) + (P - U):(P - S)\\ &= \frac{1}{1 + r} + \frac{1}{1 + r}, \text{ where } r = (R - S):(Q - P)\end{aligned}$$

$$(T - U):(Q - P) = \frac{r}{1 + r} + \frac{r}{1 + r}$$

$$(T - U):(R - S) + (T - U):(Q - P) = 2$$

3. Suppose that, in quadrilateral $PQRS$, $\overline{PR} \cap \overline{QS} = \{O\}$, where $(O - S):(Q - O) = (O - R):(P - O)$. It follows by the corollary to Theorem 8-3 that $\overline{PQ} \parallel \overline{RS}$. Since $PQRS$ is a convex quadrilateral with two parallel sides it is a trapezoid and, by Theorem 8-19, the ratio $(R - S):(Q - P)$ of its bases is a ratio in which O divides the diagonals from R to Q and from S to P. Equivalently, $\overline{SR} : \overline{PQ} = \overline{SO} : \overline{OQ} = \overline{RO} : \overline{OP}$.

4. This follows by two applications of Exercise 3(ii) of Part D on page 330; or, see Exercise 4(a) of Part E on the same page.

Answers for Part D [cont.]

5. (a) $U = P + (S - P)\frac{t}{1 + t}$, and we wish to find numbers a and b such that

$$U + (Q - P)a = Q + (R - Q)b$$

$$P + (S - P)\frac{t}{1 + t} + (Q - P)a = Q + (R - Q)b$$

$$(P - Q)(1 - a) + (S - P)\frac{t}{1 + t} = (R - Q)b$$

$$(P - Q)(1 - (a + b)] + (S - R)b = [(P - Q) + (S - P) + (R - S)]b$$

$$(P - Q)(1 - (a + b)] + (S - R)b = (S - P)(b - \frac{t}{1 + t})$$

Since $\overline{RS} \parallel \overline{PQ}$, $(P - Q)[1 - (a + b)] + (S - R)b \in [P - Q]$. So, since $\{P, Q, S\}$ is noncollinear it follows that $b = \frac{t}{1 + t}$. So, $T = Q + (R - Q)\frac{t}{1 + t}$ and, consequently, divides the interval from Q to R in $t:1$.

(b) $T - U = (Q - P) + [(R - Q) - (S - P)]\frac{t}{1 + t}$. Since $(R - Q) - (S - P) = (R - S) - (Q - P)$, $T - U = (Q - P)[1 - \frac{t}{1 + t}] + (R - S)\frac{t}{1 + t} = (Q - P)\frac{1}{1 + t} + (R - S)\frac{t}{1 + t}$.

Sample Quiz

1. Given a parallelogram, which of the following are true?
   (a) The diagonals are the same length.
   (b) The diagonals divide each other in the same ratio.
   (c) The diagonals bisect each other.
   (d) Both pairs of opposite sides are parallel.
   (e) Both pairs of opposite sides have the same length.
   Answer: ____________________

2. Given a trapezoid, which of the five statements in 1 are true?
   Answer: ____________________

Key to Sample Quiz

1. (b), (c), (d), (e) are true.
2. (b) is true.

5. Suppose that $PQRS$ is a trapezoid with bases $\overline{PQ}$ and $\overline{RS}$, and that U divides the interval from P to S in $t : 1$. Show that
   (a) the line $\overleftrightarrow{U[Q - P]}$ intersects $\overleftrightarrow{QR}$ at a point T which divides the interval from Q to R in $t : 1$, and
   (b) $T - U = (Q - P)\frac{1}{1 + t} + (R - S)\frac{t}{1 + t}$.

8.08 Two Famous Theorems

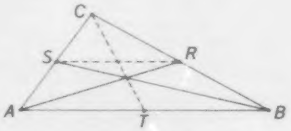


Fig. 8–17

By Theorem 8–9, if R and S are the r-point and the s-point, from C, of $\overline{BC}$ and $\overline{AC}$, respectively, then $\overline{RS} \parallel \overline{AB}$ if and only if $r = s$. Also, if $\overline{RS} \parallel \overline{AB}$ then $\overline{AR}$ and $\overline{BS}$ intersect on the median from C, and this median contains the median of $\triangle SRC$ from C. In this section we shall investigate some of the things which happen when $r \neq s$. In this case, as we know, $\overline{AR}$ and $\overline{BS}$ intersect at a point which is not on the median from C and $\overline{RS} \nparallel \overline{AB}$. As suggested in the figure, this leads to the consideration of two new points, T and T', on $\overleftrightarrow{AB}$. The relation of these points to A and B is worth investigating, and we shall look for answers to the questions:

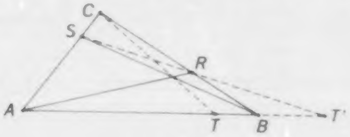


Fig. 8–18

> Where must T' be in relation to A and B in order that $\{R, S, T'\}$ be collinear?
>
> Where must T be in relation to A and B in order that $\overleftrightarrow{AR}$, $\overleftrightarrow{BS}$, and $\overleftrightarrow{CT}$ be concurrent [that is, have a point in common]?

Before answering these questions it will be helpful to consider some examples.

Exercises

Part A

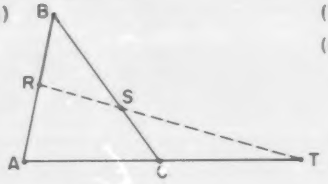
1. Given $\triangle ABC$, and points R, S, and T such that $R = A + (B - A)\frac{1}{2}$, $S = B + (C - B)\frac{2}{3}$, and $T = C + (A - C) \cdot -1$.
   (a) Draw an appropriate picture for these conditions.
   (b) Compute the ratios $(R - A) : (B - R)$, $(S - B) : (C - S)$, and $(T - C) : (A - T)$. What is the product of these ratios?
   (c) Make a conjecture about the points R, S, and T.

The "two famous theorems" are Menelaus' Theorem, page 363, and Ceva's Theorem, Exercise 1 of Part H on page 367. A proof of the latter has been indicated in the discussion of Part A on page 347.

Parts A and B can be used as a seat activity during a class period. We recommend that you permit discussion among individuals during this activity. Part C is a reasonable homework set. Because the idea of sensed distance is introduced, we recommend Part D for class discussion and solution. Parts E and F provide exercises in which students can be permitted to work with a partner. The main reason for such an arrangement is the complexity of the algebra involved in these exercises. It is not uncommon for a student working alone to make an error early in an exercise that causes the rest of his work to be in error. Parts G and H present some interesting applications that can be used as either a class activity or a homework assignment.

Answers for Part A

1. (a)



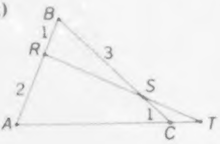
(b) 1; 2; −1/2; −1

(c) Collinear.

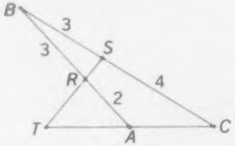
2. Given $\triangle DEF$ and points L and M such that $L = D + (E - D)\frac{3}{4}$ and $M = E + (F - E)\frac{2}{3}$.
   - **(a)** Draw an appropriate picture for these conditions.
   - **(b)** Show that $\overline{LM}$ is not parallel to $\overline{DF}$.
   - **(c)** Locate N, the point of intersection of $\overleftrightarrow{LM}$ and $\overleftrightarrow{DF}$. Compute the ratio $(N - F) : (D - N)$.
   - **(d)** Compute the ratios $(L - D) : (E - L)$ and $(M - E) : (F - M)$. What is the product of these ratios with the ratio computed in (c)?

3. In each of the following, you are given certain indicated ratios and that R, S, and T are collinear. Compute the ratio $(T - C) : (A - T)$.

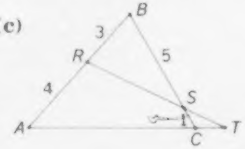
(a)



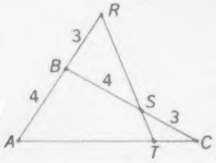
(b)



(c)



(d)



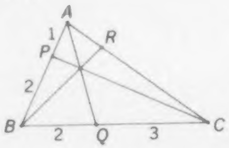
Part B

1. Given $\triangle ABC$ and points P, Q, and R such that $P = A + (B - A)3$, $Q = B + (C - B) \cdot -\frac{1}{2}$, and $R = C + (A - C)\frac{2}{3}$.
   - **(a)** Draw a careful picture of this situation.
   - **(b)** Compute the ratios $(P - A) : (B - P)$, $(Q - B) : (C - Q)$, and $(R - C) : (A - R)$. What is the product of these ratios?
   - **(c)** What can you say about the lines $\overleftrightarrow{AQ}$, $\overleftrightarrow{BR}$, and $\overleftrightarrow{CP}$?

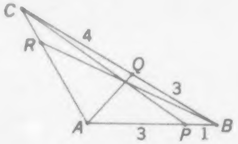
2. Given $\triangle DEF$ and points K, L, and M such that $K = D + (E - D)\frac{4}{5}$, $L = E + (F - E)\frac{1}{3}$, and $M = F + (D - F)\frac{1}{3}$.
   - **(a)** Draw a careful picture of this situation.
   - **(b)** What seems to be the case about the lines $\overleftrightarrow{DL}$, $\overleftrightarrow{EM}$, and $\overleftrightarrow{FK}$?
   - **(c)** Compute the ratios $(K - D) : (E - K)$, $(L - E) : (F - L)$, and $(M - F) : (D - M)$. What is the product of these ratios?

3. In each of the following, you are given certain indicated ratios and that the lines $\overleftrightarrow{AQ}$, $\overleftrightarrow{BR}$, and $\overleftrightarrow{CP}$ are concurrent. [Concurrent lines are lines which pass through the same point.] Compute the ratio $(R - C) : (A - R)$, and determine r such that $R = C + (A - C)r$.

(a)

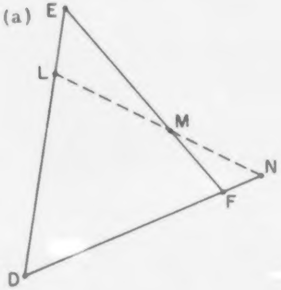


(b)



Answers for Part A [cont.]

2. (a)



(b) $M - L = (E - D)\frac{1}{4} + (F - E)\frac{2}{3}$.

So, $M - L$ is not a multiple of $F - D$ so that $\overleftrightarrow{LM} \not\parallel \overleftrightarrow{DF}$.

(c) $N \in \overleftrightarrow{LM} \cap \overleftrightarrow{DF}$ if and only if there are numbers a and b such that $N = L + (M - L)a = D + (F - D)b$. This last gives us, in turn:

$$(L - D) + (M - L)a + (D - F)b = \vec{0}$$

$$(E - D)(\tfrac{3}{4} + \tfrac{1}{4}a) + (F - E)\tfrac{2}{3}a + (D - F)b = \vec{0}$$

So, $\frac{3}{4} + \frac{1}{4}a = \frac{2}{3}a = b$. Hence,

$a = \frac{9}{5}$ and $b = \frac{6}{5}$. Therefore,

$N = D + (F - D)\frac{6}{5}$, so that

$(N - F):(D - N) = -\frac{1}{6}$.

(d) 3; 2; −1

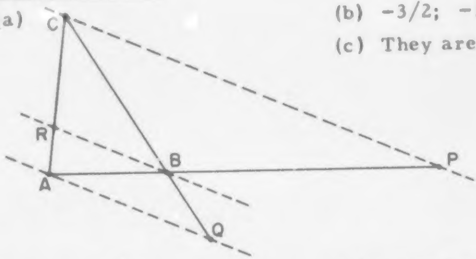
3. (a) −1/6 (b) −2 (c) −3/20 (d) 9/28

*

From the results of Exercises 1 and 2, students may suspect that, R, S, and T being collinear, $[(R - A):(B - R)][(S - B):(B - S)][(T - C):(A - T)] = -1$. The parts of Exercise 3 should encourage this suspicion.

Answers for Part B

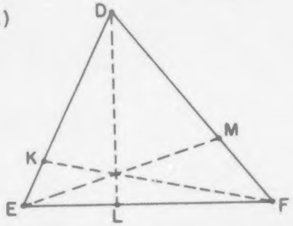
1. (a)



(b) −3/2; −1/3; 2; 1

(c) They are parallel.

2. (a)

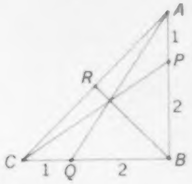


(b) They are concurrent.

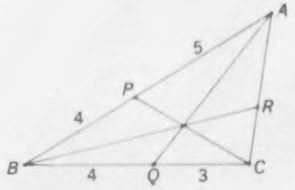
(c) 4; 1/2; 1/2; 1

3. (a) 3; 3/4 (b) 4/9; 4/13

(c)

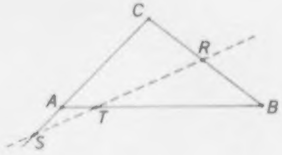


(d)



Part C

Suppose that, in $\triangle ABC$, R divides the side from B to C in $r : 1$, S divides the side from C to A in $s : 1$, T divides the side from A to B in $t : 1$. We wish to find conditions on r, s, and t which will ensure the collinearity of $\{R, S, T\}$.



1. (a) By hypothesis, $R = B + (C - B)\frac{r}{r + 1}$. Write similar expressions for 'S' and 'T'.
   (b) Using the results of part (a), show that $S - R$ is a linear combination of $C - B$ and $A - C$ and that $T - S$ is a linear combination of $A - C$ and $B - A$.
2. Show that, for any real numbers a and b,

$$(S - R)a + (T - S)b = \vec{0} \Longleftrightarrow \frac{a}{1 + r} = \frac{as + b}{1 + s} = \frac{bt}{1 + t}.$$

[*Hint*: Use the results obtained in Exercise 1(b) and apply Theorem 6-12.]

3. (a) Show that

$$\frac{a}{1 + r} = \frac{as + b}{1 + s} \Longleftrightarrow \frac{1 - rs}{1 + r}a = b.$$

   (b) Conclude that

$$(S - R)a + (T - S)b = \vec{0} \Longleftrightarrow \left(\frac{1 - rs}{1 + r}a = b \text{ and } a = \frac{(1 - rs)t}{1 + t}a\right).$$

4. Show that $\{R, S, T\}$ is collinear if and only if $rst = -1$.
5. Show that if $\{R, S, T\}$ is collinear then S divides the interval from R to T in $rs - 1 : 1 + r$.

Part D

Although we cannot, as yet, introduce the notion of distance between points of $\mathscr{E}$ we can, if we consider only points of a given line l, introduce a notion of distance between points of l. To do so, choose a

Answers for Part B [cont.]

(c) 1; 1/2 (d) 3/5; 3/8

*

Just as the results of Part A illustrate, and suggest, Menelaus' Theorem, those of Part B anticipate Ceva's Theorem.

Part C gives a proof of Menelaus' Theorem. Part D introduces a notion of sensed distances which allows the theorem to be put in more conventional form.

Answers for Part C

1. (a) $R = B + (C - B)\frac{r}{r + 1}$, $S = C + (A - C)\frac{s}{s + 1}$, $T = A + (B - A)\frac{t}{t + 1}$
   [Students should note cyclic symmetry and use it as a time-saving device in part (b) and in later exercises in this section.]
   (b) $S - R = (C - B) + (A - C)\frac{s}{s + 1} - (C - B)\frac{r}{r + 1} = (C - B)\frac{1}{r + 1}$
   $+ (A - C)\frac{s}{s + 1}$
   $T - S = (A - C)\frac{1}{s + 1} + (B - A)\frac{t}{t + 1}$
2. By Exercise 1(b), $(S - R)a + (T - S)b = (C - B)\frac{a}{r + 1} + (A - C)(\frac{as}{s + 1} + \frac{b}{s + 1}) + (B - A)\frac{bt}{t + 1}$. Since $\{A, B, C\}$ is noncollinear and $(C - B) + (A - C) + (B - A) = \vec{0}$ it follows from Theorem 6-12 that

$$(S - R)a + (T - S)b = \vec{0} \Longleftrightarrow \frac{a}{1 + r} = \frac{as + b}{1 + s} = \frac{bt}{1 + t}.$$

3. (a) [Trivial algebra; note that since $B \neq C$, $r \neq -1$ and since $C \neq A$, $s \neq -1$.]
   (b) [By part (a) and Exercise 2.]
4. $\{R, S, T\}$ is collinear if and only if '$(S - R)a + (T - S)b = \vec{0}$' is satisfied for values of 'a' and 'b' which are not both zero. By the first equation on the right side of Exercise 3(b), a solution for which $a = 0$ has $b = 0$, also. By the second equation, a solution for which $a \neq 0$ exists only if $(1 - rs)t = 1 + t$ — that is, only if $rst = -1$. Finally, if this condition is satisfied, we may evidently choose a arbitrarily and compute b from the first equation.
5. By Exercise 3(b), $(S - R):(T - S) = -\frac{b}{a} = (rs - 1):(1 + r)$.

*

The introduction of sensed distances [usually called 'directed distances'] in Part D enables us to state Menelaus' Theorem in its usual form. [Theorem 8-21; Menelaus' Theorem is the only if-part.] Ordinarily, there is available a notion of distance for all points in the plane of $\triangle ABC$; here, we must use three quite independent notions of distance — one on each of the lines $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, and $\overleftrightarrow{AB}$. But, as is pointed out in Exercise 2, this does no harm. Note that in Theorem 8-21, R, for example, may be any point of $\overleftrightarrow{BC}$. This is a slight extension of the situation dealt with in Part C, when $B \neq R \neq C$. This extension is, of course, of little importance, but it does make for a simpler hypothesis in the theorem. The extension is justified [in the case of R] in Exercises 3 and 4. Obviously, then, the restrictions of Part C on S and T may also be dropped.

zero-point, O, and a unit-point, U, on l, such that $U \neq O$. With respect to these we can, for $\{P, Q\} \subseteq l$, define the *sensed distance* from P to Q [for short, PQ] by:

$$PQ = \begin{cases} (Q - P) : (U - O) & [P \neq Q] \\ 0 & [P = Q] \end{cases}$$

Note that the sensed distance PQ depends on the choice of O and U. However, if $\{P, Q, R, S\} \subseteq l$ and $P \neq Q$ and $R \neq S$ then

$$PQ/RS = \frac{(Q - P) : (U - O)}{(S - R) : (U - O)} = (Q - P) : (S - R)$$

and, so, PQ/RS is independent of the choice of O and U.

An advantage of the notion of sensed distances can be seen in the following statement of the results of Exercise 4 of Part C.

> **Theorem 8-21** [Menelaus' Theorem and Converse]
> If, in $\triangle ABC$, $R \in \overleftrightarrow{BC}$, $S \in \overleftrightarrow{CA}$, and $T \in \overleftrightarrow{AB}$
> then $\{R, S, T\}$ is collinear
> if and only if
> (*) $\quad BR \cdot CS \cdot AT = -(RC \cdot SA \cdot TB)$.

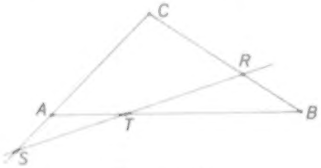


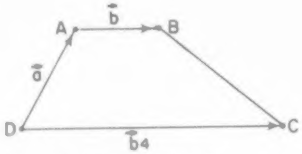
Fig. 8-19

1. Derive Menelaus' Theorem, in the case in which $R \notin \{B, C\}$, $S \notin \{C, A\}$, and $T \notin \{A, B\}$, from the result of Exercise 4 of Part C. [*Hint*: $r = (R - B) : (C - R) = BR/RC$.]
2. Given the points B, C, and R, the translations $R - B$ and $C - R$ are completely determined and so is the real number r which is their ratio. On the other hand, to determine the real numbers BR and RC we must not only know B, C, and R but, also, have chosen a sense on $\overleftrightarrow{BC}$ and a "unit interval" in [or parallel to] $\overleftrightarrow{BC}$. Explain why it is that whether (*) holds or not is independent of these two choices.
3. (a) Show that in case $R = B$ it follows that $\{R, S, T\}$ is collinear if and only if $T = B$ or $S = A$.
   (b) Show that in case $R = B$ it follows that $RC \cdot SA \cdot TB = 0$ if and only if $T = B$ or $S = A$.
   (c) Conclude that Menelaus' Theorem holds in case $R = B$.
4. As in Exercise 2, show that Menelaus' Theorem holds in case $R = C$.

Answers for Part D

1. In the case in question it follows by Exercise 4 that $\{R, S, T\}$ is collinear if and only if $rst = -1$, where $r = (R - B):(C - R)$, $s = (S - C):(A - S)$, and $t = (T - A):(B - T)$. Using sensed distances on $\overleftrightarrow{BC}$, on $\overleftrightarrow{CA}$, and on $\overleftrightarrow{AB}$, $r = BR/RC$, $s = CS/SA$, and $t = AT/TB$. So, $rst = -1$ if and only if $(BR/RC)(CS/SA)(AT/TB) = -1$, and this is the case if and only if $BR \cdot CS \cdot AT = -(RC \cdot SA \cdot TB)$.
2. Whatever nonzero translation $U - O$ we use as "unit translation" in the direction of $\overleftrightarrow{BC}$, any other is a multiple of this by some nonzero real number. So, sensed distances computed with respect to such another unit translation will be equal to the quotients of those computed with respect to $U - O$ by the same real number. So, the result of a different choice of unit translation for $\overleftrightarrow{BC}$ will be to divide each of BR and RC by the same "conversion factor". This has no effect on the equality of the products referred to in (☆).
3. (a) Suppose that $R = B$. If $T = B$ then $\{R, S, T\} \subseteq \overleftrightarrow{BS}$ and, $T \in \overleftrightarrow{AB}$, if $S = A$ then $\{R, S, T\} \subseteq \overleftrightarrow{AB}$. So, in either case, $\{R, S, T\}$ is collinear. On the other hand [still assuming that $R = B$], if $T \neq B$ then $\overleftrightarrow{RT} = \overleftrightarrow{AB}$. So, if $\{R, S, T\}$ is collinear, $S \in \overleftrightarrow{AC} \cap \overleftrightarrow{RT} = \overleftrightarrow{AC} \cap \overleftrightarrow{AB} = \{A\}$. Hence, if $\{R, S, T\}$ is collinear then $T = B$ or $S = A$.
   (b), (c) In case $R = B$, $BR = 0$ and Theorem 8-21 requires that $\{R, S, T\}$ be collinear if and only if $RC \cdot SA \cdot TB = 0$. Since $R = B \neq C$, $RC \cdot SA \cdot TB = 0$ if and only if $SA = 0$ or $TB = 0$ — that is, if and only if $S = A$ or $T = B$. But, for $R = B$, we know from part (a) that $\{R, S, T\}$ is collinear if and only if $S = A$ or $T = B$.
4. [Show, as in Exercise 3(a) that, in case $R = C$, $\{R, S, T\}$ is collinear if and only if $S = C$ or $T = A$. Then, proceed as in parts (b) and (c).]

Sample Quiz



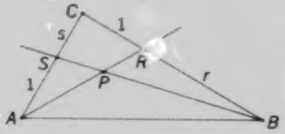
Assume that ABCD is a trapezoid, that $A - D = \vec{a}$, $B - A = \vec{b}$, and $C - D = \vec{b}4$, as shown at the right.

1. What is the ratio $B - A : C - D$?
2. Assume that $\overline{AC}$ and $\overline{BD}$ intersect in the point P. What is $P - A : C - P$?
3. Locate the point E such that AECD is a parallelogram. Express $E - D$ as a linear combination of $\vec{a}$ and $\vec{b}$.
4. What is the ratio $E - B : C - D$?
5. Assume that $\overline{BC}$ and $\overline{DE}$ intersect in the point F. What is $\overline{BF} : \overline{FC}$? What is $\overline{DF} : \overline{DE}$?
6. Assume that $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ intersect in the point Q. What is $\overline{QA} : \overline{AD}$? What is $\overline{QB} : \overline{QC}$?

Key to Sample Quiz

1. 1/4
2. 1/4
3. $\vec{a} + \vec{b}4$
4. 3/4
5. 3/4; 4/7
6. 1/3; 1/4

Part E



Suppose that, in $\triangle ABC$, R divides the interval from B to C in $r : 1$ and S divides the interval from C to A in $s : 1$. [So, neither r nor s is 0, 1, or -1.]

1. (a) Show that $R - A = (B - A) + (C - B)\frac{r}{1 + r}$.

   (b) Complete: $S - B = (C \ldots$

2. (a) To determine the conditions under which $\overleftrightarrow{AR}$ and $\overleftrightarrow{BS}$ are parallel, show that

$$(R - A)a + (S - B)b = \vec{0} \longleftrightarrow a = \frac{ar}{1 + r} + b = \frac{bs}{1 + s}.$$

   [*Hint*: Use Exercise 1 and Theorem 6–12.]

   (b) Show that $\overleftrightarrow{AR} \parallel \overleftrightarrow{BS} \longleftrightarrow 1 + r + rs = 0$. [*Hint*: The right side of the biconditional sentence in part (a) is equivalent to: $\frac{1}{1 + r}a = b$ and $a = \frac{s}{1 + s}b$.]

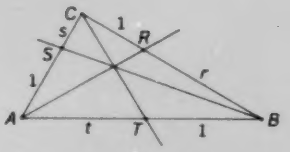
3. (a) To determine the conditions under which $\overleftrightarrow{AR}$ and $\overleftrightarrow{BS}$ intersect, show that

$$A + (R - A)a = B + (S - B)b \longleftrightarrow 1 - a = b - \frac{ar}{1 + r} = \frac{bs}{1 + s}.$$

   (b) Show that if $\overleftrightarrow{AR} \nparallel \overleftrightarrow{BS}$ then $\overleftrightarrow{AR} \cap \overleftrightarrow{BS} = \{P\}$ where

$$P = A + (R - A)\cdot\frac{1 + r}{1 + r + rs} = B + (S - B)\cdot\frac{r(1 + s)}{1 + r + rs}.$$

Part F



Suppose that, in $\triangle ABC$, R divides the side from B to C in $r : 1$, S divides the side from C to A in $s : 1$, T divides the side from A to B in $t : 1$.

1. It follows from Exercise 2(b) of Part E that

$$\overleftrightarrow{AR} \parallel \overleftrightarrow{BS} \parallel \overleftrightarrow{CT} \longleftrightarrow (1 + r + rs = 0 \text{ and } 1 + s + st = 0).$$

   Explain.

2. (a) Show that if $1 + r + rs = 0$ and $1 + s + st = 0$ then $rst = 1$ and $1 + t + tr = 0$. [*Hint*: To show that $rst = 1$, multiply on both sides of '$1 + s + st = 0$' with 'r'. Having shown that $rst = 1$, show that $1 + t + tr = 0$ by performing a similar maneuver with '$1 + r + rs = 0$'.]

Parts E and F deal with the proof of Theorem 8-22, of which Ceva's Theorem is the only if-part, with R, S, and T restricted to be on the sides of $\triangle ABC$.

Answers for Part E

1. (a) By hypothesis [and definition], $R = B + (C - B)\frac{r}{1 + r}$. Since $(B + \vec{a}) - A = (B - A) + \vec{a}$, the desired result follows.

   (b) $S - B = (C - B) + (A - C)\frac{s}{1 + s}$ [From (a), by cyclic symmetry of notation.]

2. (a) Since $R - A = (B - A) + (C - B)\frac{r}{1 + r}$ and $S - B = (C - B) + (A - C)\frac{s}{1 + s}$, $(R - A)a + (S - B)b = (B - A)a + (C - B)(\frac{ar}{1 + r} + b) + (A - C)\frac{bs}{1 + s}$. Since $(B - A, C - A)$ is linearly independent and $(B - A) + (C - B) + (A - C) = \vec{0}$ it follows by Theorem 6-12 that

$$(R - A)a + (S - B)b = \vec{0} \Longleftrightarrow a = \frac{ar}{1 + r} + b = \frac{bs}{1 + s}.$$

   (b) $\overleftrightarrow{AR} \parallel \overleftrightarrow{BS}$ if and only if $(R - A, S - B)$ is linearly dependent. By part (a), the latter is the case if and only if the equations:

$$(*) \qquad a = \frac{ar}{1 + r} + b = \frac{bs}{1 + s}$$

   are satisfied for values of 'a' and 'b' which are not both zero. Now,

$$a = \frac{ar}{1 + r} + b \Longleftrightarrow \frac{1}{1 + r}a = b, \text{ and}$$
$$a = \frac{bs}{1 + s} \Longleftrightarrow a = \frac{s}{1 + s}b.$$

   So, (*) is equivalent to:

$$(**) \qquad \frac{1}{1 + r}a = b \text{ and } a = \frac{s}{1 + s}b$$

   From the first equation, if $a = 0$ then $b = 0$. Hence, if not both a and b are zero, a must be nonzero. Since [substituting from the first equation into the second], (**) implies that

$$(***) \qquad a = \frac{s}{(1 + r)(1 + s)}a$$

   it follows that if (*) has solutions with a and b not both zero then $s = (1 + r)(1 + s)$ — that is, $1 + r + rs = 0$. Conversely, if this condition is satisfied, we may choose any value for 'a' and compute a corresponding value for 'b' from the first equation in (**). Since this equation and (***) are, together, equivalent to (**), this will yield a solution of (*). Consequently, $\overleftrightarrow{AR} \parallel \overleftrightarrow{BS} \Longleftrightarrow 1 + r + rs = 0$.

Answers for Part E [cont.]

3. (a) [The procedure is exactly like that used in Exercise 2(a). In fact, starting with the second line of the answer given above, all that is needed is to add an 'A - B' on both sides of the equation and replace 'b' by '-b'. Then, make the appropriate minor changes in what follows.]

(b) The sentence corresponding with (*) of Exercise 2(b) is:

$$(*) \qquad 1 - a = b - \frac{ar}{1 + r} = \frac{bs}{1 + s}$$

That corresponding with (**) is:

$$(**) \quad \frac{1}{1 + r}a + b = 1 \text{ and } a + \frac{s}{1 + s}b = 1$$

Multiplying on both sides of the first with '$\frac{s}{s + 1}$' and subtracing from the second yields, after slight simplification:

$$\frac{1 + r + rs}{1 + r}a = 1$$

This is solvable if and only if $1 + r + rs \neq 0$ — that is, if and only if $\overleftrightarrow{AR} \nparallel \overleftrightarrow{BS}$ — and, the first equation of (**), yields:

$$a = \frac{1 + r}{1 + r + rs} \text{ and } b = \frac{r(1 + s)}{1 + r + rs}$$

So, assuming that $\overleftrightarrow{AR} \nparallel \overleftrightarrow{BS}$, $\overleftrightarrow{AR} \cap \overleftrightarrow{BS} = \{P\}$, where

$$P = A + (R - A)\frac{1 + r}{1 + r + rs} = B + (S - B)\cdot\frac{r(1 + s)}{1 + r + rs}.$$

Answers for Part F

1. By Exercise 2(b) of Part E, $\overleftrightarrow{AR} \parallel \overleftrightarrow{BS}$ if and only if $1 + r + rs = 0$. By the symmetry of the notation we have introduced it follows that $\overleftrightarrow{BS} \parallel \overleftrightarrow{CT}$ if and only if $1 + s + st = 0$. Hence,

$$\overleftrightarrow{AR} \parallel \overleftrightarrow{BS} \parallel \overleftrightarrow{CT} \Longleftrightarrow (1 + r + rs = 0 \text{ and } 1 + s + st = 0)$$

[Since parallelism is transitive, students should realize that it follows that if $1 + r + rs = 0$ and $1 + s + st = 0$ then $1 + t + tr = 0$. This purely algebraic result is established algebraically in Exercise 2.]

2. (a) Suppose that $1 + r + rs = 0$ and $1 + s + st = 0$. From the second it follows that $r + rs + rst = 0$. Comparing this with the first assumption shows that $rst = 1$. From the first of the two assumptions, $t + rt + rst = 0$. So, since $rst = 1$, $1 + t + tr = 0$.

Answers for Part F [cont.]

2. (b) The equations are:

(1) $1 + r + rs = 0$

(2) $1 + s + st = 0$

(3) $1 + t + tr = 0$

(4) $rst = 1$

It has been shown that if (1) and (2) are satisfied then so are (3) and (4). By symmetry it obviously follows that if (2) and (3) are satisfied then so are (1) and (4), and that if (3) and (1) are satisfied then so are (2) and (4). It has also been shown that if (4) and (1) are satisfied then so is (3) and, by the immediately preceding remark, so is (2). By symmetry, if (4) and (2), or (4) and (3) are satisfied, so must be the others.

3. (a) The first equation has been obtained in Exercise 3(a) of Part E. The second is obtained from it by symmetry of notation.

(b) $\overleftrightarrow{AR} \cap \overleftrightarrow{BS} \cap \overleftrightarrow{CT} \neq \emptyset$ if and only if $P = Q$. Since $S \neq B$, $P = Q$ if and only if

$$(*) \qquad \frac{1 + s}{1 + s + st} = \frac{r(1 + s)}{1 + r + rs}$$

Since $1 + s \neq 0$ and, by the assumption of nonparallelism, $1 + s + st \neq 0 \neq 1 + r + rs$, (*) is equivalent to:

$$1 + r + rs = r(1 + s + st)$$

and, so, to '$1 = rst$'. Hence, if $\overleftrightarrow{AR} \nparallel \overleftrightarrow{BS} \nparallel \overleftrightarrow{CT}$ then

$$\overleftrightarrow{AR} \cap \overleftrightarrow{BS} \cap \overleftrightarrow{CT} \neq \emptyset \Longleftrightarrow rst = 1.$$

4. (a) From the result of Exercise 3(b) it follows (by symmetry) that if one of the lines $\overleftrightarrow{AR}$, $\overleftrightarrow{BS}$, and $\overleftrightarrow{CT}$ is not parallel to either of the other two then the lines are concurrent if and only if $rst = 1$. To *deny* that some one of the lines is not parallel to either of the others is to affirm that each of the lines is parallel to one of the others. Suppose this to be the case for lines ℓ, m, and n and that, say, $\ell \parallel m$. Since $n \parallel \ell$ or $n \parallel m$ it follows that $\ell \parallel m \parallel n$. So, what follows from Exercise 3(b) is that if it is not the case that $\overleftrightarrow{AR}$, $\overleftrightarrow{BS}$, and $\overleftrightarrow{CT}$ are parallel then $\overleftrightarrow{AR} \cap \overleftrightarrow{BS} \cap \overleftrightarrow{CT} \neq \emptyset \Longleftrightarrow rst = 1$. This conclusion is of the form 'not $p \Longrightarrow [q \Longleftrightarrow r]$'. Such a sentence has as one consequence the corresponding sentence of the form:

$$\text{not } p \Longrightarrow [r \Longrightarrow q]$$

$$r \Longrightarrow [\text{not } p \Longrightarrow q]$$

$$r \Longrightarrow (p \text{ or } q)$$

So, we have that if $rst = 1$ then $\overleftrightarrow{AR}$, $\overleftrightarrow{BS}$, and $\overleftrightarrow{CT}$ are parallel or concurrent. On the other hand, we have shown in Exercises

(b) Show that if r, s, and t satisfy any two of the equations in part (a) then they satisfy all four equations.

3. (a) It follows from Exercise 3(b) of Part E that if $\overleftrightarrow{AR} \nparallel \overleftrightarrow{BS} \nparallel \overleftrightarrow{CT}$ then $\overleftrightarrow{AR}$ and $\overleftrightarrow{BS}$ intersect in a point P and $\overleftrightarrow{BS}$ and $\overleftrightarrow{CT}$ intersect in a point Q, where

$$P = A + (R - A)\frac{1 + r}{1 + r + rs} = B + (S - B)\frac{r(1 + s)}{1 + r + rs}, \text{ and}$$
$$Q = B + (S - B)\frac{1 + s}{1 + s + st} = C + (T - C)\frac{s(1 + t)}{1 + s + st}.$$

Explain.

(b) Show that if $\overleftrightarrow{AR} \nparallel \overleftrightarrow{BS} \nparallel \overleftrightarrow{CT}$ then

$$\overleftrightarrow{AR} \cap \overleftrightarrow{BS} \cap \overleftrightarrow{CT} \neq \varnothing \longleftrightarrow rst = 1.$$

[Two or more lines which have a common point are said to be *concurrent.*]

4. (a) Show that

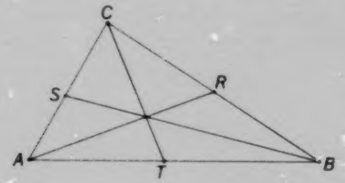
$\overleftrightarrow{AR}$, $\overleftrightarrow{BS}$, and $\overleftrightarrow{CT}$ are concurrent
or parallel if and only if $rst = 1$.

(b) Show that $\overleftrightarrow{AR}$, $\overleftrightarrow{BS}$, and $\overleftrightarrow{CT}$ are parallel if and only if

$$(rst = 1 \text{ and } (1 + r + rs = 0 \text{ or}$$
$$1 + s + st = 0 \text{ or } 1 + t + tr = 0)).$$

5. As in the case of Menelaus' Theorem, the results of Exercise 4(a) can be expressed conveniently in terms of sensed distances:

> Theorem 8-22
> If, in $\triangle ABC$, $R \in \overleftrightarrow{BC}$, $S \in \overleftrightarrow{CA}$, and $T \in \overleftrightarrow{AB}$
> then $\overleftrightarrow{AR}$, $\overleftrightarrow{BS}$, and $\overleftrightarrow{CT}$ are concurrent or parallel
> if and only if
> $BR \cdot CS \cdot AT = RC \cdot SA \cdot TB$.



(a) Derive Theorem 8-22, in the case in which $R \notin \{B, C\}$, $S \notin \{C, A\}$, and $T \notin \{A, B\}$, from the result of Exercise 4(a).

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1 and 2(a) that rst = 1 if the lines are parallel. Also, if the lines are concurrent they are not parallel [since {A, B, C} is noncollinear] and, by Exercise 3(b), rst = 1. Consequently, $\overleftrightarrow{AR}$, $\overleftrightarrow{BS}$, and $\overleftrightarrow{CT}$ are concurrent or parallel if and only if rst = 1.

(b) This follows at once from Exercises 1 and 2(b).

5. (a) [See answer for Exercise 1 of Part D on page 363.]

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(b) Show that the theorem holds in case $R = B$ and in case $R = C$. [*Hint*: The arguments are much like the corresponding ones for Menelaus' Theorem.]

6. (a) As in Exercise 3, if $\overleftrightarrow{AR} \cap \overleftrightarrow{BS} \cap \overleftrightarrow{CT} = \{P\}$ then

$$\begin{aligned} P &= A + (R - A)\frac{1 + r}{1 + r + rs} \\ &= B + (S - B)\frac{1 + s}{1 + s + st} \\ &= C + (T - C)\frac{1 + t}{1 + t + tr}. \end{aligned}$$

Explain.

(b) Show that if $\overleftrightarrow{AR}$, $\overleftrightarrow{BS}$, and $\overleftrightarrow{CT}$ are concurrent at P, where $P \notin \{A, B, C\}$, then

P divides the interval from A to R in $1 + r : rs$,
P divides the interval from B to S in $1 + s : st$, and
P divides the interval from C to T in $1 + t : tr$.

(c) Show that if $rst = 1$ then $1 + r : rs = t(1 + r) : 1$.

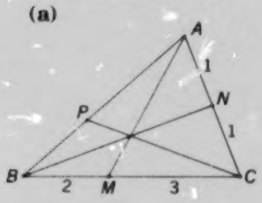
(d) Show that, under the assumption in part (b),

P divides the interval from A to R in $BC \cdot SA \cdot AT : k$,
P divides the interval from B to S in $CA \cdot TB \cdot BR : k$, and
P divides the interval from C to T in $AB \cdot RC \cdot CS : k$, where
$k = AT \cdot BR \cdot CS = SA \cdot TB \cdot RC$.

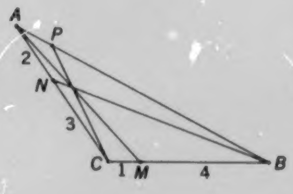
Part G

1. In each of the following, the figures are marked so as to indicate ratios in which two sides of a triangle are divided. You are to determine $AP : PB$.

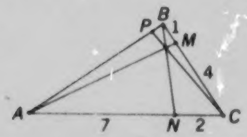
(a)



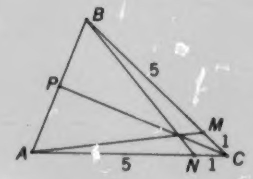
(b)



(c)



(d)



Answers for Part F [cont.]

5. (b) Suppose that R = B. In this case the lines are certainly concurrent if T = B or S = A. On the other hand, if T ≠ B [and R = B] and the lines are concurrent then $\overleftrightarrow{AR} = \overleftrightarrow{AB} = \overleftrightarrow{TB}$ and S must belong to $\overleftrightarrow{AB}$ — so, S = A. Hence, for R = B, the lines are concurrent if and only if T = B or S = A. Finally, if R = B then $\overleftrightarrow{CT}$ cannot be parallel to $\overleftrightarrow{AR}$. So, it merely weakens our result to say that, for R = B, the lines are concurrent or parallel if and only if T = B or S = A. [The remainder of the argument that the theorem holds if R = B is like that given in answer to Exercise 3(c) of Part D on page 363. That the theorem holds in case R = C is established by repeating the preceding with merely alphabetic variations.]

6. (a) The first two formulas are given in Exercise 3(a) [with 'Q' in place of 'P']. The third follows from the second — as the second from the first — by cyclic permutation.

(b) Since the u-point, from M, of $\overline{MN}$ divides the interval from M to N in u : 1 - u it is sufficient to note that

$$1 - \frac{1 + r}{1 + r + rs} = \frac{rs}{1 + r + rs}.$$

[The assumption is needed to ensure that neither r, s, nor t is zero.]

(c) Multiply both terms of the ratio by t.

(d) $$\begin{aligned} BC \cdot SA \cdot AT : AT \cdot BR \cdot CS &= (BC/BR) \cdot (SA/CS) = \frac{1 + r}{r} \cdot \frac{1}{s} \\ &= 1 + r : rs \end{aligned}$$

$$\begin{aligned} BC \cdot SA \cdot AT : SA \cdot TB \cdot RC &= (BC/BR) \cdot (AT/TB) = \frac{1 + r}{1} \cdot \frac{t}{1} \\ &= t(1 + r) : 1 \end{aligned}$$

The preceding computations, together with parts (b) and (c) establish the result concerning the division of the interval from A to R by P. The other two results follow by cyclic permutation. [Actually, the second computation — and, for that matter, part (b), also — is unnecessary since, by Theorem 8-22, if $\overleftrightarrow{AR}$, $\overleftrightarrow{BS}$, and $\overleftrightarrow{CT}$ are concurrent then SA · TB · RC = AT · BR · CS.]

Answers for Part G

1. (a) 3/2 (b) 1/6

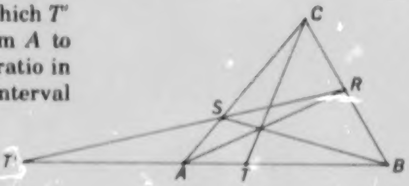
(c) 14

(d) 1 [This is, of course consistent with Theorem 8-10(a).]

2. In each of the preceding figures, let O be the point of concurrency. Compute:
   (a) $CO : OP$ **(b)** $MO : OA$ **(c)** $NO : OB$ **(d)** $AM : MC$

Part H

1. A theorem called Ceva's Theorem [and its converse] is obtained from Theorem 8-22 by replacing all '$\leftrightarrow$' by '$-$'s, and by omitting the words 'or parallel'. Prove this theorem. [*Hint*: Use Exercise 6(a) of Part F to show that the proof of concurrency requires that the points belong to the intervals.]
2. Use Ceva's Theorem to show that the medians of a triangle are concurrent.
3. Use Ceva's Theorem and Exercise 6 of Part F to prove Theorem 8-10(a).
4. Show that the ratio in which T' divides the interval from A to B is the opposite of the ratio in which T divides the interval from A to B.



8.09 Chapter Summary

Vocabulary Summary

triangle	quadrilateral
median of a triangle	diagonal of a quadrilateral
transversal	concurrent lines
simple quadrilateral	convex quadrilateral
parallelogram	trapezoid
bisect	centroid

Definitions

8-1. For $P \in \overleftrightarrow{AB}$, $A \neq P \neq B$ and $a \neq 0 \neq b$, P divides the interval from A to B in the ratio $a : b$ if and only if $(P - A) : (B - P) = a/b$.

8-2. **(a)** $PQR = \overline{PQ} \cup \overline{QR} \cup \overline{RP}$
(b) PQR is a triangle $\longleftrightarrow$ $\{P, Q, R\}$ is noncollinear

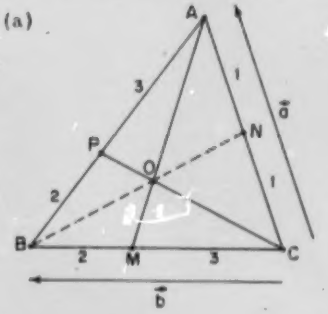
8-3. The *median* of a triangle from a given vertex is the interval whose endpoints are the given vertex and the midpoint of the opposite side.

8-4. **(a)** $PQRS = \overline{PQ} \cup \overline{QR} \cup \overline{RS} \cup \overline{SP}$

Answers for Part G [cont.]

2. [Although the questions to answer here look innocent enough, the work involved is a bit tedious. Even so, the answers should be well within reach of the students by this stage of the game.]

(a)



$O - C = (\vec{a}\frac{2}{5} + \vec{b}\frac{3}{5})p$ and

$O - A = ((\vec{b} - \vec{a})\frac{3}{5} + -\vec{a} \cdot \frac{2}{5})q$,

for some p and q. So,

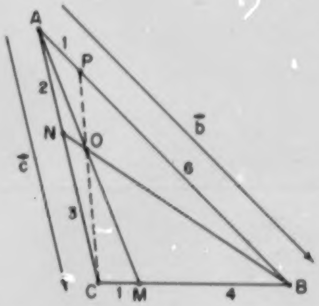
$$C - A = (\vec{b}\frac{3}{5} - \vec{a})q - (\vec{a}\frac{2}{5} + \vec{b}\frac{3}{5})p$$

or, more conveniently,

$$\vec{a}(-1 + q + \frac{2}{5}p) + \vec{b}(\frac{3}{5}p - \frac{3}{5}q) = \vec{0}$$

So, since $(\vec{a}, \vec{b})$ is linearly independent, $p = q = \frac{5}{7}$. Hence, $CO : OP = \underline{5/2}$.

(b)



$O - A = (\vec{b} \cdot \frac{1}{5} + \vec{c} \cdot \frac{4}{5})p$ and

$O - B = ((\vec{c} - \vec{b})\frac{2}{5} - \vec{b}\frac{3}{5})q$, for some p and q. So,

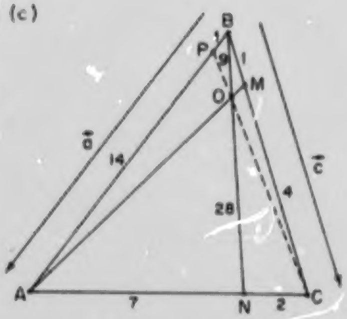
$$A - B = ((\vec{c} - \vec{b})\frac{2}{5} - \vec{b}\frac{3}{5})q - (\vec{b}\frac{1}{5} + \vec{c}\frac{4}{5})p.$$

or, equivalently:

$$\vec{b}(-1 + q + \frac{1}{5}p) + \vec{c}(\frac{4}{5}p - \frac{2}{5}q) = \vec{0}$$

Since $(\vec{b}, \vec{c})$ is linearly independent, $q = 2p$ and $p = \frac{5}{11}$. So, $p = \frac{5}{11}$ and $q = \frac{10}{11}$. Hence $MO : OA = \underline{6/5}$.

(c)



$O - B = (\vec{c}\frac{7}{9} + \vec{a}\frac{2}{9})p$ and

$O - A = ((\vec{c} - \vec{a})\frac{1}{5} - \vec{a}\frac{4}{5})q$, for some p and q. So,

$$A - B = (\vec{c}\frac{7}{9} + \vec{a}\frac{2}{9})p - (\vec{c}\frac{1}{5} - \vec{a})q$$

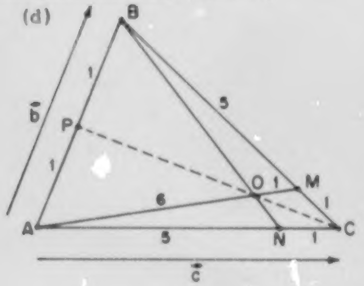
or, equivalently:

$$\vec{a}(1 - \frac{2}{9}p - q) + \vec{c}(-\frac{7}{9}p + \frac{1}{5}q) = \vec{0}$$

Since $(\vec{a}, \vec{c})$ is linearly independent, $q = \frac{35}{9}p$ and $p = \frac{9}{37}$ so that $p = \frac{9}{37}$ and $q = \frac{35}{37}$. Hence, $NO : OB = \underline{28/9}$.

Answers for Part G [cont.]

(d)



$O - A = (\vec{c}\frac{5}{6} + \vec{b}\frac{1}{6})p$ and

$O - B = ((\vec{c} - \vec{b})\frac{5}{6} - \vec{b}\frac{1}{6})q$, for some p and q. So,

$$A - B = (\vec{c}\frac{5}{6} - \vec{b})q - (\vec{c}\frac{5}{6} + \vec{b}\frac{1}{6})p$$

or, equivalently:

$$\vec{b}(-1 + q + \frac{1}{6}p) + \vec{c}(\frac{5}{6}p - \frac{5}{6}q) = \vec{0}$$

Since $(\vec{b}, \vec{c})$ is linearly independent, $q = p = \frac{6}{7}$. Hence, AM : MO = $\underline{-7}$.

Answers for Part H

1. Since R, S, and T belong to the sides of $\triangle ABC$, $0 < r < 1$, $0 < s < 1$, and $0 < t < 1$. Since r, s, and t are all positive, it follows from Exercise 6(a) of Part F on page 366 that the point P of concurrency belongs to the intervals.

2. If R, S, and T are the midpoints of the sides of $\triangle ABC$ then $r = s = t = 1$. So, $rst = 1$ and, by Ceva's Theorem, the medians are concurrent.

3. Suppose that R and S are u-points, from C, of $\overline{BC}$ and $\overline{CA}$, respectively. Then $r = BR/RC = 1 - u : u$ and $s = CS/SA = u : 1 - u$. If $T \epsilon \overline{AB}$ then $\overleftrightarrow{AR}$, $\overleftrightarrow{BS}$, and $\overleftrightarrow{CT}$ are concurrent if and only if the product of $\frac{1 - u}{u}$, $\frac{u}{1 - u}$ and the ratio in which T divides the interval from A to B is 1. This is the case if and only if the latter ratio is 1 — that is if and only if T is the midpoint of $\overline{AB}$. So, by Ceva's Theorem, $\overleftrightarrow{AR}$ and $\overleftrightarrow{BS}$ are concurrent at a point of the median from C. By Exercise 6(b), the point P of concurrency divides the interval from A to R in $1 + \frac{1 - u}{u} : 1$. This ratio is $1 : u$. P divides the interval from B to S in $1 + \frac{u}{1 - u} : \frac{u}{1 - u} - 1$. This ratio is, also, $1 : u$. P divides the interval from C to T in $1 + 1 : 1 \cdot \frac{1 - u}{u}$. This ratio is $2u : 1 - u$. From this last result it follows that P is the $2u/(1 + u)$-point, from C, of $\overline{CT}$. So, Theorem 10(a) is completely reestablished.

4. By Ceva's Theorem, the ratio in which T divides the interval from A to B is $1/(rs)$ [where 'r' and 's' have the meanings they have had throughout the foregoing exercises]; and, by Menelaus' Theorem T'' divides the interval from A to B in $-1/(rs)$.

(b) $PQRS$ is a quadrilateral $\longleftrightarrow$ each of $\{P, Q, R\}$, $\{Q, R, S\}$, $\{R, S, P\}$, $\{S, P, Q\}$ is noncollinear

8-5. (a) A quadrilateral is *simple* if and only if no two of its sides intersect.

(b) A quadrilateral is *convex* if and only if its diagonals intersect.

8-6. (a) A quadrilateral is a *trapezoid* if and only if it is simple and has two parallel sides.

(b) A quadrilateral is a *parallelogram* if and only if its opposite sides are parallel.

Other Theorems

8-1. $\overrightarrow{AB} \cap \overrightarrow{CD} \neq \varnothing \longleftrightarrow C - A \epsilon [B - A, C - D]$

8-2. If $\overline{AC}$ and $\overline{BD}$ are noncollinear parallel segments and $(D - B) : (C - A) = r$ then $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ if $r = 1$ and $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{A + (B - A) \cdot /(1 - r)\}$ if $r \neq 1$.

8-3. If A, B, C, D, and P are five points such that $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{P\}$ then (a) $\overline{AC} \parallel \overline{BD} \longrightarrow (P - D) : (P - C) = (D - B) : (C - A) = (P - B) : (P - A)$, and (b) $(P - D) : (P - C) = (P - B) : (P - A) \longrightarrow \overline{AC} \parallel \overline{BD}$.

Corollary. Under the conditions specified in the theorem, $\overline{AC} \parallel \overline{BD} \longleftrightarrow (P - D) : (P - C) = (P - B) : (P - A)$.

8-4. For $A \neq P \neq B$, (a) P divides the interval from A to B in $a : b \longrightarrow P = A + (B - A)\frac{a}{a + b}$ $[P \epsilon \overleftrightarrow{AB},\ a \neq 0 \neq b]$ (b) $P = A + (B - A)\frac{a}{a + b} \longrightarrow P$ divides the interval from A to B in $a : b$ $[a + b \neq 0]$

Corollary. For $A \neq P \neq B$, (a) P divides the interval from A to B in $s : 1 \longrightarrow P = A + (B - A)\frac{s}{s + 1}$ $[P \epsilon \overleftrightarrow{AB},\ s \neq 0]$ (b) $P = A + (B - A)t \longrightarrow P$ divides the interval from A to B in $t : 1 - t$.

8-5. For $P \epsilon \overleftrightarrow{AB}$ and $A \neq P \neq B$, $P \epsilon \overline{AB}$ or $P \epsilon -\overrightarrow{AB}$ or $P \epsilon -\overrightarrow{BA}$ according as the ratio in which P divides the interval from A to B is positive, or between 0 and -1, or less than -1.

8-6. (a) The ratio of two intervals which are intercepted on one of two parallel lines by concurrent transversals of both these lines is the same as that of the corresponding intervals which are intercepted by these transversals on the other.

(b) The ratio of two intervals which are intercepted by parallel lines on one transversal of these lines is the same as that of the corresponding intervals which are intercepted by these lines on any other transversal.

8-7. The interval whose endpoints are the midpoints of two sides of a triangle is parallel to the third side and its ratio to the third side is 1/2.

Corollary. A line through the midpoint of one side of a triangle is parallel to a second side if and only if it contains the midpoint of the third side.

8-8. The three medians of a triangle intersect at a point which divides each of them, from vertex to midpoint of the opposite side, in 2 : 1.

8-9. **(a)** The interval whose endpoints are the r-points of two sides of a triangle, from their common endpoint, is parallel to the third side, and its ratio to the third side is r.

(b) A line through the r-point of one side of a triangle, from one of its vertices, is parallel to the side opposite that vertex if and only if it contains the r-point, from that vertex, of the third side.

8-10. **(a)** Intervals from two vertices of a triangle to r-points of the opposite sides [from their common vertex] intersect at the $\frac{2r}{r+1}$-point of the median from that vertex. The point of intersection divides each of the two intervals, from vertex to opposite side in $1 : r$ and divides the median, from vertex to side, in $2r : 1 - r$

(b) Lines through two vertices of a triangle which intersect at the s-point of the median from the third vertex intersect the opposite sides at their $\frac{s}{2-s}$-points from this vertex.

8-11. If, in $\triangle ABC$ and $\triangle A'B'C'$, $\overline{AB} \parallel \overline{A'B'}$, $\overline{BC} \parallel \overline{B'C'}$, and $\overline{CA} \parallel \overline{C'A'}$ then (a) $(B' - A') : (B - A) = (C' - B') : (C - B) = (A' - C') : (A - C)$, and (b) for $A \neq A'$, $B \neq B'$, and $C \neq C'$, the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ are parallel or concurrent.

8-12. [The Twice-Around Theorem] If, in $\triangle ABC$, G and D are in $\overline{BC}$, E and H are in $\overline{CA}$, and I and F are in $\overline{AB}$, and $\overline{DE} \parallel \overline{BA} \parallel \overline{GH}$, $\overline{EF} \parallel \overline{CB} \parallel \overline{HI}$, $\overline{FG} \parallel \overline{AC}$, then $\overline{ID} \parallel \overline{AC}$.

8-13. If, in $\triangle ABP$, D and F are in $\overline{BP}$, C and E are in $\overline{PA}$, $\overline{EF} \parallel \overline{BC}$, and $\overline{CD} \parallel \overline{AF}$, then $\overline{DE} \parallel \overline{AB}$.

8-14. If $\vec{a}$, $\vec{b}$, and $\vec{r}$ are position vectors of A, B, and R [with respect to any point O] then, for $A \neq B$ and $0 \neq r \neq 1$, R is the point which divides the interval from A to B in $r : 1 - r$ if and only if $\vec{r} = \vec{a}(1 - r) + \vec{b}r$.

8-15. $\vec{a}$, $\vec{b}$, and $\vec{c}$ are position vectors of collinear points if and only if there exist numbers x, y, and z, not all 0, such that $\vec{a}x + \vec{b}y + \vec{c}z = \vec{0}$ and $x + y + z = 0$.

Corollary. If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are position vectors of noncollinear points, and a, b, and c are numbers such that $a + b + c = 0$, then $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$ if and only if $[a = 0, b = 0, \text{ and } c = 0]$.

8-16. Quadrilateral $ABCD$ is a parallelogram if and only if $\{A, B, C\}$ is noncollinear and $B - A = C - D$.

8-17. A quadrilateral is a parallelogram if and only if its diagonals bisect each other.

8-18. **(a)** $PQRS$ is a trapezoid with bases $\overline{PQ}$ and $\overline{RS}$ if and only if $\overline{PQ}$ and $\overline{RS}$ are noncollinear parallel intervals such that $(Q - P) : (R - S) > 0$.

(b) If, in trapezoid $PQRS$, $\overline{PS} \nparallel \overline{QR}$ then $\overleftrightarrow{PS}$ and $\overleftrightarrow{QR}$ intersect at a point which divides both the interval from P to S and the interval from Q to R in $-(\overline{PQ} : \overline{RS})$.

8-19. **(a)** A trapezoid is convex.

(b) If $PQRS$ is a trapezoid with bases $\overline{PQ}$ and $\overline{RS}$ then the intersection of its diagonals divides each of them, from P to R and from Q to S, respectively, in $\overline{PQ} : \overline{RS}$.

8-20. **(a)** The midpoints of successive sides of a simple quadrilateral are the successive vertices of a parallelogram.

(b) The intervals joining the midpoints of opposite sides of a simple quadrilateral bisect each other.

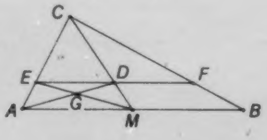
8-21. [Menelaus' Theorem and Converse] If, in $\triangle ABC$, $R \in \overleftrightarrow{BC}$, $S \in \overleftrightarrow{CA}$, and $T \in \overleftrightarrow{AB}$ then $\{R, S, T\}$ is collinear if and only if $BR \cdot CS \cdot AT = -(RC \cdot SA \cdot TB)$.

8-22. If, in $\triangle ABC$, $R \in \overleftrightarrow{BC}$, $S \in \overleftrightarrow{CA}$, and $T \in \overleftrightarrow{AB}$ then $\overleftrightarrow{AR}$, $\overleftrightarrow{BS}$, and $\overleftrightarrow{CT}$ are concurrent or parallel if and only if $BR \cdot CS \cdot AT = RC \cdot SA \cdot TB$.

[For Ceva's Theorem, see Exercise 1, Part H, page 367.]

Chapter Test

1. Given $\triangle ABC$, assume that M is the midpoint of $\overline{AB}$, that $\overline{EF} \parallel \overline{AB}$, and that $\overline{CF} : \overline{FB} = 5/2$. Complete each of the following. If not possible, say so.



(a) $(E - C) : (A - E) = ?$

(b) $(F - E) : (B - A) = ?$

(c) $EDMA$ is a ?

(d) $(E - C) : (F - C) = ?$

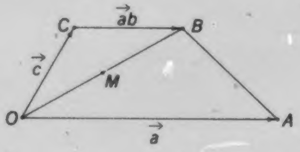
(e) $\overline{DE} : \overline{MA} = ?$

(f) $\overline{EG} : \overline{GM} = ?$

(g) $C = F + (B - A)$ ____

(h) A divides the segment from C to E in ________.

2. Given quadrilateral $OABC$, with $A - O = \vec{a}$, $C - O = \vec{c}$, and $B - C = \vec{a}b$, for some nonzero b. Also, M is the midpoint of $\overline{OB}$. Answer these questions.



(a) $OABC$ is a parallelogram if $b =$ ________.

(b) $OABC$ is a trapezoid with base $\overline{BC}$ longer than $\overline{OA}$ if $b =$ ______.

(c) For what values of 'b' is it the case that (interval) $\overline{OC}$ and (line) $\overleftrightarrow{AM}$ intersect?

(d) For what values of 'b' is it the case that (interval) $\overline{BC}$ and (line) $\overleftrightarrow{AM}$ intersect?

(e) For what values of 'b' is it the case that $C \in \overleftrightarrow{AM}$?

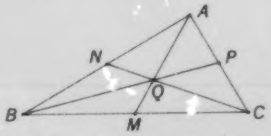
(f) Assume that P is the midpoint of $\overline{OC}$ and that Q is the midpoint of $\overline{CB}$. Express $Q - P$ as a linear combination of $\vec{a}$ and $\vec{c}$.

3. Suppose that $(\vec{a}, \vec{b})$ is linearly independent and that

$$(*) \quad \vec{a}(p - q) - \vec{b}(q - p) + (\vec{a} - \vec{b})(2p + 1) = (\vec{b} - \vec{c})(q - 2).$$

Compute the values of 'p' and 'q' for which (*) is satisfied.

4. Given $\triangle ABC$, with P, M, and N the midpoints of sides $\overline{AC}$, $\overline{CB}$, and $\overline{BA}$, respectively. Assume that the medians of $\angle ABC$ intersect in the point Q.



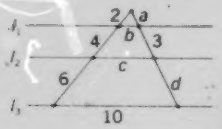
(a) Consider $\triangle AQC$. Is the point of intersection of its medians on the line $\overleftrightarrow{BP}$? Explain your answer.

(b) Consider $\triangle NBC$. Is the point of intersection of its medians on $\overleftrightarrow{BP}$? Explain.

(c) If either of your answers in (a) or (b) is 'Yes.', give the ratio in which that point divides the segment from B to P. If both answers are 'No.', ignore this part.

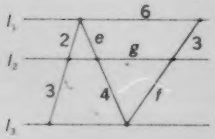
5. In each of the following pictures, you are given the lengths of some segments. Use this together with the other given information to help you to answer the questions.

(a)



Given: $l_1 \parallel l_2 \parallel l_3$
Compute: a; b; c; d

(b)



Given: $l_1 \parallel l_2 \parallel l_3$
Compute: e; f; g

Key to Chapter Test

1. (a) 5/2 (b) 5/7 (c) trapezoid (d) not possible
(e) 5/7 (f) 5/7 (g) not possible (h) −[illegible]/2

2. (a) 1 (b) [any number greater than 1]
(c) all values less than 1 (d) all values greater than 1
(e) 1 (f) $\vec{a} \cdot b/2 + \vec{c} \cdot 1/2$

3. (*) is equivalent to:

$$\vec{a}(p - q + 2p + 1 + q - 2) + \vec{b}(p - q - 2p - 1 + 2 - q) = \vec{0}$$

or, more conveniently, to:

$$\vec{a}(3p - 1) + \vec{b}(-p - 2q + 1) = \vec{0}$$

So, $p = \frac{1}{3} = q$.

4. (a) Yes, for $\overline{QP} \subseteq \overleftrightarrow{BP}$ and $\overline{QP}$ contains the point of intersection of the medians of $\triangle AQC$.

(b) No, for $\overline{BQ}$ contains no point of the median from B of $\triangle NBC$. So, neither does $\overleftrightarrow{BP}$.

(c) The point of intersection of the medians of $\triangle AQC$ divides the segment from B to P in the ratio 8 : 1.

5. (a) $a = \frac{3}{2}$; $b = \frac{5}{2}$; $c = 5$; $d = \frac{9}{2}$

(b) $e = \frac{8}{3}$; $f = \frac{9}{2}$; $g = \frac{18}{5}$

Chapter Nine

Planes in $\mathscr{E}$

9.01 Coplanar Points

So far, we have used our postulates about points and translations to introduce the notion of collinearity of points and, in terms of this, notions about lines and their subsets, and about triangles and quadrilaterals. In a similar fashion, we now wish to add to our structure of formal geometric ideas some notions about coplanar points and planes.

In our intuitive consideration of geometry in Chapter 1 it was convenient to describe a plane as being a set of points which is the union of two closed half-planes. Now, as in the case with lines, we wish to define 'plane' in terms of 'point' and 'translation'. Our earlier description will turn out to be a theorem. So will the other results about planes which we reached intuitively in Chapter 1. As before, this will be an important test of the adequacy of our definitions. If it turned out that our definitions didn't enable us to reach the conclusions which seem intuitively sound, we might have reason to suspect that something was wrong with them.

Some of the work with linearly dependent and independent translations and with lines may have given you a hint as to how we shall define the word 'plane'. As in the case of lines, we shall decide what it means, in terms of translations, to say that four points are coplanar. Then we can say that a plane is a set of points which consists of all the points which are coplanar with any three of its noncollinear members.

In the following exercises, we shall examine our intuitive notions about planes and linear dependence to see if we can find a suitable definition for a set of coplanar points in terms of translations determined by those points.

The early parts of this chapter on planes are very similar to the treatment of lines in Chapter 7. Some similarities are pointed out in the text and in the commentary. It should be worthwhile to read through, in class, Definitions 7-1 through 7-7 and Theorems 7-1 through 7-10. Spot reading in section 7.01 through section 7.05 may also be in order, as a rapid review. You can best judge what is appropriate for your class.

Exercises

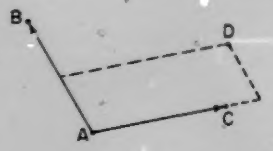
1. **(a)** Given a point – say, A – is there a plane which contains A? Is there more than one such plane?
   (b) Given two points – say, A and B – is there a plane which contains both A and B? Is there more than one such plane?
   (c) Given any set of collinear points, is there a plane which contains them? How many such planes are there?
   (d) Given any set of collinear points and a point which is not a member of this set, is there a plane which contains them all? How many such planes are there?
   (e) If $\{A, B, C\}$ is noncollinear, is there a plane which contains A, B, and C? How many such planes are there?
   (f) Describe a set $\{A, B, C, D\}$ of four points such that no plane contains all of them.
2. Suppose that A, B, C, and D are four points contained in a plane. That is, suppose that A, B, C, and D are coplanar points. Suppose, also, that $\{A, B, C\}$ is noncollinear.
   (a) Draw an appropriate picture for these conditions and estimate values for 'b' and 'c' such that $D - A = (B - A)b + (C - A)c$.
   (b) Is $(B - A, C - A, D - A)$ linearly dependent or not? Explain your answer.
   (c) Suppose that E is a point such that $(B - A, C - A, E - A)$ is linearly dependent. Do you think that A, B, C, and E are contained in a plane?
   (d) Suppose that F is a point in the same plane with A, B, and C. Do you think that $(B - A, C - A, F - A)$ is linearly dependent or not?
   (e) Is it possible to locate a point – say, G – which is not coplanar with A, B, and C? If so, describe the position of G with respect to the plane you drew in (a). What would you say about $(B - A, C - A, G - A)$ in this case?
   (f) Is there a translation – say, $\vec{a}$ – which maps A onto a point which you would not consider to be coplanar with A, B, and C? If so, use your picture from (a) and a pencil to help you describe such a translation. What would you say about $(B - A, C - A, \vec{a})$?
3. Suppose that E, F, G, and H are four coplanar points, and that $\{E, F, G\}$ is collinear. Draw a picture to illustrate these conditions and answer the following.
   (a) Is $(G - E, F - E)$ linearly dependent or not? Explain.
   (b) Is $(G - E, F - E, H - E)$ linearly dependent or not? Explain.
   (c) Given any point – say, P – show that $(G - E, F - E, P - E)$ is linearly dependent.
   (d) Given any point – say, Q – do you think that E, F, G, and Q are coplanar? Explain.

Answers for most of the exercises — both here and on pages 374-5 — must be based on intuition, since we do not yet have a formal definition of 'coplanar'. Appropriate intuitions should have been developed in section 1.05. Some can, however, be answered on formal grounds which include Definition 7-1 of 'collinear' and theorems concerning linear dependence. Note that the word 'coplanar' is introduced in its usual meaning in Exercise 2. As with 'collinear', we shall adopt this meaning after we have defined 'plane'. Up to then, and after Definition 9-1, 'collinear' will have the rather special and restricted meaning given it in that definition.

The purpose of these exercises is, mainly, to stimulate discussion. For this reason the exercises are best treated as a class activity. Recall that it has been helpful to use pencils and pieces of cardboard as models of lines and planes.

Answers for Exercises

1. (a) Yes.; Yes.
   (b) Yes.; Yes.
   (c) Yes.; Lots. [Infinitely many; as many as there are points on a line skew to the line containing the given set.]
   (d) Yes.; Just one if the point is not collinear with the other given points; otherwise, see part (c).
   (e) Yes.; Exactly one.
   (f) Choose, for example, noncollinear points A, B, and C in a horizontal plane and choose D above or below this plane.
2. (a)

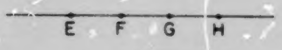


$b = 1/2$, $c = 5/4$

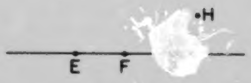
[Students will, of course, have a variety of drawings.]

   (b) (B - A, C - A, D - A) is linearly dependent.
   (c) Yes.
   (d) Linearly dependent.
   (e) Yes. Given that the plane of A, B, and C is horizontal, choose G above or below this plane; (B - A, C - A, G - A) is linearly independent.
   (f) (B - A, C - A, $\vec{a}$) is linearly independent.

3.



or:



(a) (G - E, F - E) is linearly dependent because {E, F, G} is collinear.

(b) (G - E, F - E, H - E) is linearly dependent by part (a), since a sequence is linearly dependent if it has a linearly dependent subsequence.

(c) [See answer for part (b).]

(d) Yes. There are lots of planes containing {E, F, G} and at least one of them contains Q. Or: In Chapter 1 we decided that, given any line and any point, there is a plane which contains them.

4. Suppose that P, Q, and R are three points.
   (a) Is $(P - P, Q - P, R - P)$ linearly dependent or not? Explain.
   (b) Is there a plane which contains P, Q, and R? Might there be more than one such plane?
5. Suppose that L, M, and N are noncollinear points.
   (a) Draw a picture showing a point—say, P—such that L, M, N, and P are not coplanar.
   (b) Given the point P of part (a), would you say that L, M, and P are coplanar? If so, is N in this plane? Explain your answers.

In the exercises above, we used our intuitions about points and lines to try to develop a feeling for some of the relationships among points in a plane and linear dependence. The results of these exercises suggest that the following definition agrees with our intuitions about these relationships.

> **Definition 9–1** $\{A, B, C, D\}$ is *coplanar*
>
> $$\Longleftrightarrow$$
>
> $(B - A, C - A, D - A)$ is linearly dependent

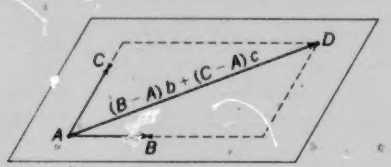


Fig. 9-1

In this definition, we speak of the set of points $\{A, B, C, D\}$ being coplanar. As in the case of collinearity, and since $\{A, B, C, D\} = \{B, D, C, A\}$, we would be in trouble with this definition were it not the case that

$$(B - A, C - A, D - A) \text{ is linearly dependent}$$
$$\Longleftrightarrow$$
$$(C - B, D - B, A - B) \text{ is linearly dependent.}$$

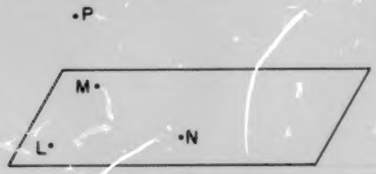
That this is the case can be verified by an argument which is very similar to that suggested on page 277 in connection with the definition of 'collinear'.

Exercises

1. Prove that
   (a) any pair of points is coplanar, and
   (b) any three points are coplanar.

Answers for Exercises [cont.]

4. (a) This sequence is linearly dependent because its first term $P - P$, is $\vec{0}$.
   (b) Yes.; There will be more than one such plane if and only if $\{P, Q, R\}$ is collinear.
5. (a)



   (b) Yes. [Intuitively, any three points are contained in a plane.] No. [If N was in the same plane as L, M, and P, then all four points would be in the same plane. But, P was chosen so that it was not in the same plane as L, M, and N.]

The displayed theorem and successive instances of it obtained by cyclic permutation supply the justification which is needed because of the fact that

$$\{A, B, C, D\} = \{B, C, D, A\} = \{C, D, A, B\} = \{D, A, B, C\}.$$

[Going one step further, it is readily seen that the if-part of the theorem follows from its only if-part.] The other facts which need to be taken account of — for example, that $\{A, B, C, D\} = \{A, C, B, D\}$ — are taken account of by recalling that any permutation of a linearly dependent sequence of translations is, also, linearly dependent.

As a result of the preceding analysis, all that needs to be proved is that

$$(B - A, C - A, D - A) \text{ is linearly dependent}$$
$$\Longrightarrow$$
$$(C - B, D - B, A - B) \text{ is linearly dependent.}$$

The proof is like that of the similar Theorem 6-14 [See page 263.] Suppose $(B - A)a + (C - A)b + (D - A)c = \vec{0}$. Using Postulate 3 twice it follows that $(C - B)b + (D - B)c + (A - B)\cdot -(a + b + c) = \vec{0}$. If, now, $b = c = -(a + b + c) = 0$ then $a = b = c = 0$. Hence, if not all of a, b, and c are zero then not all of b, c, and $-(a + b + c)$ are zero. Consequently, if $(B - A, C - A, D - A)$ is linearly dependent then so is $(C - B, D - B, A - B)$.

Definition 9-1 gives a basis for "formal answers" for exercises 1, 2, 4(c) and 5.

Answers for Exercises

1. (a) $\{A, B\} = \{A, B, B, B\}$ is coplanar if and only if $(B - A, B - A, B - A)$ is linearly dependent. Since any sequence with a repeated term is linearly dependent, $\{A, B\}$ is coplanar. [For a variation, see answer for part (b).]
   (b) $\{A, B, C\} = \{A, B, C, A\}$ is coplanar if and only if $(B - A, C - A, A - A)$ is linearly dependent. Since $A - A = \vec{0}$ and any sequence with a term $\vec{0}$ is linearly dependent, $\{A, B, C\}$ is coplanar.

2. Suppose that P, Q, and R are collinear points.
   (a) Choose any point – say, S – of $\overleftrightarrow{QR}$. Show that $\{P, Q, R, S\}$ is coplanar.
   (b) Choose any point – say, T. Show that $\{P, Q, R, T\}$ is coplanar.
3. By Exercise 2, it should be clear that the set consisting of any three [or, two] points of a line together with any point in space is a coplanar set. Can you find two points in space which, together with two points of a given line, constitute a set of four points which are not coplanar? Try to picture four such points.
4. On your paper, draw a picture of three noncollinear points R, S, and T.
   (a) Picture a point P such that $\{R, S, T, P\}$ is coplanar.
   (b) Picture a point Q such that $Q \neq P$ and $\{R, S, T, Q\}$ is coplanar.
   (c) Is it the case that $\{R, S, Q, P\}$ is coplanar? Explain your answer.
   (d) Describe this set of points:

$$\{X: R, S, T, \text{and } X \text{ are coplanar points}\}$$

5. On your paper, draw a picture of three collinear points U, V, and W. Describe this set of points:

$$\{X: U, V, W, \text{and } X \text{ are coplanar}\}$$

6. Suppose that π is a plane, and that π contains noncollinear points A, B, and C.
   (a) Given that $D \in \pi$, what can be said about $\{A, B, C, D\}$?
   (b) Given that D is a point such that $\{A, B, C, D\}$ is coplanar, what can be said about D and the plane π?

9.02 Planes

Now that we have a formalized notion of coplanar points, we are in a position to make use of this notion, together with out intuitive ideas about what planes are (or, ought to be), in formulating a definition of the term *plane*.

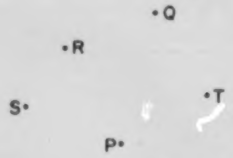
In order to agree with our intuitive notions, we want to be sure that, among other things, any plane π is such that

(i) π is a subset of $\mathscr{E}$ and π contains at least three noncollinear points.

Certainly, this condition (i) is not enough to pin down exactly what we mean by *plane*, for we expect a plane to contain many more than three noncollinear points. And, furthermore, you probably can think of many sets which contain three noncollinear points and which aren't anything like what you think of as a plane. Draw pictures of at least two such sets.

Answers for Exercises [cont.]

2. (a) Since $S \in \overleftrightarrow{QR}$, $\{Q, R, S\}$ is collinear — that is, $(R - Q, S - Q)$ is linearly dependent. It follows that $(R - Q, S - Q, P - Q)$ is linearly dependent and, so, that $\{Q, R, S, P\}$ is coplanar. But, $\{P, Q, R, S\} = \{Q, R, S, P\}$.
   (b) The assumption that $\{P, Q, R\}$ is collinear was not used in part (a). So, with 'P' for 'Q', 'Q' for 'R', 'R' for 'S', and 'T' for 'R', the same argument shows that $\{P, Q, R, T\}$ is coplanar.
3. Choose two points on a line which is skew to the given line. [See, also, Exercise 1 of Part B on page 299 for a suggestion of possible answers.]
4. (a), (b)



   (c) Yes.

   In case $\{R, S, Q\}$ is collinear it follows by Exercise 2(b) that $\{R, S, Q, P\}$ is coplanar. Suppose, then, that $\{R, S, Q\}$ is noncollinear. Since $\{R, S, T\}$ is noncollinear and $\{R, S, T, P\}$ and $\{R, S, T, Q\}$ are coplanar it follows by Theorem 6-13 that $P - R$ and $Q - R$ belong to $[S - R, T - R]$. So, $P - Q \in [S - R, T - R]$. Since $\{R, S, Q\}$ is noncollinear and $\{R, S, Q, T\}$ is coplanar it follows that $T - R \in [S - R, Q - R]$. So, $[S - R, T - R] \subseteq [S - R, Q - R]$. Since $S - R = (Q - R) + (S - Q)$ and $Q - R = -(R - Q)$ it follows that $[S - R, Q - R] \subseteq [R - Q, S - Q]$. So, $P - Q \in [R - Q, S - Q]$ from which it follows that $\{Q, R, S, P\}$ is coplanar.

   (d) The set in question is a plane; it is the unique plane containing R, S, and T.
5. The set in question is $\mathscr{E}$. [See Exercise 2(b).]
6. [If our intuitive notions concerning planes are correct, and Definition 9-1 is suitable, then]
   (a) $\{A, B, C, D\}$ is coplanar (b) $D \in \pi$

The results in the previous exercises should suggest that the notion of coplanar points can be used to help us define the term *plane*. One way to use this notion is as follows:

(ii) Given that A, B, C, and D are points of π, then A, B, C, and D are coplanar.

We saw that (i) wasn't enough to fully describe what is generally thought of as a plane. Neither are (i) and (ii) together. Describe, and draw pictures of, at least two sets of points each of which satisfies the conditions (i) and (ii) but which are not what you consider to be planes.

Here is another insight into the relationship between coplanarity and being a plane:

(iii) Given that A, B, and C are noncollinear points of π and that $\{A, B, C, D\}$ is coplanar, then D is a point of π.

It should be intuitively clear that what one thinks of as a plane satisfies all three of these conditions, and any set which satisfies (i), (ii), and (iii) is a plane. So, the above discussion suggests the following definition:

> **Definition 9–2** π is a plane if and only if
> (a) π is a subset of $\mathscr{E}$ which contains at least three noncollinear points, and
> (b) $\forall_X\forall_Y\forall_Z\,[(\{X, Y, Z\} \subseteq \pi$ and $\{X, Y, Z\}$ is noncollinear$) \Longrightarrow \forall_W\,[W \in \pi \Longleftrightarrow \{X, Y, Z, W\}$ is coplanar$]]$

Notice that part (a) of this definition says what (i) says, and that part (b) of this definition says what (ii) *and* (iii) say. In words, part (b) says, "For each three noncollinear points of a plane π, any fourth point is in π if and only if it and the three given points are coplanar."

From now on, we shall use 'π' (pi) and 'σ' (sigma), with or without subscripts, as variables whose domain is the set of planes of $\mathscr{E}$. We shall also use 'π' and 'σ' as indices on quantifiers. Read '$\forall_\pi$' as 'for each plane π', etc.

Exercises

Part A

As in the case of Definition 7–2 of 'line', Definition 9–2 can be reformulated in various ways. In two simple reformulations the clause:

$$\forall_W\,[W \in \pi \longleftrightarrow \{X, Y, Z, W\} \text{ is coplanar}]$$

in part (b) is replaced, in one case by:

$$\forall_W\,[W \in \pi \longleftrightarrow (Y - X, Z - X, W - X) \text{ is linearly dependent}]$$

TC 375,376

The discussion leading to the adoption of Definition 9-2 is similar to that on pages 278 and 279 which leads to the adoption of Definition 7-2. Both discussions are, of course, based on intuitive notions and each serves only as motivation for adoption of the corresponding definition.

Condition (i) is satisfied by any set which consists of just three noncollinear points, by one which consists of the points of a line together with a point not on the line, by a plane with a hole in it, by $\mathscr{E}$ itself, and by many other sets which are not planes. Of the sets mentioned specifically in the preceding sentence, condition (ii) rules out only $\mathscr{E}$. Condition (ii) is satisfied only by sets which are subsets of planes. Condition (ii) tends to require π to be "small". Conditions (i) and (iii) [on page 376] force π to be "large" among sets which satisfy (ii).

Definition 9-2 is, obviously, analogous to Definition 7-2 on page 279.

As indicated in the exercises, Definition 9-2 can be modified as was Definition 7-2 in Exercise 1 of Part A on page 279.

Answers for Part A

1. By Definition 9-1, the sentences:

$$D \in \pi \Longleftrightarrow \{A, B, C, D\} \text{ is coplanar}$$

$$D \in \pi \Longleftrightarrow (B - A, C - A, D - A) \text{ is linearly dependent}$$

are equivalent. [For the rest, see the answer in TC 280(1) for Exercise 1 of Part A.]

TC 377 (1)

2. By Theorem 6-13 [and the definitions of '$[\vec{a}, \vec{b}]$' and 'linearly dependent'], if $(\vec{a}, \vec{b})$ is linearly independent then $(\vec{a}, \vec{b}, \vec{c})$ is linearly dependent if and only if $\vec{c} \in [\vec{a}, \vec{b}]$. So, for $\{A, B, C\}$ noncollinear,

$$(B - A, C - A, D - A) \text{ is linearly dependent}$$
$$\Longleftrightarrow$$
$$D - A \in [B - A, C - A].$$

Hence, under the same assumption, the sentences:

$$\forall_W\,[W \in \pi \Longleftrightarrow (B - A, C - A, W - A) \text{ is linearly dependent}]$$

$$\forall_W\,[W \in \pi \Longleftrightarrow W - A \in [B - A, C - A]]$$

are equivalent. So, the conditional sentences which have these as consequents and have:

$$(\{A, B, C\} \subseteq \pi \text{ and } \{A, B, C\} \text{ is noncollinear})$$

as their common antecedent are equivalent. Consequently, the two modifications of Definition 9-2 are equivalent.

Answers for Part B

1. Suppose that $\{A, B, C\}$ is a noncollinear subset of π. Since π is a plane it follows, by part (b) of Definition 9-2, that $D \in \pi$ if and only if $\{A, B, C, D\}$ is coplanar. So, in this case, $\{A, B, C, D\}$ is coplanar. On the other hand, it has been shown, in Exercise 2(a) on page 375, that in case $\{A, B, C\}$ is collinear then $\{A, B, C, D\}$ is coplanar [whether or not $D \in \pi$].

and, in the other, by:

$$\forall_W [W \in \pi \longleftrightarrow W - X \in [Y - X, Z - X]]$$

1. Tell why the first of these replacements yields a statement equivalent to Definition 9–2.
2. Tell why the two replacements yield equivalent statements. [*Hint*: If $(\vec{a}, \vec{b}, \vec{c})$ is linearly dependent and $(\vec{a}, \vec{b})$ is linearly independent, what can you infer about $\vec{c}$? If $\vec{c} \in [\vec{a}, \vec{b}]$, what can you infer about $(\vec{a}, \vec{b}, \vec{c})$?]

Part B

1. Suppose that A, B, C, and D are points of a plane π. Show that $\{A, B, C, D\}$ is coplanar. [*Hint*: Consider two cases—that in which $\{A, B, C\}$ is noncollinear and that in which $\{A, B, C\}$ is collinear.]
2. (a) Show that if A, B, and C are noncollinear points of a plane π then

$$\pi = \{X: X - A \in [B - A, C - A]\}.$$

   (b) Does it follow *from the result in part (a)* that, given noncollinear points A, B, and C, there is at least one plane which contains these points?

   (c) Your answer for part (b) should have been either 'Yes.' or 'No.'. Change one word in part (b) to obtain a reasonable question which has the other answer.
3. Suppose that l is a line, π is a plane, and $l \subseteq \pi$. Show that $\pi \neq l$.
4. Suppose that each of two lines, l and m, is contained in a plane π and, also, contained in a plane σ. Show that $\pi = \sigma$.

Part C

Exercise 2(a) of Part B tells us that there is at most one plane containing the noncollinear points A, B, and C, and gives us a convenient description of the only subset of $\mathcal{E}$ which has a chance of being such a plane. As at the corresponding place in Chapter 7, it is worthwhile to introduce a definition:

Definition 9–3

$$\overleftrightarrow{ABC} = \{X: \exists_x \exists_y\, X = A + (B - A)x + (C - A)y\}$$

[Read '$\overleftrightarrow{ABC}$' as 'cross ABC'.] In words, $\overleftrightarrow{ABC}$ is the set of all images of A under the linear combinations of $B - A$ and $C - A$.

1. (a) Explain why Definition 9–3 is equivalent to:

$$\overleftrightarrow{ABC} = \{X: X - A \in [B - A, C - A]\}$$

   *(b) Is it the case that, however A, B, and C are chosen,

$$\overleftrightarrow{ABC} = \{X: (B - A, C - A, X - A) \text{ is linearly dependent}\}?$$

[Exercise 1 goes half way to showing that, with our definition of 'plane', Definition 9-1 is consistent with the usual definition according to which points are coplanar if and only if they belong to some plane. What should be shown is that {A, B, C, D} is coplanar [according to Definition 9-1] if and only if $\exists_\pi \{A, B, C\} \subseteq \pi$. Exercise 1 takes care of the if-part. The only if-part will be established by Theorem 9-1 on page 378 in case {A, B, C} is noncollinear [or, in any case three of A, B, C, and D are noncollinear]. This leaves the case in which A = B = C = D and that in which {A, B, C, D} is a subset of some line. [See Exercise 4 of Part A on page 280, and the discussion of it on TC 280(1, 2).] To handle this case — and the corresponding case for 'collinear', we need to know that each point belongs to some line and that each line is a subset of some plane. These results follow from postulates on dimension which are adopted in Chapter 10.]

2. (a) This is an immediate consequence of part (b) of the second modification of Definition 9-2. [Compare with Part B on pages 280-281.]

   (b) No. [Compare with Exercise 2(a) of Part B on page 281.]

   (c) Change the word 'least' to 'most'. [See Exercise 2(b) of Part B on page 281.]

3. By part (a) of Definition 9-2, π contains three noncollinear points. Given three such points, at most two can belong to ℓ. So, one of them [at least] belongs to π but not to ℓ. Since there exists a point of π which does not belong to ℓ, $\pi \neq \ell$. [Of course, the assumption that $\ell \subseteq \pi$ is irrelevant to the argument and to the result; but, if $\ell \not\subseteq \pi$ then, trivially, $\ell \neq \pi$ — an uninteresting case.]

4. $\ell \cup m$ contains three noncollinear points. As noted in Exercise 2(c), three such points are contained in at most one plane. Hence, $\pi = \sigma$.

Answers for Part C

[Definition 9-3 is obviously analogous to Definition 7-3.]

1. (a) $D = A + (B - A)a + (C - A)b$ if and only if $D - A = (B - A)a + (C - A)b$. Hence, $\exists_x \exists_y\, D = A + (B - A)x + (C - A)y$ if and only if $\exists_x \exists_y\, D - A = (B - A)x + (C - A)y$ — that is, if and only if $D - A \in [B - A, C - A]$. Consequently, by Definition 9-3, $D \in \overline{ABC}$ if and only if $D \in \{X: X - A \in [B - A, C - A]\}$. In other words, $\overline{ABC} = \{X: X - A \in [B - A, C - A]\}$. [Compare this with Exercise 2(d) of Part C on page 281.]

   (b) No. This is the case if {A, B, C} is noncollinear [see Exercise 2 of Part A]; but, if {A, B, C} is collinear then {X: (B - A, C - A, X - A) is linearly dependent} = $\mathcal{E}$ [Exercise 5 on page 375], while $\overline{ABC}$ is a line or consists of a single point. [See Exercise 4, below.]

2. (a) Draw a picture to show three noncollinear points P, Q, and R and translations $Q - P$ and $R - P$.
   (b) In your picture, locate the points C, D, E, F, G, and H such that
   $C = P + (Q - R) + (R - P)$,
   $D = P + (Q - P)\frac{1}{2} + (R - P)\cdot -1$,
   $E = P + (R - P)2$,
   $F = P + (Q - P)\cdot -2$,
   $G = P + (Q - P)\cdot -1 + (R - P)2$,
   $H = P + (R - P)\cdot -1 + (Q - P)\cdot -\frac{1}{2}$.
   (c) Why is each point described in part (b) a point in $\overline{PQR}$?
3. Show that $\{A, B, C\} \subseteq \overline{ABC}$.
4. What kind of set is $\overline{ABC}$ if $\{A, B, C\}$ is collinear?
5. If $\overline{ABC}$ is a plane, what may you say about $\{A, B, C\}$?

9.03 The Plane Containing Three Noncollinear Points

We have seen that if there is a plane which contains given noncollinear points A, B, and C then it is $\overline{ABC}$. Since, according to our intuition, there is such a plane, we should be able to prove:

> **Theorem 9–1** For $\{A, B, C\}$ noncollinear, $\overline{ABC}$ is the plane which contains A, B, and C.

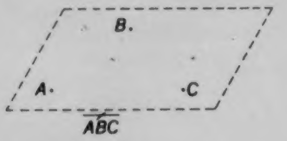


Fig. 9-2

Since we already know that $\{A, B, C\} \subseteq \overline{ABC}$, all we need to do to prove Theorem 9–1 is to show that, for $\{A, B, C\}$ noncollinear, $\overline{ABC}$ is a plane. Since we already know that part (a) of Definition 9–2 is satisfied [Why?], all that concerns us is part (b). To show that the set $\overline{ABC}$ satisfies this part of the definition, we must show that

if $\{P, Q, R\}$ is a noncollinear subset of $\overline{ABC}$ then

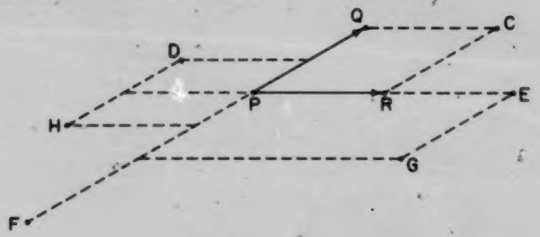
$$D \in \overline{ABC} \longleftrightarrow D - P \in [Q - P, R - P].$$

[Explain.] Recalling Exercise 1(a) of Part C, the most reasonable way to attempt to carry this out is to assume that

(*) $\{P, Q, R\}$ is a noncollinear subset of $\overline{ABC}$

Answers for Part C [cont.]

2. (a), (b)



   (c) For each of the points, there are real numbers as required by Definition 9-3.

3. $A = A + (B - A)0 + (C - A)0$; $B = A + (B - A)1 + (C - A)0$; $C = A + (B - A)0 + (C - A)1$

4. Suppose that $\{A, B, C\}$ is collinear. If $C = A$ then, by Definitions 9-3 and 7-3, $\overline{ABC} = \overline{AB}$. If $C \neq A$ then, since $\{A, B, C\}$ is collinear, $B - A \in [C - A]$ [Theorem 6-13] and, by Exercise 1(a) and Definition 7-3, $\overline{ABC} = \overline{AC}$. In either case, $\overline{ABC}$ either is a line or consists of a single point.

5. By Exercise 3 of Part B, a plane is not a line, and by (a) of Definition 9-2, no plane consists of a single point. So, by Exercise 4, if $\overline{ABC}$ is a plane then $\{A, B, C\}$ is noncollinear.

Theorem 9-1 is analogous to Theorem 7-1. Just as Theorem 7-1 implies that two points belong to a unique line, Theorem 9-1 implies that three noncollinear points belong to a unique plane.

By Exercise 3 of Part C, $\{A, B, C\} \subseteq \overline{ABC}$ so, for $\{A, B, C\}$ noncollinear, $\overline{ABC}$ contains three noncollinear points [and, so, satisfies part (a) of Definition 9-2.]

The explanation asked for in line 3b is to the effect that one "may" universally generalize on a variable in the consequent of a conditional sentence if this variable does not occur in the antecedent. [For this rule of logic, see Exercise 2 of Part C on page 251.]

and, with the help of this assumption, show that

$$(1)\quad D - A \in [B - A, C - A] \Longleftrightarrow D - P \in [Q - P, R - P].$$

Since, by (*), $P \in \overrightarrow{ABC}$ it follows that

$$P - A \in [B - A, C - A].$$

So, if either of $D - A$ and $D - P$ belongs to $[B - A, C - A]$ then so does the other. [Explain.] Hence, in place of (1), it will be sufficient to derive:

$$(2)\quad D - P \in [B - A, C - A] \Longleftrightarrow D - P \in [Q - P, R - P]$$

Since, in (2), D may be any point in $\mathscr{E}$, $D - P$ may be any translation in $\mathscr{T}$. So, deriving (2) amounts to deriving:

$$(3)\qquad [Q - P, R - P] = [B - A, C - A]$$

To see how to use (*) in deriving (3), we need to see what (*) tells us about the translations $Q - P$ and $R - P$. In the first place, by the definition of 'collinear', (*) tells us that

$$(*)\qquad (Q - P, R - P) \text{ is linearly independent.}$$

In the second place, by Exercise 1(a) of Part C, (*) tells us that $P - A$, $Q - A$, and $R - A$ all belong to $[B - A, C - A]$. From this it follows that

$$(**)\, Q - P \in [B - A, C - A] \text{ and } R - P \in [B - A, C - A].$$

So, we shall have proved Theorem 9–1 if we are able to derive (3) from (*) and (**). In doing so we shall be dealing with the translations $Q - P$, $R - P$, $B - A$, and $C - A$ rather than with the specific points P, Q, R, A, B, and C. So, we may expect to find it more convenient to prove a general theorem about translations similar to those of Chapter 6. Looking at (*), (**), and (3) suggests that the theorem we need may be stated as follows:

Lemma

$$(\overrightarrow{c}, \overrightarrow{d}) \text{ is linearly independent and } \{\overrightarrow{c}, \overrightarrow{d}\} \subseteq [\overrightarrow{a}, \overrightarrow{b}])$$
$$\Longrightarrow [\overrightarrow{c}, \overrightarrow{d}] = [\overrightarrow{a}, \overrightarrow{b}]$$

[A *lemma* is a theorem whose main use is to simplify the proofs of other theorems.] This result should seem intuitively reasonable to

The explanation asked for preceding (2) is that since $D - P = (D - A) - (P - A)$ and $D - A = (D - P) + (P - A)$ and since $[B - A, C - A]$, being a vector space, is closed under subtraction and addition, it follows that if $D - A$ and $P - A$ belong to this space then so does $D - P$ and if $D - P$ and $P - A$ belong to it then so does $D - A$.

As to (**), the argument just used shows that, for example, if $P - A$ and $Q - A$ belong to $[B - A, C - A]$ then so does $P - Q$ — for, $P - Q = (P - A) - (Q - A)$.

This lemma is stated [and proved] on TC 299(2) where it is shown to be related to Exercise 5 of Part A on page 299. The proof given there is suggested to students in Exercise 2 of Part D on page 384. Another proof is suggested in Exercise 1. The third proof suggested here makes use of properties of determinants developed in the Background Exercises on pages 273 and 274. As pointed out in the text following the lemma, what needs to be proved is that if $(\vec{c}, \vec{d})$ is linearly independent then the equations:

$$(*)\qquad \begin{cases} \vec{c} = \vec{a}c_1 + \vec{b}c_2 \\ \vec{d} = \vec{a}d_1 + \vec{b}d_2 \end{cases}$$

can be solved for '$\vec{a}$' and '$\vec{b}$'. We note, first, that if $(\vec{a}, \vec{b})$ is linearly dependent then either $[\vec{a}, \vec{b}] = [\vec{a}]$ or $[\vec{a}, \vec{b}] = [\vec{b}]$ and, so, if (*) holds then either $\{\vec{c}, \vec{d}\} \subseteq [\vec{a}]$ or $\{\vec{c}, \vec{d}\} \subseteq [\vec{b}]$. And, in either case, $(\vec{c}, \vec{d})$ is linearly dependent. Since we are supposing that this is not the case — and that (*) holds — $(\vec{a}, \vec{b})$ is linearly independent. Now, by pages 273 and 274 [with the roles there assigned to $(\vec{c}, \vec{d})$ and $(\vec{a}, \vec{b})$ interchanged], if $(\vec{a}, \vec{b})$ is linearly independent and (*) holds then $(\vec{c}, \vec{d})$ is linearly independent if and only if $c_1d_2 - c_2d_1 \neq 0$. But, this is just the condition needed in order to be able to solve (*) for '$\vec{a}$' and '$\vec{b}$'. In short, if $(\vec{c}, \vec{d})$ is linearly independent then (*) can be solved for '$\vec{a}$' and '$\vec{b}$'.

Incidentally, the proof of Theorem 7-1 can be carried out along the lines indicated for the proof of Theorem 9-1. The analogue of the lemma which is needed for this proof of Theorem 7-1 is:

$$(\vec{c} \neq \vec{0} \text{ and } \vec{c} \in [\vec{a}]) \Longrightarrow [\vec{c}] = [\vec{a}]$$

The proof of this, like that just given for the lemma, involves showing that if $\vec{c} \neq \vec{0}$ then the equation '$\vec{c} = \vec{a}c$' can be solved for '$\vec{a}$'. This is fairly obvious, since, if $\vec{a}c \neq \vec{0}$ then $c \neq 0$. It is essentially this that is used, on page 283 in the "Derivation of (ii)".

Finally, note that the argument given above for the lemma shows that, assuming (*), $(\vec{c}, \vec{d})$ is linearly independent if and only if $(\vec{a}, \vec{b})$ is linearly independent and $c_1d_2 - c_2d_1 \neq 0$.

you. In the first place, if $\vec{c}$ and $\vec{d}$ are linear combinations of $\vec{a}$ and $\vec{b}$ – that is if

$$\begin{aligned} \vec{c} &= \vec{a}c_1 + \vec{b}c_2 \\ \vec{d} &= \vec{a}d_1 + \vec{b}d_2 \end{aligned} \tag{4}$$

for some values of 'c_1', 'c_2', 'd_1', and 'd_2' – then, clearly, any linear combination of $\vec{c}$ and $\vec{d}$ is also a linear combination of $\vec{a}$ and $\vec{b}$ – that is, $[\vec{c}, \vec{d}] \subseteq [\vec{a}, \vec{b}]$. And if, in addition, $(\vec{c}, \vec{d})$ is linearly independent then you might expect to be able to solve equations (4) to show that $\vec{a}$ and $\vec{b}$ are linear combinations of $\vec{c}$ and $\vec{d}$. The lemma can, indeed, be proved in this way. Another proof is developed in Part D of the exercises which follow. Pending this we shall take Theorem 9-1 as proved and investigate some of its consequences.

One immediate corollary of Theorem 9-1 is:

> There is one and only one plane which
> contains three given noncollinear points.

More briefly:

> **Corollary**
> Three noncollinear points determine [uniquely] a plane.

Exercises

Part A

1. (a) Show that if P and Q are two points of a plane π then there is a third point – say, R – such that $\pi = \overleftrightarrow{PQR}$.
   (b) Show that if $P \in \pi$ then there are points – say, Q and R – such that $\pi = \overleftrightarrow{PQR}$.
   (c) Show that, for any plane π, there are points – say, P, Q, and R – such that $\pi = \overleftrightarrow{PQR}$.
2. Prove:

> **Theorem 9-2**
> $(\{D, E, F\} \subseteq \overleftrightarrow{ABC}$ and $\{D, E, F\}$ is noncollinear$)$
> $$\longrightarrow \overleftrightarrow{ABC} = \overleftrightarrow{DEF}$$

3. Show that
   (a) $\overleftrightarrow{ABC} = \overleftrightarrow{ACB}$ and **(b)** $\overleftrightarrow{ABC} = \overleftrightarrow{BCA}$.

In order to insure that students understand the uses of Theorem 9-1, we recommend treating Part A of the exercises in class, followed by Parts B and C for homework. Part D can be rather complicated for some students so it is probably better to do these exercises in class also. Parts E and F make another nice homework set.

Answers for Part A

1. (a) Since π is a plane it contains three noncollinear points. For any three such points, at least one is not a point of $\overleftrightarrow{PQ}$. So there is a point of π — say, R — such that $\{P, Q, R\}$ is noncollinear. Since $\overleftrightarrow{PQR}$ is, then, the only plane which contains these points, $\pi = \overleftrightarrow{PQR}$.
   (b) As in (a) there are three noncollinear points in π — say, A, B, and C. If $P \notin \overleftrightarrow{AB}$ then let $Q = A$ and $R = B$. If $P \in \overleftrightarrow{AB}$ and $P \neq A$ then $P \notin \overleftrightarrow{AC}$. In this case take $Q = A$ and $R = C$. If $P = A$, take $Q = B$ and $R = C$. In all cases, $\{P, Q, R\}$ is a noncollinear subset of π and, as in part (a), $\pi = \overleftrightarrow{PQR}$.
   (c) Let P, Q, and R be any three noncollinear points of π. Then, as in (a), $\pi = \overleftrightarrow{PQR}$.
2. Suppose that $\{D, E, F\}$ is a noncollinear subset of $\overleftrightarrow{ABC}$. It follows that $\overleftrightarrow{ABC}$ is not a line or a set consisting of a single point. So, by the result of Exercise 4 of Part C on page 378, $\{A, B, C\}$ is not collinear. Hence, $\overleftrightarrow{ABC}$ is a plane. But, by Theorem 9-1, $\overleftrightarrow{DEF}$ is the only plane containing $\{D, E, F\}$. Hence, $\overleftrightarrow{ABC} = \overleftrightarrow{DEF}$.
3. (a) Since $[B - A, C - A] = [C - A, B - A]$ it follows by Exercise 1(a) of Part C on page 377 that $\overleftrightarrow{ABC} = \overleftrightarrow{ACB}$.
   (b) Since $C - A = (B - A) + (C - B)$ and $B - A = -(A - B)$, $[B - A, C - A] \subseteq [C - B, A - B]$. Similarly, $[C - B, A - B] \subseteq [B - A, C - A]$. So, $[B - A, C - A] = [C - B, A - B]$ and, as in part (a), $\overleftrightarrow{ABC} = \overleftrightarrow{BCA}$.

4. On your paper, draw a picture of the plane $\overline{MNP}$ and of points A, B, and C which are contained in $\overline{MNP}$. [*Question*: How must the points M, N, and P be related in order that $\overline{MNP}$ is a plane?]

(a) In your picture, locate each of the points D, E, F, G, H, and I, where

$$D = A + (B - C) \cdot -1, \qquad E = A + (B - C)\tfrac{1}{2},$$
$$F = M + (B - C) \cdot -1, \qquad G = M + (N - P)\tfrac{1}{2},$$
$$H = A + (C - P)\tfrac{1}{2}, \qquad I = P + (B - A).$$

(b) Which of the points given in (a) are in $\overline{MNP}$?

(c) Show that, for each x, $A + (B - A)x$ is a point in $\overline{MNP}$. What does this tell you about $\overleftrightarrow{AB}$ and $\overline{MNP}$?

5. Prove the following.

(a) If $\{A, B\} \subseteq \pi$ then $\overleftrightarrow{AB}$ is a subset of π.

(b) Each triangle is a subset of the plane containing its vertices.

6. Suppose that l_1 and l_2 are two lines which intersect in the point P.

(a) Show that there are three noncollinear points among the points of l_1 and l_2.

(b) Show that l_1 and l_2 are contained in exactly one plane.

7. Suppose that l is a line and Q is a point not on l.

(a) Show that $l \cup \{Q\}$ contains at least three noncollinear points.

(b) Show that l and Q are contained in exactly one plane.

(c) How many lines are parallel to l and contain Q? Show that each such line is a subset of the plane determined by l and Q.

8. Suppose that l and m are two parallel lines. How many planes contain both of these lines? Justify your answer.

*

In the preceding exercises you have proved several theorems which are worth recording:

> **Theorem 9-3**
> A plane contains the line determined by any two of its points.
>
> **Theorem 9-4**
> A line and a point not on that line determine a plane.
>
> **Theorem 9-5**
> Two intersecting lines determine a plane.
>
> **Theorem 9-6**
> Two parallel lines determine a plane.

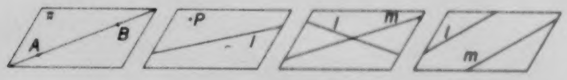
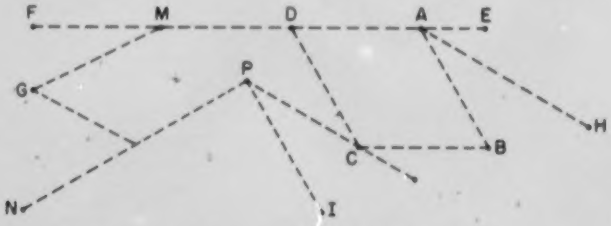


Fig. 9-3

Answers for Part A [cont.]

4. (a) [{M, N, P} must be noncollinear for $\overline{MNP}$ to be a plane.]



(b) All the points given in (a) belong to $\overline{MNP}$.

(c) Since $\{A, B\} \subseteq \overline{MNP}$, $A - M \in [N - M, P - M]$ and $B - M \in [N - M, P - M]$. Since $B - A = (B - M) - (A - M)$ it follows that $B - A \in [N - M, P - M]$ and, so, that $[B - A] \subseteq [N - M, P - M]$. Hence, if $C - A \in [B - A]$ then $C - M = (C - A) + (A - M) \in [N - M, P - M]$. So, for each x, if $C = A + (B - A)x$ then $C \in \overline{MNP}$. In brief, $\overleftrightarrow{AB} \subseteq \overline{MNP}$.

5. (a) By Exercise 1(c), $\pi = \overline{PQR}$, for properly chosen points P, Q, and R. By Exercise 4(c), if $\{A, B\} \subseteq \overline{PQR}$ then $\overleftrightarrow{AB} \subseteq \overline{PQR}$. Hence, if $\{A, B\} \subseteq \pi$ then $\overleftrightarrow{AB} \subseteq \pi$.

(b) $\triangle ABC \subseteq \overleftrightarrow{AB} \cup \overleftrightarrow{BC} \cup \overleftrightarrow{CA}$ and, by part (a), each of these lines is a subset of the plane $\overline{ABC}$ which contains noncollinear points A, B, and C.

6. (a) Let Q and R be points of ℓ_1 and ℓ_2, respectively, which are different from P. Then $R \notin \ell_1 = \overleftrightarrow{PQ}$ and, so, P, Q, and R are three noncollinear points.

(b) $\overline{PQR}$ is the only plane which contains P, Q, and R; so, it is the only plane which can contain ℓ_1 $[= \overleftrightarrow{PQ}]$ and ℓ_2 $[= \overleftrightarrow{PR}]$. But, since $\{P, Q\} \subseteq \overline{PQR}$, $\ell_1 \subseteq \overline{PQR}$, and, similarly $\ell_2 \subseteq \overline{PQR}$.

7. (a) Let P and R be two points of ℓ. Since $Q \notin \ell = \overleftrightarrow{PR}$, {P, Q, R} is noncollinear.

(b) Since $\{P, R\} \subseteq \overline{PQR}$, $\ell = \overleftrightarrow{PR} \subseteq \overline{PQR}$. Since $Q \in \overline{PQR}$, $\ell \cup \{Q\} \subseteq \overline{PQR}$. But, any plane which contains $\ell \cup \{Q\}$ contains {P, Q, R} and the only such plane is $\overline{PQR}$.

(c) There is just one line, $\overleftrightarrow{Q[\ell]}$ which contains Q and is parallel to ℓ. A point C belongs to this line if and only if $C - Q \in [\ell] = [P - R]$. Since $[P - R] \subseteq [P - Q, R - Q]$, any such point C belongs to $\overline{QPR}$. But, $\overline{QPR} = \overline{PQR}$. Hence, $\overleftrightarrow{Q[\ell]}$ is a subset of the plane $\overline{PQR}$ determined by ℓ and Q.

8. Let $Q \in m$. Since $m \parallel \ell$, $m = \overleftrightarrow{Q[\ell]}$ and, by Exercise 7, is a subset of the plane determined by ℓ and Q. No other plane can contain both ℓ and m because no other contains ℓ and Q.

[Note that, for example, Theorem 9–4 is short for:

> Given a line and a point not on it, there is one and only one plane to which the given point belongs and of which the given line is a subset.

Restate Theorems 9–5 and 9–6 in this more explicit form. Note, also, that it follows from Theorem 9–6 that any two parallel lines are coplanar.]

Part B

1. A *plane quadrilateral* is a quadrilateral which is a subset of some plane. Show that
   - **(a)** trapezoids are plane quadrilaterals [What about parallelograms?],
   - **(b)** convex quadrilaterals are plane quadrilaterals,
   - **(c)** nonsimple quadrilaterals are plane quadrilaterals,
   - **(d)** if a quadrilateral is not a plane quadrilateral then it is both simple and nonconvex.
2. Is the converse of 1(d) a theorem? Justify your answer.
3. Show that lines which are transversals of the same two parallel lines are coplanar.

*

Although if two lines are parallel then they have no point in common, the converse is not true. [Explain.] So, given two lines, one cannot [in general] conclude that they either are parallel or have a nonempty intersection. In spite of this, we discovered in Chapter 8 many situations in which we were able to draw this conclusion. [Theorem 8–2 is a case in point.] In each case we were able to do this because we had additional information about the lines in question. It is now easy to see that this additional information implied that the given lines were coplanar. [Check this in the case of Theorem 8–2.] This suggests that we might be able to prove:

Theorem 9–7
Two nonparallel coplanar lines intersect.

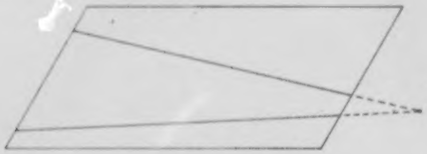


Fig. 9–4

Theorem 9-5:

> Given two intersecting lines, there is one and only one plane of which both are subsets.

Theorem 9-6:

> Given two parallel lines, there is one and only one plane of which both are subsets.

Answers for Part B

1. (a) The lines containing two bases of a trapezoid are subsets of a plane. Since the vertices of the trapezoid belong to these lines, the lines containing the other sides are subsets of this plane. Since the trapezoid is a subset of the union of these four lines, it is, also, a subset of the plane in question. [Parallelograms are trapezoids.]
   - (b) The lines containing the diagonals of a convex quadrilateral are subsets of some plane. It follows that the vertices of the quadrilateral belong to this plane and, so, as in part (a), the quadrilateral is a subset of the plane.
   - (c) [Like (b), but start with lines containing two intersecting sides.]
   - (d) This follows [by contraposition] from parts (b) and (c).

*

That two parallel lines have no common point follows from Exercise 1(e) of Part C on page 293. Lines with no point in common may be skew.

The lines referred to in Theorem 8-2 are all subsets of the plane determined by the parallel lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$.

2. No. [There are simple nonconvex quadrilaterals which are plane quadrilaterals.]

3. Two parallel lines determine a [unique] plane and any line which is a transversal of both is a subset of that plane. All such transversals will thus be coplanar.

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Answers for Part C

1. Show that $C - A \in [B - A, C - D]$.
2. $C \in \overleftrightarrow{CD}$ but, by hypothesis, $C \notin \overleftrightarrow{AB}$. It follows that $\{A, B, C\}$ is a noncollinear subset of any plane which contains $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ and, hence, that the only such plane is $\overline{ABC}$.
3. That $D - A \in [B - A, C - A]$.
4. That $C - D \in [B - A, C - A]$.
5. No. But we also know, by hypothesis that $\overleftrightarrow{CD} \nparallel \overleftrightarrow{AB}$. So, $[C - D] \neq [B - A]$ and it follows, since $C - D \neq \vec{0}$, that $C - D \notin [B - A]$. [This is fairly obvious and easily proved, but, if a reference is needed, it is to Exercise 4 of Part B on page 262.] Since $C - D \notin [B - A]$ and $B - A \neq \vec{0}$ it follows by Theorem 6-13 that $(B - A, C - D)$ is not linearly dependent. But, by the preceding Exercise 4, $(B - A, C - D, C - A)$ is linearly dependent. So, by Theorem 6-13, $C - A \in [B - A, C - D]$. [Hence, by Theorem 8-1, $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} \neq \emptyset$.] [A different argument is given in Exercise 2(a) of Part D, below.]

Part C

Suppose that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are coplanar lines such that $C \notin \overleftrightarrow{AB}$ and $\overleftrightarrow{CD} \not\parallel \overleftrightarrow{AB}$.

1. What does Theorem 8–1 suggest as a way of proving that $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} \neq \emptyset$?
2. Show that $\overleftrightarrow{AB} \neq \overleftrightarrow{CD}$, and that the plane containing $\overleftrightarrow{AB} \cup \overleftrightarrow{CD}$ is $\overline{ABC}$.
3. It follows from Exercise 2 that $D \in \overline{ABC}$. What does this tell you about $D - A$?
4. It is also the case that $C - A \in [B - A, C - A]$. From this and Exercise 3, what can you conclude about $C - D$?
5. Compare your answers for Exercises 1 and 4. Can you derive the former from the latter alone? If you can, do so. If not, do you have any additional information from which, together with the result of Exercise 4, you can derive your answer for Exercise 1?
6. Use what you have established in Exercises 1–5 to prove Theorem 9–7. [No further algebra is needed.]
7. Prove:

> **Theorem 9–8**
> Two lines are parallel if and only if they are coplanar and have no common point.

[Theorem 9–8 is often used as a definition of parallelism for lines.]

8. Show that a line which is coplanar with two parallel lines and is a transversal of one of them is also a transversal of the other.

Part D

We have still to prove the lemma to which we reduced the proof of Theorem 9–1:

$$((\vec{c}, \vec{d}) \text{ is linearly independent and } \{\vec{c}, \vec{d}\} \subseteq [\vec{a}, \vec{b}]) \longrightarrow [\vec{c}, \vec{d}] = [\vec{a}, \vec{b}]$$

[Incidentally, we shall have other uses for this lemma later.] The following exercises (and hints) outline two proofs of the lemma.

1. Suppose that $(\vec{c}, \vec{d})$ is linearly independent and $\{\vec{c}, \vec{d}\} \subseteq [\vec{a}, \vec{b}]$.
   - **(a)** Why does it follow that $[\vec{c}, \vec{d}] \subseteq [\vec{a}, \vec{b}]$?
   - **(b)** What remains to be derived from the assumption? What does your answer for part (a) suggest as to how to go about deriving this?
   - **(c)** The linear independence of $(\vec{c}, \vec{d})$ should remind you of an earlier theorem which will allow you to carry out the suggestion of part (b) in one case.
   - **(d)** Complete the proof of the lemma by showing that if $\{\vec{c}, \vec{d}\} \subseteq [\vec{a}, \vec{b}]$ then $(\vec{c}, \vec{d}, \vec{a})$ is linearly dependent.

6. Suppose that ℓ and m are two coplanar nonparallel lines. Let A and B be two points of ℓ, C a point of m which is not on ℓ, and D a point of m different from C. Then $\ell = \overleftrightarrow{AB}$, $m = \overleftrightarrow{CD}$, and the hypotheses for Exercises 1 - 5 are satisfied. So, by Exercise 5, it follows that $\ell \cap m \neq \emptyset$.

7. By Theorem 9-6, two parallel lines are coplanar and, by Theorem 7-6 [or Exercise $\overline{1(e)}$ of Part C on page 293], they have no common point. On the other hand, by Theorem 9-7, coplanar lines which have no common point cannot be nonparallel.

8. Suppose that ℓ and m are two parallel lines and let π be the plane which contains them. Suppose that n is a line which is contained in π and intersects ℓ at a single point. It follows that $n \not\parallel \ell$ and [by transitivity of parallelism] $n \not\parallel m$. It also follows, since $\ell \cap m = \emptyset$, that $n \neq m$. So, by Theorem 9-7, $n \cap m \neq \emptyset$ [and, of course, consists of a single point].

Answers for Part D

1. (a) Since, by hypothesis, $\vec{c}$ and $\vec{d}$ are linear combinations of $\vec{a}$ and $\vec{b}$, so is any linear combination of $\vec{c}$ and $\vec{d}$. [It may be well to call attention to the fact that this follows from Postulates 4_7, 4_8, the ''twist principle'' '$(\vec{a} + \vec{b}) + (\vec{c} + \vec{d}) = (\vec{a} + \vec{c}) + (\vec{b} + \vec{d})$', and Postulate 4_6.]

   (b) That $[\vec{a}, \vec{b}] \subseteq [\vec{c}, \vec{d}]$.; Show that $\{\vec{a}, \vec{b}\} \subseteq [\vec{c}, \vec{d}]$.

   (c) In case $(\vec{c}, \vec{d}, \vec{a})$ is linearly dependent it follows by Theorem 6-13 that, since $(\vec{c}, \vec{d})$ is linearly independent, $\vec{a} \in [\vec{c}, \vec{d}]$. Similarly, if $(\vec{c}, \vec{d}, \vec{b})$ is linearly dependent then $\vec{b} \in [\vec{c}, \vec{d}]$.

   (d) Suppose that $\vec{c} = \vec{a}c_1 + \vec{b}c_2$ and $\vec{d} = \vec{a}d_1 + \vec{b}d_2$. It follows that $\vec{c}d_2 - \vec{d}c_2 = \vec{a}(a_1d_2 - d_1c_2)$. Now, supposing that $(\vec{c}, \vec{d}, \vec{a})$ is linearly independent, it follows that $c_2 = d_2 = 0$ and so, by our original assumption, $\{\vec{c}, \vec{d}\} \subseteq [\vec{a}]$, and $(\vec{c}, \vec{d})$ is linearly dependent. But, supposing that $(\vec{c}, \vec{d}, \vec{a})$ is linearly independent, $(\vec{c}, \vec{d})$ is linearly independent. From this contradiction it follows that $(\vec{c}, \vec{d}, \vec{a})$ is not linearly independent. [Similarly, $(\vec{c}, \vec{d}, \vec{b})$ is not linearly independent. Hence, if $\{\vec{c}, \vec{d}\} \subseteq [\vec{a}, \vec{b}]$ then $(\vec{c}, \vec{d}, \vec{a})$ and $(\vec{c}, \vec{d}, \vec{b})$ are linearly dependent. So, by part (c), if, in addition, $(\vec{c}, \vec{d})$ is linearly independent then $\{\vec{a}, \vec{b}\} \subseteq [\vec{c}, \vec{d}]$. It follows that $[\vec{a}, \vec{b}] \subseteq [\vec{c}, \vec{d}]$ and so, by part (a), that $[\vec{c}, \vec{d}] = [\vec{a}, \vec{b}]$.]

2. Another proof of the lemma starts from the argument you may have used in Exercise 5 of Part C. The result established there is stated in part (a), below. Prove each of the following.
   (a) $(\vec{c} \in [\vec{a}, \vec{b}] \text{ and } \vec{c} \notin [\vec{a}]) \longrightarrow \vec{b} \in [\vec{a}, \vec{c}]$
   (b) $(\vec{c} \in [\vec{a}, \vec{b}] \text{ and } \vec{c} \notin [\vec{a}]) \longrightarrow [\vec{a}, \vec{c}] = [\vec{a}, \vec{b}]$
   (c) $(\{\vec{c}, \vec{d}\} \subseteq [\vec{a}, \vec{b}], \vec{c} \notin [\vec{a}], \text{and } \vec{d} \notin [\vec{c}]) \longrightarrow [\vec{c}, \vec{d}] = [\vec{a}, \vec{b}]$ [*Hint*: Use part (b) twice – the second time with '$\vec{c}$' for '$\vec{a}$'.]
   (d) $((\vec{c}, \vec{d}) \text{ is linearly independent and } \{\vec{c}, \vec{d}\} \subseteq [\vec{a}, \vec{b}]) \longrightarrow [\vec{c}, \vec{d}] = [\vec{a}, \vec{b}]$ [*Hint*: Use part (c) twice – the second time with '$\vec{a}$' and '$\vec{b}$' interchanged. This will take care of all cases except that in which $\vec{c} \in [\vec{a}]$ and $\vec{c} \in [\vec{b}]$. Show that this case cannot occur.]

3. Prove:
   (a) $((\vec{c}, \vec{d}) \text{ is linearly independent and } \{\vec{c}, \vec{d}\} \subseteq [\vec{a}, \vec{b}]) \longrightarrow (\vec{a}, \vec{b})$ is linearly independent.
   (b) $\{\vec{c}, \vec{d}, \vec{e}\} \subseteq [\vec{a}, \vec{b}] \longrightarrow (\vec{c}, \vec{d}, \vec{e})$ is linearly dependent. [*Hint*: Use the lemma proved in Exercises 1 and 2.]

Part E

In these exercises we shall be concerned with the translations which map a given plane π into itself. Since, by Definition 7–7 [see page 295] the image of π under the translation $\vec{a}$ is $\pi + \vec{a}$, the translations we are interested in are those for which $\pi + \vec{a} \subseteq \pi$. Before reading further, your experience with lines may suggest to you what we are about to discover, and why it is of interest.

1. Suppose that $\pi + \vec{a} \subseteq \pi$. Show that there are points – say, Q and R which belong to π and are such that $\vec{a} = R - Q$.
2. Suppose that P, Q, and R are points of π. Show that $P + (R - Q) \in \pi$. [*Hint*: Consider the case in which $Q = R$ and the case in which $Q \neq R$.]
3. Prove:

$$\pi + \vec{a} \subseteq \pi \longleftrightarrow \exists_Y \exists_Z\, (Y \in \pi \text{ and } Z \in \pi \text{ and } \vec{a} = Z - Y)$$

Part F

Suppose that $\{A, B, C\}$ is a noncollinear subset of a plane π.

1. Show that if P and Q belong to π then $Q - P \in [B - A, C - A]$.
2. Show that if $\vec{a} \in [B - A, C - A]$ then there is a point – say, P – such that $P \in \pi$ and $P - A = \vec{a}$.
3. Conclude from Exercises 1 and 2 that

$$[B - A, C - A] = \{\vec{x}\colon \exists_Y \exists_Z\, (Y \in \pi \text{ and } Z \in \pi \text{ and } \vec{x} = Z - Y)\}$$

4. Show that if $P \in \pi$ and $\vec{a} \in [B - A, C - A]$ then $P + \vec{a} \in \pi$.

Answers for Part D [cont.]

2. (a) Suppose that $\vec{c} \in [\vec{a}, \vec{b}]$ and $\vec{c} \notin [\vec{a}]$. It follows that there are numbers — say, c_1 and c_2 — such that $\vec{c} = \vec{a}c_1 + \vec{b}c_2$ and $c_2 \neq 0$. So, $\vec{b} = \vec{a}\cdot(-c_1/c_2) + \vec{c}\cdot/c_2$, and $\vec{b} \in [\vec{a}, \vec{c}]$.
   (b) Suppose that $\vec{c} \in [\vec{a}, \vec{b}]$ and $\vec{c} \notin [\vec{a}]$. Since $\vec{a} \in [\vec{a}, \vec{b}]$ and $\vec{c} \in [\vec{a}, \vec{b}]$, $[\vec{a}, \vec{c}] \subseteq [\vec{a}, \vec{b}]$. On the other hand, by part (a), $\vec{b} \in [\vec{a}, \vec{c}]$. So, since $\vec{a} \in [\vec{a}, \vec{c}]$, $[\vec{a}, \vec{b}] \subseteq [\vec{a}, \vec{c}]$. Hence, $[\vec{a}, \vec{c}] = [\vec{a}, \vec{b}]$.
   (c) Suppose that $\{\vec{c}, \vec{d}\} \subseteq [\vec{a}, \vec{b}]$, $\vec{c} \notin [\vec{a}]$, and $\vec{d} \notin [\vec{c}]$. Since $\vec{c} \in [\vec{a}, \vec{b}]$ and $\vec{c} \neq \vec{a}$ it follows by part (b) that $[\vec{a}, \vec{c}] = [\vec{a}, \vec{b}]$. Since $\vec{d} \in [\vec{a}, \vec{b}] = [\vec{a}, \vec{c}] = [\vec{c}, \vec{a}]$ and $\vec{d} \notin [\vec{c}]$ it follows by part (b) that $[\vec{c}, \vec{d}] = [\vec{c}, \vec{a}] = [\vec{a}, \vec{b}]$.
   (d) Suppose that $(\vec{c}, \vec{d})$ is linearly independent and $\{\vec{c}, \vec{d}\} \subseteq [\vec{a}, \vec{b}]$. It follows that $\vec{d} \notin [\vec{c}]$ and so, by part (c), if $\vec{c} \notin [\vec{a}]$ then $[\vec{c}, \vec{d}] = [\vec{a}, \vec{b}]$. Similarly, if $\vec{c} \notin [\vec{b}]$ then $[\vec{c}, \vec{d}] = [\vec{b}, \vec{a}] = [\vec{a}, \vec{b}]$. Since $(\vec{c}, \vec{d})$ is linearly independent, $\vec{c} \neq \vec{0}$. So, if $\vec{c} \in [\vec{a}]$ and $\vec{c} \in [\vec{b}]$ then $[\vec{a}] = [\vec{c}] = [\vec{b}]$. Since $[\vec{a}] = [\vec{b}]$, $[\vec{a}, \vec{b}] = [\vec{a}]$. Since $\{\vec{c}, \vec{d}\} \subseteq [\vec{a}, \vec{b}]$ it follows that if $\vec{c} \in [\vec{a}]$ and $\vec{c} \in [\vec{b}]$ then $\{\vec{c}, \vec{d}\} \subseteq [\vec{a}]$. But, since $(\vec{c}, \vec{d})$ is linearly independent, $(\vec{c}, \vec{d}) \not\subseteq [\vec{a}]$. Hence, it is not the case that $\vec{c} \in [\vec{a}]$ and $\vec{c} \in [\vec{b}]$.

3. (a) If $(\vec{a}, \vec{b})$ is linearly dependent then either $[\vec{a}, \vec{b}] = [\vec{a}]$ or $[\vec{a}, \vec{b}] = [\vec{b}]$ and, in either case, $[\vec{a}, \vec{b}]$ cannot contain linearly independent vectors $\vec{c}$ and $\vec{d}$.
   (b) Suppose that $\{\vec{c}, \vec{d}, \vec{e}\} \subseteq [\vec{a}, \vec{b}]$. In case $(\vec{c}, \vec{d})$ is linearly dependent, so is $(\vec{c}, \vec{d}, \vec{e})$. In case $(\vec{c}, \vec{d})$ is linearly independent then, by the lemma, $[\vec{c}, \vec{d}] = [\vec{a}, \vec{b}]$. In this case, since $\vec{e} \in [\vec{a}, \vec{b}]$, $\vec{e} \in [\vec{c}, \vec{d}]$ and, so, $(\vec{c}, \vec{d}, \vec{e})$ is linearly dependent.

Answers for Part E

[The purpose of these exercises is to prepare the way for section 9.04.]

1. Let Q be any point of π. Since $\pi + \vec{a} \subseteq \pi$, $Q + \vec{a} \in \pi$. Let $R = Q + \vec{a}$. Then Q and R are points of π and $\vec{a} = (Q + \vec{a}) - Q = R - Q$.

2. In case $Q = R$, $P + (R - Q) = P \in \pi$. Suppose that $Q \neq R$. By Theorem 9-3, $\overleftrightarrow{QR} \subseteq \pi$. In case $P \in \overleftrightarrow{QR}$, so does $P + (R - Q)$. In case $P \notin \overleftrightarrow{QR}$ it follows that $\pi = \overline{PQR}$ and $P + (R - Q) = P + (Q - P) \cdot -1 + (R - P)1 \in \pi$.

3. The only if-part is established in Exercise 1. The if-part is established in Exercise 2. [By Exercise 2, if Q and R are any points of π then $\pi + (R - Q) \subseteq \pi$.]

Answers for Part F

$\{A, B, C\}$ is a noncollinear subset of π, $\pi = \overline{ABC}$.

1. $P \in \pi \Longleftrightarrow P - A \in [B - A, C - A]$ and $Q \in \pi \Longleftrightarrow Q - A \in [B - A, C - A]$. Hence, if P and Q are points of π then $Q - P = (Q - A) - (P - A) \in [B - A, C - A]$.

2. Suppose that $\vec{a} \in [B - A, C - A]$. Let P be any point of π. It follows that $A + \vec{a} \in \pi$. So, if $P = A + \vec{a}$ then $P \in \pi$ and $P - A = (A + \vec{a}) - A = \vec{a}$.

3. Since $A \in \pi$ it follows by Exercise 2 that each member of $[B - A, C - A]$ is a difference of points of π. By Exercise 1, any difference of points of π belongs to $[B - A, C - A]$.

4. Suppose that $P \in \pi$ and $\vec{a} \in [B - A, C - A]$. Since $P \in \pi$, $P - A \in [B - A, C - A]$. So, since $\vec{a} \in [B - A, C - A]$, $(P - A) + \vec{a} \in [B - A, C - A]$. But, $(P - A) + \vec{a} = (P + \vec{a}) - A$ and, since $(P + \vec{a}) - A \in [B - A, C - A]$, $P + \vec{a} \in \pi$.

9.04 Directions of Planes

In Chapter 7 we chose to define the direction of a line l as a set of translations. As it turned out, a translation $\vec{a}$ belongs to the direction of l if and only if it maps l onto itself. It seems intuitively reasonable to say that a translation is in the direction of a given plane if and only if it maps that plane onto itself. As you have seen in the exercises of Part E on page 384, these translations are just those from a first point of the given plane to a second. Modeling our notation after that which we used for lines, we shall adopt:

> Definition 9–4
> $[\pi] = \{\vec{x}: \exists_Y \exists_Z(Y \in \pi \text{ and } Z \in \pi \text{ and } \vec{x} = Z - Y)\}$

and read '$[\pi]$' as 'the direction of π'. In words, Definition 9–4 says that the direction of π is the set of all translations determined by points of π.

In Part F you have proved:

> Theorem 9–9
> $\{A, B, C\}$ is a noncollinear subset of π
> $$\Longrightarrow [\pi] = [B - A, C - A]$$

and:

> Theorem 9–10
> $(P \in \pi \text{ and } \vec{a} \in [\pi]) \Longrightarrow P + \vec{a} \in \pi$

What are the analogous theorems about lines?

Exercises

Part A

1. Is there any translation which belongs to the direction of every plane? Do you think that there is more than one such translation? Explain.
2. **(a)** Suppose that $\vec{a}$ and $\vec{b}$ belong to $[\pi]$ and that $(\vec{a}, \vec{b})$ is linearly independent. What else can you infer about $[\pi]$, $\vec{a}$, and $\vec{b}$?
   (b) Suppose that $\vec{a}$, $\vec{b}$, and $\vec{c}$ belong to $[\pi]$. What can you infer about $(\vec{a}, \vec{b}, \vec{c})$?
   (c) Suppose that $[\pi] = [\vec{a}, \vec{b}]$. What can you infer about $(\vec{a}, \vec{b})$?
   (d) Suppose that $[\pi] = [\vec{a}, \vec{b}, \vec{c}]$. What can you infer about $(\vec{a}, \vec{b}, \vec{c})$? What else?

The treatment here of directions of planes is analogous to that of directions of lines in section 7.04. In particular, Definition 9-4 is analogous to Definition 7-4, Theorem 9-9 to Theorem 7-3, and Theorem 9-10 to Theorem 7-4.

Answers for Part A

1. Yes.; $\vec{0}$ belongs to the direction of every plane. If $\vec{a} \neq \vec{0}$ and belongs to the direction of every plane then, by Theorem 9-10, every plane containing a given point P would contain $P + \vec{a}$ and, by Theorem 9-3, would contain the line $P[\vec{a}]$. However, on intuitive grounds, given any line ℓ through P, there is a plane through P which does not contain this line. So, on the basis of this intuition, there is no non-$\vec{0}$ translation which belongs to the direction of every plane. [Of course, if the intuitive considerations governing our development of geometry were concerned with a space of fewer than three dimensions, we would obtain a different answer.]

2. (a) $[\pi] = [\vec{a}, \vec{b}]$. For, π contains three noncollinear points and, if A, B, and C are three such points it follows by Theorem 9-3 that $[\pi] = [B - A, C - A]$. But, by the lemma on page 379 [proved in Part D on page 383], if $(\vec{a}, \vec{b})$ is linearly independent and $\{\vec{a}, \vec{b}\} \subseteq [B - A, C - A]$ then $[\vec{a}, \vec{b}] = [B - A, C - A]$.
   (b) $(\vec{a}, \vec{b}, \vec{c})$ is linearly dependent. [See Exercise 3(b) on page 384.]
   (c) $(\vec{a}, \vec{b})$ is linearly independent. Since π contains three noncollinear points, $[\pi]$ contains two linearly independent translations. So, if $[\pi] = [\vec{a}, \vec{b}]$, it follows by Exercise 3(a) on page 384 that $(\vec{a}, \vec{b})$ is linearly independent.
   (d) By part (b), $(\vec{a}, \vec{b}, \vec{c})$ is linearly dependent. It follows that one of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is a linear combination of the others and, so, that $[\vec{a}, \vec{b}, \vec{c}]$ is either $[\vec{a}, \vec{b}]$, $[\vec{b}, \vec{c}]$, or $[\vec{c}, \vec{a}]$. So, by part (c), $(\vec{a}, \vec{b})$, $(\vec{b}, \vec{c})$, or $(\vec{c}, \vec{a})$ is linearly independent.

Answers for Part B

1. (a) Yes, if they are collinear [and there are noncoplanar points].; No, by Theorem 9-1.
   (b) Suppose that $\pi \subseteq \sigma$ and let A, B, and C be three noncollinear points of π. [That there are such follows from Definition 9-2.] Since $\pi \subseteq \sigma$, these points also belong to σ. So, by Theorem 9-1, $\pi = \overleftrightarrow{ABC} = \sigma$.

2. (a) Since $(B - A, C - A) = ((B + \vec{a}) - (A + \vec{a}), (C + \vec{a}) - (A + \vec{a}))$, $(B - A, C - A)$ is linearly dependent if and only if $((B + \vec{a}) - (A + \vec{a}), (C + \vec{a}) - (A + \vec{a}))$ is linearly dependent. Hence, $\{A, B, C\}$ is collinear if and only if $\{A + \vec{a}, B + \vec{a}, C + \vec{a}\}$ is collinear.
   (b) In answer for part (a), replace 'dependent' by 'independent' and 'collinear' by 'noncollinear'. [Or, make use of the logical equivalence of sentences of the forms '$p \Longleftrightarrow q$' and 'not $p \Longleftrightarrow$ not q'.]

Part B

1. **(a)** Can two planes have three points in common? Three noncollinear points? Explain.
   (b) Suppose that $\pi \subseteq \sigma$, where π and σ are planes. Show that $\pi = \sigma$.
2. **(a)** Show that $\{A, B, C\}$ is collinear if and only if $\{A + \vec{a}, B + \vec{a}, C + \vec{a}\}$ is collinear.
   (b) Repeat part (a) with 'collinear' replaced by 'noncollinear'.
3. **(a)** Show that, for any plane π and any translation $\vec{a}$, $\pi + \vec{a}$ is a plane. [*Hint*: There are noncollinear points—say, A, B, and C—such that $\pi = \overleftrightarrow{ABC}$. [Explain.] You may find Theorem 7–9 helpful.]
   (b) Compare $[\pi]$ and $[\pi + \vec{a}]$.
   (c) Show that $\vec{a} \in [\pi] \longleftrightarrow \pi + \vec{a} = \pi$. [*Hint*: Use a result you obtained on page 384.]

Part C

1. **(a)** Suppose that $(\vec{a}, \vec{b})$ is linearly independent. Describe the set $\{X: X - A \in [\vec{a}, \vec{b}]\}$. [*Hint*: Postulate 2(b) may be of use.]
   (b) What is the direction of the plane described in part (a)?
2. Suppose that $A \in \pi$.
   (a) Show that there are points—say, B and C—such that $[\pi] = [B - A, C - A]$.
   (b) Show that $\{X: X - A \in [\pi]\} = \pi$.
   (c) Show that if $D \in \mathscr{E}$ then $\{X: X - D \in [\pi]\}$ is a plane. What is its direction?
3. **(a)** Show that if π and σ are planes with the same direction and $\pi \cap \sigma \neq \varnothing$ then $\pi = \sigma$.
   (b) What follows from part (a) concerning two planes which have the same direction?

9.05 Planes with a Given Direction

In Exercise 1 of Part C you have seen that if $(\vec{a}, \vec{b})$ is linearly independent then there are planes whose direction is $[\vec{a}, \vec{b}]$. In fact, for any point A, $\overleftrightarrow{A(A + \vec{a})(A + \vec{b})}$ is such a plane. This suggests that we refer to $[\vec{a}, \vec{b}]$ as a *bidirection*—the bidirection determined by [or, simply: of] $\vec{a}$ and $\vec{b}$. In analogy with our previous use of 'direction' as applied to single translations, we shall speak of $[\vec{a}, \vec{b}]$ as a bidirection even in case $(\vec{a}, \vec{b})$ is linearly dependent. When we wish to imply that $(\vec{a}, \vec{b})$ is linearly independent we shall call $[\vec{a}, \vec{b}]$ a *proper bidirection*.

In analogy with Definition 7–5, we adopt:

Definition 9–5 (a) $\overleftrightarrow{A[\vec{a}, \vec{b}]} = \{X: X - A \in [\vec{a}, \vec{b}]\}$
(b) $\overleftrightarrow{A[\pi]} = \{X: X - A \in [\pi]\}$

3. (a) Since π is a plane it follows from Definition 9-2 that π contains three noncollinear points — say, A, B, and C. So, by Theorem 9-1, $\pi = \overleftrightarrow{ABC}$. By Theorem 7-9, $D \in \pi + \vec{a}$ if and only if $D - \vec{a} \in \overleftrightarrow{ABC}$. By Definition 9-3, the latter is the case if and only if $(D - \vec{a}) - A \in [B - A, C - A]$. Since $(D - \vec{a}) - A = D - (A + \vec{a})$, $B - A = (B + \vec{a}) - (A + \vec{a})$, and $C - A = (C + \vec{a}) - (A + \vec{a})$, it follows that $D \in \pi + \vec{a}$ if and only if $D - (A + \vec{a}) \in [(B + \vec{a}) - (A + \vec{a}), (C + \vec{a}) - (A + \vec{a})]$ — that is, if and only if $D \in \overleftrightarrow{(A + \vec{a})(B + \vec{a})(C + \vec{a})}$. So, $\pi + \vec{a} = \overleftrightarrow{(A + \vec{a})(B + \vec{a})(C + \vec{a})}$. Since $\{A, B, C\}$ is noncollinear, so is $\{A + \vec{a}, B + \vec{a}, C + \vec{a}\}$. Hence, $\pi + \vec{a}$ is a plane.

   (b) $[\pi] = [\pi + \vec{a}]$, by the preceding and Theorem 9-9.

   (c) By Exercise 3 of Part E on page 384 and Definition 9-4, $\pi + \vec{a} \subseteq \pi$ if and only if $\vec{a} \in [\pi]$. By Exercise 1(b), above, since $\pi + \vec{a}$ is a plane, $\pi + \vec{a} \subseteq \pi$ if and only if $\pi + \vec{a} = \pi$. Hence, $\vec{a} \in [\pi]$ if and only if $\pi + \vec{a} = \pi$.

Answers for Part C

1. (a) Since $\vec{a} = (A + \vec{a}) - A$ and $\vec{b} = (A + \vec{b}) - A$ it follows by Definition 9-3 that $\{X: X - A \in [\vec{a}, \vec{b}]\} = \overleftrightarrow{A(A + \vec{a})(A + \vec{b})}$. Since $(\vec{a}, \vec{b})$ is linearly independent, $\{A, A + \vec{a}, A + \vec{b}\}$ is noncollinear. So, the set in question is a plane.

   (b) By Theorem 9-9, the direction of the plane in question is $[\vec{a}, \vec{b}]$.

2. (a) By Exercise 1(b) of Part A on page 380, points B and C can be chosen in π such that $\pi = \overleftrightarrow{ABC}$. $\{A, B, C\}$ is, of course, noncollinear and so, by Theorem 9-9, $[\pi] = [B - A, C - A]$.

   (b) In the notation of part (a), $\pi = \overleftrightarrow{ABC} = \{X: X - A \in [B - A, C - A]\} = \{X: X - A \in [\pi]\}$.

   (c) $\{X: X - D \in [\pi]\} = \{X: (X - A) - (D - A) \in [\pi]\} = \{X: (X - (D - A)) - A \in [\pi]\} = \{X: X - (D - A) \in [\pi]\} = \pi + (D - A)$. By Exercise 3(a) of Part B, this set is a plane. By Exercise 3(b), its direction is that of π.

3. (a) Suppose that π and σ are planes such that $[\pi] = [\sigma]$ and that $A \in \pi \cap \sigma$. By Exercise 2(b), $\pi = \{X: X - A \in [\pi]\}$ and $\sigma = \{X: X - A \in [\sigma]\}$. So, $\pi = \sigma$.

   (b) Two planes which have the same direction have no common point.

This section is analogous to part of section 7.05. In particular, Definition 9-5 is analogous to Definition 7-5, Theorem 9-11 to Theorem 7-5. Parallelism is treated in section 9.06 which is analogous to some of the remainder of section 7.05.

Note that Exercises 1 and 2 of Part C show that $\overleftrightarrow{A[\vec{a}, \vec{b}]}$ and $\overleftrightarrow{A[\pi]}$ are planes through A which have the directions specified in Theorem 9-11. Exercise 3(a) of Part C shows that there are no other such planes.

From the exercises of Part C we have:

Theorem 9-11
(a) For $(\vec{a}, \vec{b})$ linearly independent, $\overline{A[\vec{a}, \vec{b}]}$ is the plane through A with the bidirection $[\vec{a}, \vec{b}]$.
(b) $\overline{A[\pi]}$ is the plane through A with the direction of π.

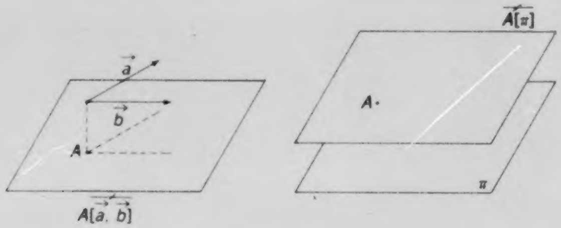


Fig. 9-5

Exercises

Part A

1. On defining 'proper bidirection' we should check to see that if $[\vec{c}, \vec{d}] = [\vec{a}, \vec{b}]$ and $[\vec{c}, \vec{d}]$ is a proper bidirection then so is $[\vec{a}, \vec{b}]$. What earlier result tells us that this is the case?
2. Prove each part of Theorem 9-11 by referring to earlier exercises.
3. Show, similarly, that $A \in \pi$ if and only if $\pi = \overline{A[\pi]}$.
4. Suppose that $\vec{c}$ is a non-$\vec{0}$ translation belonging to a proper bidirection $[\vec{a}, \vec{b}]$. Show that either $[\vec{c}, \vec{a}] = [\vec{a}, \vec{b}]$ or $[\vec{c}, \vec{b}] = [\vec{a}, \vec{b}]$. [*Hint*: This might suggest to you the lemma used in proving Theorem 9-1.]
5. Suppose that $\vec{c}$ and $\vec{d}$ are non-$\vec{0}$ members of a given bidirection and that $[\vec{c}] \neq [\vec{d}]$. Show that the given bidirection is $[\vec{c}, \vec{d}]$ and that it is a proper bidirection.

Part B

1. **(a)** Show that if $l \subseteq \pi$ then $[l] \subseteq [\pi]$.
   (b) Is the converse of the result in part (a) a theorem?
2. Suppose that $[l] \subseteq [\pi]$ and that $l \cap \pi \neq \varnothing$. Show that $l \subseteq \pi$. [*Hint*: One very easy solution is based on Exercise 3 of Part A and a corresponding theorem on lines.]
3. **(a)** Suppose that $[l] \subseteq [\pi]$. Show that there is a plane which contains l and whose direction is that of π.
   (b) How many such planes are there?

You can save time by treating both Parts A and B as in-class exercises. The text at the beginning of Section 9.06 as well as Part A which follows it can usually be covered in the same class period. Then, Part B of Section 9.06 can be assigned for homework.

Answers for Part A

1. If $[\vec{c}, \vec{d}]$ is a proper direction and $[\vec{c}, \vec{d}] = [\vec{a}, \vec{b}]$ then $(\vec{c}, \vec{d})$ is linearly independent and $\{\vec{c}, \vec{d}\} \subseteq [\vec{a}, \vec{b}]$. It follows by Exercise 3(a) of Part D on page 384 that $(\vec{a}, \vec{b})$ is linearly independent and, so, that $[\vec{a}, \vec{b}]$ is a proper bidirection.
2. (a) Suppose that $(\vec{a}, \vec{b})$ is linearly independent. By Exercise 1(a) of Part C on page 386 [and Definition 9-5(a)], $\overline{A[\vec{a}, \vec{b}]}$ is a plane through A whose direction is $[\vec{a}, \vec{b}]$. By Exercise 3(a) of Part C, there is no other such plane.
   (b) [Like (a), but use Exercise 3(c) in place of Exercise 2(a).]
3. By Exercise 2(b) of Part C on page 386, if $A \in \pi$ then $\pi = \overline{A[\pi]}$. Since $A \in \overline{A[\pi]}$ it follows that if $\pi = \overline{A[\pi]}$ then $A \in \pi$.
4. Suppose that $[\vec{a}, \vec{b}]$ is a proper bidirection. Since $(\vec{a}, \vec{b})$ is linearly independent, $[\vec{a}] \cap [\vec{b}] = \{\vec{0}\}$. [For, if $\vec{a}a = \vec{b}b$ then $a = b = 0$.] Suppose, now, that $\vec{c} \neq \vec{0}$. It follows that either $\vec{c} \notin [\vec{a}]$ or $\vec{c} \notin [\vec{b}]$. Since $(\vec{a}, \vec{b})$ is linearly independent it follows that neither $\vec{a}$ nor $\vec{b}$ is $\vec{0}$. So, by Theorem 6-13, either $(\vec{a}, \vec{c})$ or $(\vec{c}, \vec{b})$ is linearly independent. Assuming that $\vec{c} \in [\vec{a}, \vec{b}]$ it follows that $\{\vec{c}, \vec{a}\}$ and $\{\vec{b}, \vec{c}\}$ are subsets of $[\vec{a}, \vec{b}]$. So, by the lemma, $[\vec{c}, \vec{a}] = [\vec{a}, \vec{b}]$ or $[\vec{b}, \vec{c}] = [\vec{a}, \vec{b}]$. [The last four sentences can be replaced by: So, by Exercise 2(b) of Part D on page 384, either $[\vec{c}, \vec{a}] = [\vec{a}, \vec{b}]$ or $[\vec{b}, \vec{c}] = [\vec{a}, \vec{b}]$.]
5. Since $\vec{c} \neq \vec{0}$ and $[\vec{c}] \neq [\vec{d}]$ it follows that $\vec{c} \notin [\vec{d}]$. Similarly, $\vec{d} \notin [\vec{c}]$. Hence [by Theorem 6-2], $(\vec{c}, \vec{d})$ is linearly independent. By the lemma for Theorem 9-1 it follows that the bidirection in question is $[\vec{c}, \vec{d}]$. Since $(\vec{c}, \vec{d})$ is linearly independent, it is a proper bidirection.

Answers for Part B

1. (a) This is an immediate consequence of Definitions 7-4 and 9-4.
   (b) No.
2. Suppose that $A \in l \cap \pi$. Then $l = \overleftrightarrow{A[l]}$ and $\pi = \overline{A[\pi]}$. Since $[l] \subseteq [\pi]$ it follows from Definitions 7-5(b) and 9-5(b) that $l \subseteq \pi$.
3. (a) Let $A \in l$. Then $l = \overleftrightarrow{A[l]} \subseteq \overline{A[\pi]}$ and $\overline{A[\pi]}$ is a plane whose direction is that of π.
   (b) Only one, since there is only one plane containing A whose direction is that of π.

9.06 Parallelism of Planes and Lines

In analogy with Definition 7-6 we adopt:

Definition 9-6

(a) $\pi_1 \parallel \pi_2 \longleftrightarrow [\pi_1] = [\pi_2]$

(b) $l \parallel \pi \longleftrightarrow [l] \subseteq [\pi]$

(c) $\pi \parallel l \longleftrightarrow l \parallel \pi$

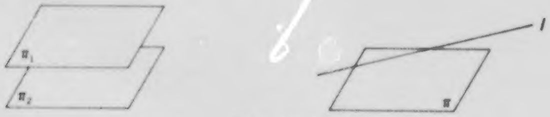


Fig. 9-6

So, parallel planes are planes with the same direction and a line is parallel to a plane if and only if the direction of the line is a subset of the direction of the plane.

In view of Definition 9-6(a), you have already proved:

Theorem 9-12 Any translation maps any plane onto a parallel plane.

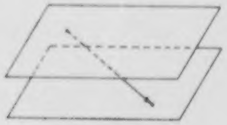


Fig. 9-7

[What earlier exercise is relevant here?] Similarly, in consequence of an earlier theorem, we have:

Theorem 9-13 There is one and only one plane through a given point and parallel to a given plane.

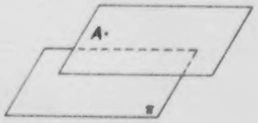


Fig. 9-8

[Which theorem yields Theorem 9-13?] How many lines are there through a given point and parallel to a given plane? Use pencils and cardboard to illustrate this situation. Given a line and a plane, do you think that you can count on there being a plane which contains the given line and is parallel to the given plane? Explain. Where have you proved the following theorem?

Theorem 9-12 is proved in Exercises 3(a) and 3(b) of Part B on page 386.

Theorem 9-13 follows from Theorem 9-11(b) [and Definition 9-6(a)].

There are infinitely many lines through a point A parallel to a plane π. In fact they are just the lines $A[\vec{a}]$ for $\vec{a} \in [\pi]$, and there are as many members of $[\pi]$ as there are ordered pairs of real numbers.

The if-part of Theorem 9-14 is proved in Exercise 3(a) of Part B on page 387. If there is a plane σ containing ℓ and parallel to π then, by Exercise 1(a), $[\ell] \subseteq [\sigma] = [\pi]$ and, so, $\ell \parallel \pi$. This establishes the only if-part of Theorem 9-14.

Theorem 9-14 There is a plane containing a given line and parallel to a given plane if and only if the given line and plane are parallel.

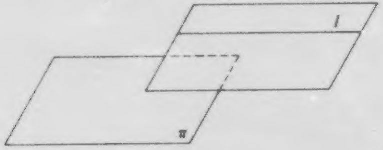


Fig. 9–9

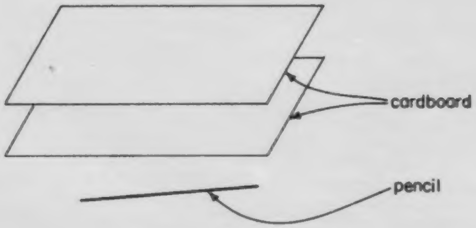
Exercises

Part A

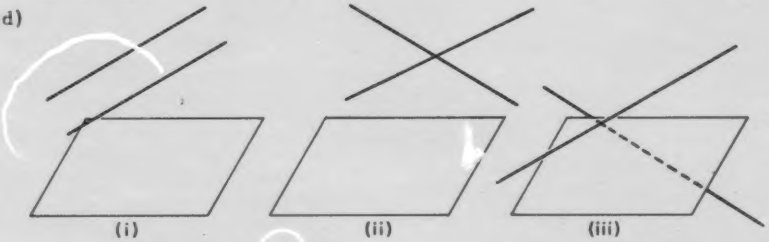
1. Hold a pencil and a piece of cardboard so that they represent a line parallel to a plane.
   (a) Could the line and plane intersect in exactly one point? Exactly two points?
   (b) With a second piece of cardboard, demonstrate that two planes can be parallel to the same line and also be parallel to each other. Draw an appropriate picture.
   (c) Demonstrate that two nonparallel planes can be parallel to the same line.
   (d) Demonstrate that two lines can be parallel to a given plane and (i) be parallel lines; (ii) be intersecting lines; (iii) be skew lines. Draw an appropriate picture.
2. Hold two pencils so that they represent two intersecting lines.
   (a) How many planes can contain the lines represented by the pencils?
   (b) Hold a piece of cardboard in such a way that it represents a plane parallel to each of the two lines. How many such planes are there?
   (c) What can you say about the direction of any of the planes found in (b)? Draw an appropriate picture of two such planes.
   (d) What would you expect to be the intersection of any two of the planes found in (b)?
3. Hold two pencils so that they represent two parallel lines.
   (a) How many planes can contain the lines represented by the pencils?
   (b) Hold a piece of cardboard so that it represents a plane parallel to each of the two given parallel lines and (i) is parallel to the plane containing the parallel lines; (ii) is not parallel to the plane containing the parallel lines; (iii) contains one of the lines but not the other. Draw appropriate pictures for (i), (ii), and (iii).

Answers for Part A

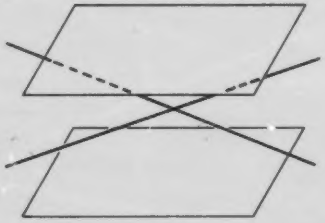
1. (a) No.; No.
   (b)



   (c) [For example, each of two planes containing a line is parallel to the line, and to any parallel line, but the planes are not parallel to each other.]
   (d)

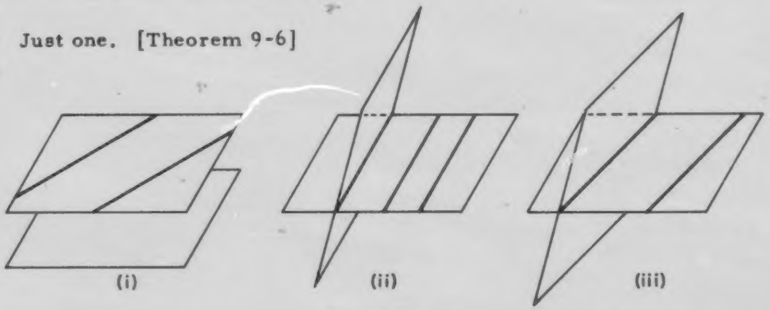


2. (a) Just one. [Theorem 9-5] (b) Infinitely many.
   (c) The directions of two such planes are the same.



   (d) The intersection of two such planes is ∅. [Part (c) and Exercise 3(b) of Part C on page 386.]

3. (a) Just one. [Theorem 9-6]
   (b)



(c) Hold the cardboard so that it is parallel to both of the given lines and intersects the plane of these two lines. What can you say about the line of intersection of the two planes and the two given lines? Draw an appropriate picture.

Part B

1. Prove each of the following theorems.
   - (a) $\pi \parallel \pi$
   - (b) $(\pi_1 \parallel \sigma \text{ and } \pi_2 \parallel \sigma) \longrightarrow \pi_1 \parallel \pi_2$
   - (c) $\pi_2 \parallel \pi_1 \longrightarrow \pi_1 \parallel \pi_2$ [Show that this follows from (a) and (b).]
   - (d) $(\pi_1 \parallel \sigma \text{ and } \sigma \parallel \pi_2) \longrightarrow \pi_1 \parallel \pi_2$ [Show that this follows from (b) and (c).]
   - (e) $(l \parallel m \text{ and } m \parallel \pi) \longrightarrow l \parallel \pi$
   - (f) $(l \parallel \pi \text{ and } \pi \parallel \sigma) \longrightarrow l \parallel \sigma$
   - (g) $\sigma \parallel \pi \longrightarrow (\sigma = \pi \text{ or } \sigma \cap \pi = \emptyset)$ [*Hint*: This is a corollary of what theorem?]
   - (h) $l \parallel \pi \longrightarrow (l \subseteq \pi \text{ or } l \cap \pi = \emptyset)$ [*Hint*: Refer to earlier exercises.]
   - (i) $l \subseteq \pi \longrightarrow l \parallel \pi$
   - (j) $\sigma = \pi \longrightarrow \sigma \parallel \pi$
2. (a) Do you think that the converses of parts (g) and (h) are true on intuitive grounds?
   - (b) Do you think that these converses are theorems?
   - (c) As you know, there are skew lines. Do you believe that there might be "skew planes"?
3. Here are four theorems about parallel lines and planes. Relate each of them to results stated in Exercise 1.
   - (a) Two parallel planes have no point in common.
   - (b) Lines which are parallel are, also, parallel to the same planes.
   - (c) A line is a subset of a plane if and only if it is parallel to the plane and contains a point of the plane.
   - (d) Each line contained in one of two parallel planes is parallel to the other plane.
4. Explain why 3(a) is a consequence of Theorem 9–13.
5. Prove:

> **Theorem 9–15** There is one and only one plane which contains a given point and is parallel to each of two given nonparallel lines.

[*Hint*: How might you describe the direction of such a plane?]

6. (a) Show that if each of two intersecting lines is parallel to a given plane then the plane determined by these lines is parallel to the given plane.
   - (b) Can you solve part (a) if 'intersecting' is replaced by 'coplanar'?
7. (a) Suppose that a line is parallel to each of two planes. Does it follow that the planes are parallel?
   - (b) Suppose that each of two lines is parallel to a given plane. Does it follow that the lines are parallel?
   - (c) Draw pictures to illustrate your answers for (a) and (b).

Answers for Part A [cont.]

(c) The line of intersection of the two planes is parallel to each of the given lines. [The figure (ii) of part (b) is appropriate.]

If Part B is used as a homework assignment as suggested earlier, we recommend not making Part C part of this same assignment. Part C can be a class activity for the next day.

Answers for Part B

1. (a) Since $[\pi] = [\pi]$, $\pi \parallel \pi$. [Definition 9-6(a)]
   - (b) If $[\pi_1] = [\sigma]$ and $[\pi_2] = [\sigma]$ then $[\pi_1] = [\pi_2]$. So, by Definition 9-6(a), if $\pi_1 \parallel \sigma$ and $\pi_2 \parallel \sigma$ then $\pi_1 \parallel \pi_2$. [This property of parallelism of planes is, you will notice, slightly different from transitivity [(d)]. It is called skew-transitivity. As is indicated in the notes for parts (c) and (d), reflexivity [(a)] and skew-transitivity imply symmetry and transitivity.]
   - (c) By (b), if $\pi_1 \parallel \pi_1$ and $\pi_2 \parallel \pi_1$ then $\pi_1 \parallel \pi_2$. Since, by (a), $\pi_1 \parallel \pi_1$ it follows that if $\pi_2 \parallel \pi_1$ then $\pi_1 \parallel \pi_2$.
   - (d) If $\pi_1 \parallel \sigma$ and $\sigma \parallel \pi_2$ then, by (c), $\pi_1 \parallel \sigma$ and $\pi_2 \parallel \sigma$ and so, by (b), $\pi_1 \parallel \pi_2$.
   - (e) If $[\ell] = [m]$ and $[m] \subseteq [\pi]$ then $[\ell] \subseteq [\pi]$. So, by Definitions 7-6 and 9-6(b), if $\ell \parallel m$ and $m \parallel \pi$ then $\ell \parallel \pi$.
   - (f) If $[\ell] \subseteq [\pi]$ and $[\pi] = [\sigma]$ then $[\ell] \subseteq [\sigma]$. So, by Definitions 9-6(a) and (b), if $\ell \parallel \pi$ and $\pi \parallel \sigma$ then $\ell \parallel \sigma$.
   - (g) Suppose that $\sigma \parallel \pi$ and that $A \in \sigma \cap \pi$. It follows by Theorem 9-13 that $\sigma = \pi$. Hence, if $\sigma \parallel \pi$ then $(\sigma = \pi$ or $\sigma \cap \pi = \emptyset)$.
   - (h) Suppose that $\ell \parallel \pi$ and $\ell \cap \pi \neq \emptyset$. By Exercise 2 of Part B on page 387 [and Definition 9-6(b)], $\ell \subseteq \pi$. Hence, if $\ell \parallel \pi$ then $(\ell \subseteq \pi$ or $\ell \cap \pi = \emptyset)$.
   - (i) If $\ell \subseteq \pi$ then [by Exercise 1(a) of Part B on page 387] $[\ell] \subseteq [\pi]$ and so [by Definition 9-6(b)], $\ell \parallel \pi$.
   - (j) If $\pi = \sigma$ then $[\pi] = [\sigma]$ and [by Definition 9-6(a)] $\sigma \parallel \pi$.
2. (a) To establish the converse of (g), all we need do, in view of part (j), is to show that if $\sigma \cap \pi = \emptyset$ then $\sigma \parallel \pi$. On intuitive grounds, this seems reasonable. [But it is not true in a space of four or more dimensions.] Similar remarks apply to the converse of (h).
   - (b) [Whatever students may think these converses are not, as yet, theorems. They will become so, in Chapter 10, after the adoption of a postulate restricting $\mathcal{E}$ to be at most 3-dimensional.]
   - (c) Not in 3-dimensional space.

Answers for Part B [cont.]

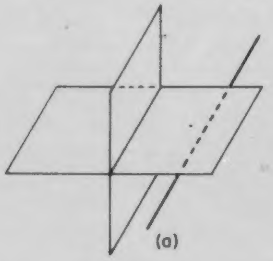
3. (a) This is a consequence of Exercise 1(g).
   (b) This is a consequence of Exercise 1(e).
   (c) If-part from Exercise 1(h); only if-part from Exercise 1(i) and the fact that $\emptyset$ is not a line.
   (d) This is a consequence of Exercises 1(i) and (f).

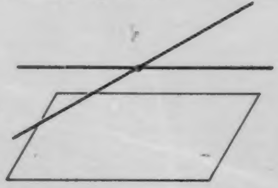
4. Two parallel planes have the same direction and, by Theorem 9-13, at most one of them can contain a given point. Hence, two such planes have no common point.

5. Consider a point, A, and two lines, ℓ and m, such that $\ell \nparallel m$. Suppose that $[\ell] = [\vec{a}]$ and $[m] = [\vec{b}]$. It follows that $\vec{a} \neq \vec{0} \neq \vec{b}$ and, since $\ell \nparallel m$, $[\vec{a}] \neq [\vec{b}]$. The direction of any plane parallel to both ℓ and m must include $[\vec{a}]$ and $[\vec{b}]$. So, by Exercise 5 of Part A on page 387, the direction of any such plane must be $[\vec{a}, \vec{b}]$, and $[\vec{a}, \vec{b}]$ is a bidirection. Moreover, any plane which has this direction is parallel to each of ℓ and m. Since $[\vec{a}, \vec{b}]$ is a bidirection it follows by Theorem 9-5(a) that there is one and only one plane — the plane $A[\vec{a}, \vec{b}]$ — which contains A and has this direction.

6. (a) Suppose that ℓ and m are two intersecting lines and $[\ell] = [\vec{a}]$ and $[m] = [\vec{b}]$. Then, as in Exercise 5, the direction of both planes is $[\vec{1}, \vec{b}]$. Hence, the planes are parallel.
   (b) No. If the lines are parallel then all we know about the direction of a plane parallel to both lines is that it contains all common directions of both lines. This goes only "half-way" to specifying the direction of the given plane.

7. (a) No. The intersection of the directions of the two planes must contain the direction of the line; but this intersection may coincide with the latter, in which case the planes will not be parallel.
   (b) No. For example, the lines might be two intersecting lines and the plane might be the plane determined by them.
   (c)


(a)


(b)

Answers for Part B [cont.]

8. Two lines which are contained in different parallel planes have no point in common [Exercise 3(a)]. By Theorem 9-8, two such lines which are coplanar are parallel. By Theorem 9-6, two lines which are parallel are coplanar.

   [The if-part may be rephrased as: Lines which are the intersections of a plane with two parallel planes are parallel. Note, in discussing this theorem, that we are not yet in a position to prove that the intersection of two intersecting planes is a line; nor can we prove that two nonparallel planes intersect.]

9. (a) Suppose that $\sigma \parallel \pi$, that ℓ intersects π and σ at A and B, respectively, that $m \parallel \ell$, and that $C \in m \cap \pi$. It follows that $B - A \in [\ell] = [m]$ and, so, that $C + (B - A) \in m$. Also, $C - A \in [\pi] = [\sigma]$ and, so, $B + (C - A) \in \sigma$. Since $C + (B - A) = B + (C - A)$, $m \cap \sigma \neq \emptyset$. On the other hand, if $m \cap \sigma$ contained more than one point it would follow that $m \subseteq \sigma$ and, so, that $m \parallel \sigma$ and, so, $\ell \parallel \sigma$. But, since ℓ is a transversal of σ, $\ell \nparallel \sigma$. Consequently, $m \cap \sigma$ consists of a single point, and m is a transversal of σ. Similarly, m is a transversal of π. [We are not yet in a position to establish (a) without the assumption that $m \cap \pi \neq \emptyset$. Why this is can be seen by an analogy. Suppose given a transversal of two parallel lines. The three lines are coplanar and any line which is parallel to the transversal and which is in the same plane is easily shown [Theorem 9-7] to be a transversal of the given parallel lines. To insure that the fourth line is in the plane of the given three it is sufficient [Exercise 3(c)] to require that it intersect one of the given parallels. However, there are, in space, lines parallel to the given transversal which do not intersect either of the given parallels. The line through any point not in the plane but parallel to the given transversal has the desired property. Entirely analogously, in a space of more than three dimensions one may have a transversal of two parallel planes and a line parallel to it which intersects neither. The transversal and the parallel planes all lie in one 3-dimensional subspace and the line through any point not in this subspace but parallel to the transversal has the desired property. In Chapter 10 we adopt a postulate which ensures that $\mathcal{E}$ is at most 3-dimensional. Using this it is possible to prove that any parallel to a transversal of two parallel planes is, also, a transversal of these planes. In fact, any line not parallel to a plane can then be proved to be a transversal of that plane. [This is an analogue of Theorem 9-7.]

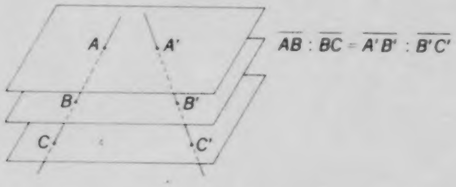
8. Prove:

> **Theorem 9-16** Two lines which are contained in different parallel planes are parallel if and only if they are coplanar.

[*Hint*: What do you know about coplanar lines?]

9. A line which has just one point in common with a plane is called a *transversal* of the plane.
   - **(a)** Show that if a line is a transversal of each of two parallel planes then any line parallel to it which intersects one of the planes is a transversal of both.
   - **(b)** Prove:

> **Theorem 9-17** The ratio of two intervals which are intercepted by parallel planes on one transversal of these planes is the same as that of the corresponding intervals which are intercepted by these planes on any other transversal.


A
A'
B
B'
C
C'
$\overline{AB} : \overline{BC} = \overline{A'B'} : \overline{B'C'}$

Part C

1. Given a plane π and a point $P \notin \pi$, show that
   - **(a)** there is a plane which contains no points of π, and
   - **(b)** there is a plane, different from π, whose intersection with π contains a line.
2. Suppose that $\sigma \neq \pi$.
   - **(a)** Show that $\sigma \cap \pi$ either is empty, or consists of a single point, or is a line.
   - **(b)** Do you think that $\sigma \cap \pi$ can consist of a single point?
   - **(c)** Do you think that if $\sigma \cap \pi = \varnothing$ then $\sigma \parallel \pi$?
   - **(d)** Do you think that if l is a line such that $l \cap \pi = \varnothing$ then $l \parallel \pi$?
   - **(e)** Do you think that a line which is a transversal of one of two parallel planes must intersect the other?
3. Parts (c) – (e) of Exercise 2 suggest theorems which we have not yet proved. As a matter of fact, although the statements in question are intuitively reasonable they are not yet theorems. They will become theorems, however, in the next chapter after we adopt an additional postulate. To see what is involved here, consider the following question:

Is '$l \cap m = \varnothing \longrightarrow l \parallel m$' a theorem?

Answers for Part B [cont.]

9. (b) Using the notation of the figure, if $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are concurrent or parallel then they determine a plane π which, by Theorem 9-16, intersects the given planes in parallel lines. $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are transversals of these parallel lines, and the desired conclusion follows from Theorem 8-6(b). For the general case, in which $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are not coplanar, introduce as a third transversal the line through A′ which is parallel to $\overleftrightarrow{AB}$. By part (a) this line is a transversal, and, by the preceding argument, the ratio of the intervals intercepted on it is the same as the ratio of those intercepted on $\overleftrightarrow{A'B'}$ and is also the same as the ratio of the intervals intercepted on $\overleftrightarrow{AB}$. [Note that the figure illustrates a special case. In general, there will be four parallel planes and the intervals which are in question on a given transversal will not have a common end point. Note, also, that by using Exercise 6 of Part D on page 325 one can obtain more precise information. In the case illustrated in the figure, it follows that $(B - A):(C - B) = (B' - A'):(C' - B')$.]

*

There is a corollary to Theorem 9-17 which is of basic importance in volume 2. Suppose that π_1, π_2, σ_1, and σ_2 are parallel planes and that ℓ, m, and n are transversals of these planes and $\ell \parallel m$. Suppose that ℓ intersects π_1 and π_2 at A and B, respectively, that m intersects σ_1 and σ_2 at C and D, respectively, and that n intersects π_1, π_2, σ_1, and σ_2 at A′, B′, C′, and D′. It follows that $(B' - A'):(D' - C') = (B - A):(D - C)$.

To establish this suppose, first, that $\ell = m$. In this case the desired conclusion follows at once from Theorem 9-17 [if this theorem is made more precise as indicated in the note to the answer for Exercise 9(b)]. In case $\ell \neq m$, let C″ and D″ be the points at which ℓ intersects σ_1 and σ_2, respectively. By the case just settled, $(B' - A'):(D' - C') = (B - A):(D'' - C'')$. Since $\ell \parallel m$ and $\ell \neq m$, ℓ and m determine a plane [Theorem 9-6] and the lines $\overleftrightarrow{CC''}$ and $\overleftrightarrow{DD''}$ are contained in this plane [Theorem 9-3]. Since $\sigma_1 \parallel \sigma_2$, $\overleftrightarrow{CC''} \parallel \overleftrightarrow{DD''}$ [Theorem 9-16]. It follows that C″D″DC is a parallelogram and, so, that $D'' - C'' = D - C$. Hence, $(B' - A'):(D' - C') = (B - A):(D - C)$.

Answers for Part C

1. (a) $\overline{P[\pi]}$ is such a plane. [Theorem 9-11(b), Definition 9-6(a), and Exercise 3(a) of Part B]

   (b) Let Q and R be two points of π. Then $\overline{PQR}$ is such a plane. [Theorems 9-3 and 9-1]
2. (a) If $\sigma \cap \pi$ contains two points then, by Theorem 9-3, it contains a line. If it contains a line and a point not on the line then, contrary to assumption, $\sigma = \pi$.

   (b)-(e) [Intuitions of physical space would motivate a negative answer to (b) and positive answers for the others. All four situations can occur in spaces of more than three dimensions.]

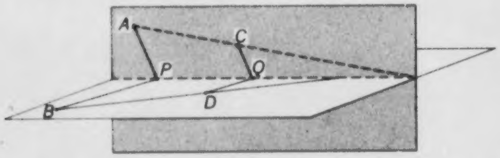
(a) How would you answer this question now?
(b) How would you answer it *if* we were to adopt a postulate according to which any four points of $\mathscr{E}$ are coplanar?

4. As Exercise 3 suggests, our present postulates allow for the possibility that $\mathscr{E}$ is so "large" that queer-seeming things can happen in it. For example, our present postulates are satisfied in spaces which contain pairs of planes which have a single point in common. In spite of this, we can prove the following theorem:

> If a given line is parallel to each of two intersecting planes then the intersection of the planes is a line which is parallel to the given line.

Prove this theorem.

5. Suppose that, as shown in the figure, $(A - P, B - P, Q - P)$ is linearly independent, $\overleftrightarrow{CQ} \parallel \overleftrightarrow{AP}$, and $\overleftrightarrow{DQ} \parallel \overleftrightarrow{BP}$.



(a) What tells you that A, P, and Q determine a plane?
(b) How do you know that $\overleftrightarrow{APQ} \neq \overleftrightarrow{BPQ}$?
(c) Show that $\overleftrightarrow{APB}$ and $\overleftrightarrow{CQD}$ are parallel planes.
(d) Can $\overleftrightarrow{AC}$ be parallel to $\overleftrightarrow{PQ}$? Explain.
(e) If $\overleftrightarrow{AC} \parallel \overleftrightarrow{PQ}$, what can you say about $\overleftrightarrow{AC} \cap \overleftrightarrow{BD}$? Why?
(f) If $\overleftrightarrow{AC} \cap \overleftrightarrow{BD} \neq \emptyset$, what can you say about $(D - Q) : (B - P)$ and $(C - Q) : (A - P)$? About $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$?

*9.07 Half-planes

As you will recall from Chapter 1, half-planes are analogous to half-lines, and closed half-planes are analogous to rays. Both half-planes and closed half-planes have edges rather than vertices. A closed half-plane contains its edge, a half-plane does not. A half-plane is determined when one knows a point which belongs to it and knows the line which is its edge.

Fig. 9–10

Answers for Part C [cont.]

3. (a) No, because our postulates are consistent with the existence of skew lines.
   (b) With this postulate to work with the statement in question would follow from Theorem 9-8. [The statement of this exercise should, of course, be compared with Exercise 2(c).]

4. Suppose that $\ell \parallel \pi$, $\ell \parallel \sigma$, $\sigma \neq \pi$ and $A \in \pi \cap \sigma$. It follows that $A[\ell] \subseteq \pi \cap \sigma$ [Exercise 3(c) of Part B] and, since $\sigma \neq \pi$, that $\pi \cap \sigma =$ [illegible]] [Exercise 2(a), above].

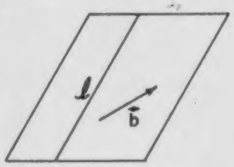
   [A more familiar theorem is: A line which is parallel to each of two nonparallel planes is parallel to their line of intersection. However, at this stage we cannot prove that nonparallel planes intersect; and it is only during the proof, given above, that we find that 'their line of intersection' makes sense.]

5. (a) Since it follows from the hypothesis that $(A - P, Q - P)$ is linearly independent, $\{P, A, Q\}$ is noncollinear. So, A, P, and Q determine a plane — the plane $\overleftrightarrow{APQ}$.
   (b) If $\overleftrightarrow{APQ} = \overleftrightarrow{BPQ}$ then, by (a), $\{P, A, B, Q\}$ is coplanar. This latter is not the case because $(A - P, B - P, Q - P)$ is linearly independent.
   (c) $[\overleftrightarrow{APB}] = [A - P, B - P]$ and $[\overleftrightarrow{CQD}] = [C - Q, D - Q]$. Since $\overleftrightarrow{CQ} \parallel \overleftrightarrow{AP}$, $[C - Q] = [A - P]$, and since $\overleftrightarrow{DQ} \parallel \overleftrightarrow{BP}$, $[D - Q] = [B - P]$. So, $[A - P, B - P] = [C - Q, D - Q]$. Hence, $[\overleftrightarrow{APB}] = [\overleftrightarrow{CQP}]$ and, by definition $\overleftrightarrow{APB} \parallel \overleftrightarrow{CQD}$.
   (d) Yes. This will be the case if and only if $C - Q = A - P$. [Exercise 2(a) of Part A on page 385]
   (e) If $\overleftrightarrow{AC} \parallel \overleftrightarrow{PQ}$ then $\overleftrightarrow{AC} \cap \overleftrightarrow{PQ} = \emptyset$. Since $\overleftrightarrow{AC} \subseteq \overleftrightarrow{APQ}$ and $\overleftrightarrow{BD} \subseteq \overleftrightarrow{BPQ}$, $\overleftrightarrow{AC} \cap \overleftrightarrow{BD} \subseteq \overleftrightarrow{APQ} \cap \overleftrightarrow{BPQ} = \overleftrightarrow{PQ}$. So, if $\overleftrightarrow{AC} \parallel \overleftrightarrow{PQ}$ then $\overleftrightarrow{AC} \cap \overleftrightarrow{BD} = \emptyset$.
   (f) By Theorem 8-2, the ratios are the same. By Theorem 8-3, $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$.

TC 393 (1)

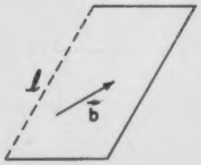
Answers for Exercises

1. (a)



This set is a plane. In fact, if $A \in \ell$ and $[\ell] = [\vec{a}]$, it is the plane $A[\vec{a}, \vec{b}]$.

(b)



The set in question is a half-plane.

We shall study half-planes more thoroughly in Volume 2. Here, we shall only suggest how they may be defined and how to establish the line-plane separation property stated on page 31.

Exercises

1. Suppose that l is a line and that $\vec{b} \notin [l]$. Picture, and describe, each of the following sets.
   (a) $\{X: \exists_{Y \in l}\, X - Y \in [\vec{b}]\}$ (b) $\{X: \exists_{Y \in l}\, X - Y \in [\vec{b}]^+\}$
   (c) $\{X: \exists_{Y \in l}\, X - Y \in [-\vec{b}]^+\}$ (d) $\{X: \exists_{Y \in l}\, X - Y = \vec{0}\}$
2. Which, if any, of the sets of Exercise 1 has l as a subset?
3. (a) What is the intersection of the sets of parts (a) and (d) of Exercise 1? Of parts (b) and (d)? Explain.
   (b) Do the sets of parts (b) and (c) of Exercise 1 have a point in common? [*Hint*: If B_1 and B_2 are points of l, $A - B_1 \in [\vec{b}]^+$, and $A - B_2 \in [-\vec{b}]^+$, what contradictory conclusions can you draw concerning $B_2 - B_1$?]
   (c) What is the union of the sets of parts (b), (c), and (d) of Exercise 1?
4. Suppose that $l \subseteq \pi$.
   (a) Can you find a translation $\vec{b}$ such that $\vec{b} \in [\pi]$ but $\vec{b} \notin [l]$?
   (b) If $\vec{b} \in [\pi]$ but $\vec{b} \notin [l]$, what is $\{X: \exists_{Y \in l}\, X - Y \in [\vec{b}]\}$?
   (c) Suppose that $\vec{b}_1$ and $\vec{b}_2$ are translations which belong to $[\pi]$ but not to $[l]$. Compare $\{X: \exists_{Y \in l}\, X - Y \in [\vec{b}_1]^+\}$ and $\{X: \exists_{Y \in l}\, X - Y \in [\vec{b}_2]^+\}$.
   (d) State and prove a result suggested by part (c).

9.08 Chapter Summary

Vocabulary Summary

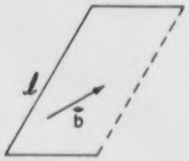
coplanar points
coplanar lines
bidirection
plane quadrilateral
plane
direction of a plane
proper bidirection
transversal

Definitions

9-1. $\{A, B, C, D\}$ is coplanar $\longleftrightarrow$ $(B - A, C - A, D - A)$ is linearly dependent.

9-2. π is a plane if and only if (a) π is a subset of $\mathscr{E}$ which contains at least three noncollinear points, and (b) $\forall_X \forall_Y \forall_Z\, [(\{X, Y, Z\} \subseteq \pi$ and $\{X, Y, Z\}$ is noncollinear$] \longrightarrow \forall_W\, (W \in \pi \longleftrightarrow \{X, Y, Z, W\}$ is coplanar$)]$

(c)



This set is a half-plane, the opposite of that in part (b).

(d)

This set is ℓ.

2. The first and fourth contain ℓ.

3. (a) ℓ [since $\vec{0} \in [\vec{b}]$; $\emptyset$ [since $\vec{0} \notin [\vec{b}]^+$ and $\vec{b} \notin [\ell]$].

   (b) No. If A belongs to both sets then there are points of ℓ — say, B_1 and B_2 — such that $A - B_1 = \vec{b}b_1$ and $A - B_2 = -\vec{b}b_2$, where both b_1 and b_2 are positive. It follows that $B_2 - B_1 = \vec{b}(b_2 + b_1)$. Since $B_2 - B_1 \in [\ell]$ and $\vec{b} \notin [\ell]$ it follows that $b_2 + b_1 = 0$. Since b_1 and b_2 are positive, this is not the case. Hence, no point belongs to both sets.

   (c) The plane described in 1(a).

4. (a) Yes. $[\ell] \subseteq [\pi]$ and $[\pi]$, since it is a bidirection, contains vectors not in $[\ell]$.

   (b) π

   (c) The sets described are either the same or opposite half-planes.

   (d) $\{X: \exists_{Y \in \ell}\, X - Y \in [\vec{b}_1]^+\}$ and $\{X: \exists_{Y \in \ell}\, X - Y \in [\vec{b}_2]^+\}$ are the same or have no common point. [As the proof shows, no conditions are needed on $\vec{b}_1$ and $\vec{b}_2$ other than that they are non-$\vec{0}$. So, this result shows that no two half-planes with edge ℓ have a common point, and, with $\vec{b}_1 \in [\ell]$, that no such half-plane contains any point of ℓ.]

   Let $\mathfrak{H}_1 = \{X: \exists_{Y \in \ell}\, X - Y \in [\vec{b}_1]^+\}$ and $\mathfrak{H}_2 = \{X: \exists_{Y \in \ell}\, X - Y \in [\vec{b}_2]^+\}$. Suppose that $\mathfrak{H}_1 \cap \mathfrak{H}_2 \neq \emptyset$. In particular suppose that $B_1 + \vec{b}_1 b_1 = B_2 + \vec{b}_2 b_2$, where B_1 and B_2 belong to ℓ and b_1 and b_2 are positive. Suppose, also, that $C = B + \vec{b}_1 b$, where $B \in \ell$ and $b > 0$. Since $\vec{b}_1 = (B_2 - B_1) \cdot /b_1 + \vec{b}_2(b_2/b_1)$ it follows that $C = B + (B_2 - B_1)(b/b_1) + \vec{b}_2(bb_2/b_1)$. Since $B + (B_2 - B_1)(b/b_1) \in \ell$ and $(bb_2)/b_1 > 0$ it follows that $C \in \mathfrak{H}_2$. Hence, if $\mathfrak{H}_1 \cap \mathfrak{H}_2 \neq \emptyset$ then $\mathfrak{H}_1 \subseteq \mathfrak{H}_2$ and, by symmetry, $\mathfrak{H}_2 \subseteq \mathfrak{H}_1$. Consequently, either $\mathfrak{H}_1 = \mathfrak{H}_2$ or $\mathfrak{H}_1 \cap \mathfrak{H}_2 = \emptyset$.

9-3. $\overline{ABC} = \{X : \exists_x \exists_y\, X = A + (B - A)x + (C - A)y\}$

9-4. $[\pi] = \{\vec{x} : \exists_Y \exists_Z\, (Y \in \pi \text{ and } Z \in \pi \text{ and } \vec{x} = Z - Y)\}$

9-5. **(a)** $\overline{A[\vec{a}, \vec{b}]} = \{X : X - A \in [\vec{a}, \vec{b}]\}$

(b) $\overline{A[\pi]} = \{X : X - A \in [\pi]\}$

9-6. **(a)** $\pi_1 \parallel \pi_2 \longleftrightarrow [\pi_1] = [\pi_2]$

(b) $l \parallel \pi \longleftrightarrow [l] \subseteq [\pi]$

(c) $\pi \parallel l \longleftrightarrow l \parallel \pi$

Other Theorems

Lemma. $((\vec{c}, \vec{d})$ is linearly independent and $\{\vec{c}, \vec{d}\} \subseteq [\vec{a}, \vec{b}]) \longrightarrow [\vec{c}, \vec{d}] = [\vec{a}, \vec{b}]$

9-1. For $\{A, B, C\}$ noncollinear, $\overline{ABC}$ is the plane which contains A, B, and C.

Corollary. Three noncollinear points determine [uniquely] a plane.

9-2. $(\{D, E, F\} \subseteq \overline{ABC}$ and $\{D, E, F\}$ is noncollinear$) \longrightarrow \overline{ABC} = \overline{DEF}$

9-3. A plane contains the line determined by any two of its points.

9-4. A line and a point not on that line determine a plane.

9-5. Two intersecting lines determine a plane.

9-6. Two parallel lines determine a plane.

9-7. Two nonparallel coplanar lines intersect.

9-8. Two lines are parallel if and only if they are coplanar and have no common point.

9-9. $\{A, B, C\}$ is a noncollinear subset of $\pi \longrightarrow [\pi] = [B - A, C - A]$

9-10. $(P \in \pi$ and $\vec{a} \in [\pi]) \longrightarrow P + \vec{a} \in \pi$

9-11. **(a)** For $(\vec{a}, \vec{b})$ linearly independent, $\overline{A[\vec{a}, \vec{b}]}$ is the plane through A with the bidirection $[\vec{a}, \vec{b}]$.

(b) $\overline{A[\pi]}$ is the plane through A with the direction of π.

9-12. Any translation maps any plane onto a parallel plane.

9-13. There is one and only one plane through a given point and parallel to a given plane.

9-14. There is a plane containing a given line and parallel to a given plane if and only if the given line and plane are parallel.

9-15. There is one and only one plane which contains a given point and is parallel to each of two given nonparallel lines.

9-16. Two lines which are contained in different parallel planes are parallel if and only if they are coplanar.

9-17. The ratio of two intervals which are intercepted by parallel planes on one transversal of these planes is the same as that of the corresponding intervals which are intercepted by these planes on any other transversal.

Chapter Test

1. True or false?
   - **(a)** If a line is parallel to a plane, it is parallel to every line in the plane.
   - **(b)** If two planes are parallel, then every line which is a transversal of one of the given planes is a transversal of the other one also.
   - **(c)** If two planes are parallel, then every plane which intersects, but is different from, one of the given planes intersects the other one also.
   - **(d)** If two lines are parallel, then every line which intersects one of the given lines, but is different from it, intersects the other one also.
   - **(e)** If two lines are parallel and one is a transversal to a given plane then so is the other.
   - **(f)** If two planes cut a third plane in such a way that their lines of intersection with the third plane are parallel, then the two given planes are parallel.
   - **(g)** If $ABCD$ is a convex quadrilateral, then $\{A, B, C, D\}$ is coplanar.
   - **(h)** If $ABCD$ is a simple quadrilateral, then $\{A, B, C, D\}$ is coplanar.

2. Suppose that $(\vec{a}, \vec{b}, \vec{q})$ is linearly independent, and that $A - P = \vec{a}$, $B - P = \vec{b}$, and $Q - P = \vec{q}$.
   - **(a)** Draw an appropriate picture for the given conditions.
   - **(b)** Is $\overline{APQ}$ a plane or not? Explain your answer.
   - **(c)** Is B in $\overline{APQ}$ or not? Explain your answer.
   - **(d)** Let $C = Q + \vec{a}2$ and $D = Q + \vec{b}d$, for some d. For what values of 'd' will it be the case that $\overleftrightarrow{BD}$ and $\overleftrightarrow{AC}$ have a point in common? Explain your answer.
   - **(e)** Given the information in (d), for what values of 'd' will it be the case that $\overleftrightarrow{BD}$ and $\overleftrightarrow{AC}$ are parallel? Explain your answer.

3. Given that $\overline{ABC}$ is a plane.
   - **(a)** What can you say about the points A, B, and C?
   - **(b)** What is the direction of $\overline{ABC}$?
   - **(c)** Show that $\overline{ABC} + \vec{d}$ is parallel to $\overline{ABC}$.
   - **(d)** Which of these points are in $\overline{ABC} + \vec{d}$, given that $\vec{d} \notin [\overline{ABC}]$?
     - (i) $A + (C - B) + \vec{d}$
     - (ii) $A + ((B + \vec{d}) - (A + \vec{d}))$
     - (iii) $(B + \vec{d}) + (B - A) + \vec{d}$
     - (iv) $(C + \vec{d}) + ((B + \vec{d}) - (A + \vec{d}))$

Background Topic

On pages 273 and 274 you considered vectors $\vec{a}$ and $\vec{b}$ such that

$$\vec{a} = \vec{c}a_1 + \vec{d}a_2 \text{ and } \vec{b} = \vec{c}b_1 + \vec{d}b_2,$$

where $(\vec{c}, \vec{d})$ is linearly independent, and showed that

$$(*) \qquad (\vec{a}, \vec{b}) \text{ is linearly independent} \Longleftrightarrow \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

[Recall that the definition of the determinant of the pair $((a_1, b_1), (a_2, b_2))$ is:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

See page 175.] Note that it follows immediately from the definition that

$$(1) \qquad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}. \text{ [Explain.]}$$

In view of this, (*) amounts to the following:

$$(2) \qquad \begin{gathered} (\vec{u}_1a_1 + \vec{u}_2a_2, \vec{u}_1b_1 + \vec{u}_2b_2) \text{ is linearly dependent} \\ \Longleftrightarrow \\ \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0 \end{gathered} \qquad [(\vec{u}_1, \vec{u}_2) \text{ is linearly independent}]$$

We shall need similar results concerning the linear dependence of $(\vec{a}, \vec{b})$, where

$$(**) \quad \vec{a} = \vec{u}_1a_1 + \vec{u}_2a_2 + \vec{u}_3a_3 \text{ and } \vec{b} = \vec{u}_1b_1 + \vec{u}_2b_2 + \vec{u}_3b_3$$

and $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is linearly independent. First, let's experiment a bit with (2).

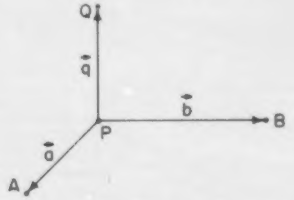
Part A

1. Suppose that $(\vec{u}_1, \vec{u}_2)$ is linearly independent. In each of the following cases, *use (2)* to determine whether or not $(\vec{a}, \vec{b})$ is linearly dependent.
   (a) $\vec{a} = \vec{u}_14 + \vec{u}_2 \cdot -2, \vec{b} = \vec{u}_1 \cdot -6 + \vec{u}_2 \cdot 3$
   (b) $\vec{a} = \vec{u}_16 + \vec{u}_27, \vec{b} = \vec{u}_13 + \vec{u}_24$
   (c) $\vec{a} = \vec{u}_1 \cdot 27 + \vec{u}_2 \cdot 18, \vec{b} = \vec{u}_1 \cdot 21 + \vec{u}_2 \cdot 14$
   (d) $\vec{a} = \vec{u}_26, \vec{b} = \vec{u}_1 \cdot -3$
   (e) $\vec{a} = \vec{u}_12 + \vec{u}_2 \cdot -3, \vec{b} = \vec{0}$
2. Recall that if $\vec{a} \neq \vec{0}$ then $(\vec{a}, \vec{b})$ is linearly dependent if and only if $\vec{b} \in [\vec{a}]$ Use this and (2) to prove the real number theorem:

Key to Chapter Test

1. (a) False. (b) True. (c) True. (d) False.
(e) True. (f) False. (g) True. (h) False.

2. (a)



(b) Yes. Since $\vec{a}$ and $\vec{q}$ are linearly independent translations, {A, P, Q} is noncollinear.

(c) No. For if $B \in \overline{\widehat{APQ}}$ then $(\vec{a}, \vec{b}, \vec{q})$ is linearly dependent. Since $(\vec{a}, \vec{b}, \vec{q})$ is not linearly dependent, $B \notin \overline{\widehat{APQ}}$.

(d) Let $R \in \overleftrightarrow{BD} \cap \overleftrightarrow{AC}$. Then, for some p and q, $R = B + (D - B)p = A + (C - A)q$. So, $(B - A) + (D - B)p + (A - C)q = \vec{0}$. Since $B - A = \vec{b} - \vec{a}$, $D - B = (Q - B) + \vec{b}d = (\vec{q} - \vec{b}) + \vec{b}d$ and $A - C = (A - Q) - \vec{a}2 = (\vec{a} - \vec{q}) - \vec{a}2$, it follows that

$$(\vec{b} - \vec{a}) + [(\vec{q} - \vec{b}) + \vec{b}d]p + [(\vec{a} - \vec{q}) - \vec{a}2]q = \vec{0}$$

or, more conveniently, that

$$\vec{a}(-1 - q) + \vec{b}[1 + (d - 1)p] + \vec{q}(p - q) = \vec{0}.$$

Since $(\vec{a}, \vec{b}, \vec{q})$ is linearly independent, $p = q = -1$ and $(d - 1)p = -1$. So, $d = 2$.

Hence, the only value of 'd' for which $\overleftrightarrow{BD} \cap \overleftrightarrow{AC} \neq \emptyset$ is $\underline{2}$.

(e) $\overleftrightarrow{BD} \parallel \overleftrightarrow{AC}$ if and only if $D - B = (C - A)t$, for some t. From (d), we know that $D - B = \vec{q} + \vec{b}(d - 1)$ and $C - A = \vec{q} + \vec{a}$. So, the given lines are parallel if and only if $\vec{q} + \vec{b}(d - 1) = (\vec{q} + \vec{a})t$ for some t, that is, if and only if, for some t, $\vec{q}(1 - t) + \vec{b}(d - 1) \vec{a} \cdot -t = \vec{0}$. The latter equation holds if and only if $t = 1$ and $-t = 0$ which is clearly impossible. So, the given lines are never parallel.

3. (a) They are noncollinear

(b) [B - A, C - A]

(c) $[\overline{\widehat{ABC}} + \vec{d}] = [(B + \vec{d}) - (A + \vec{d}), (C + \vec{d}) - (A + \vec{d})]$
$= [B - A, C - A]$
$= [\overline{\widehat{ABC}}]$

So, $\overline{\widehat{ABC}} + \vec{d} \parallel \overline{\widehat{ABC}}$.

(d) The following points are in $\overline{\widehat{ABC}} + \vec{d}$:

(i) $A + (C - B) + \vec{d}$
(iv) $(C + \vec{d}) + ((B + \vec{d}) - (A + \vec{d}))$

[Reason for (i): $A + (C - B) \in \overline{\widehat{ABC}}$; reason for (iv): Given expression equivalent to '$C + (B - A) + \vec{d}$', and $C + (B - A) \in \overline{\widehat{ABC}}$.]

(3) $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0 \Longleftrightarrow \exists_x\, (b_1 = a_1x \text{ and } b_2 = a_2x)$

[a_1 and a_2 not both 0]

[*Hint*: Let $(\vec{u}_1, \vec{u}_2)$ be any linearly independent pair of vectors and let $\vec{a} = \vec{u}_1a_1 + \vec{u}_2a_2$ and $\vec{b} = \vec{u}_1b_1 + \vec{u}_2b_2$.]

3. **(a)** Use the result of Exercise 2 to find several ordered pairs (x, y) which satisfy the equation:

$2x - 3y = 0$ [*Hint*: Let $a_1 = 2$ and $a_2 = 3$.]

(b) Describe *all* solutions (x, y) of the equation in part (a).

Part B

Suppose that $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is linearly independent and that

$$\vec{a} = \vec{u}_1a_1 + \vec{u}_2a_2 + \vec{u}_3a_3 \text{ and } \vec{b} = \vec{u}_1b_1 + \vec{u}_2b_2 + \vec{u}_3b_3.$$

1. Express $\vec{a}a + \vec{b}b$ as a linear combination of $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$.
2. What three equations must a and b satisfy in order that $\vec{a}a + \vec{b}b = \vec{0}$?
3. What is one obvious solution (a, b) of all three equations in Exercise 2?
4. What kind of common solution must the equations in Exercise 2 have if $(\vec{a}, \vec{b})$ is linearly dependent?

*

In the situation described in Part B,

$$\vec{a}a + \vec{b}b = \vec{u}_1(a_1a + b_1b) + \vec{u}_2(a_2a + b_2b) + \vec{u}_3(a_3a + b_3b)$$

and so, since $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is linearly independent, $\vec{a}a + \vec{b}b = \vec{0}$ if and only if (a, b) is a solution of the system of equations:

$$(***) \qquad \begin{cases} a_1a + b_1b = 0 \\ a_2a + b_2b = 0 \\ a_3a + b_3b = 0 \end{cases}$$

Hence, $(\vec{a}, \vec{b})$ is linearly dependent if and only if (***) has a solution other than (0, 0). That is, $(\vec{a}, \vec{b})$ is linearly dependent if and only if (***) has a *nontrivial* solution.

Now, any solution of (***) is also a solution of each of the three systems:

$$\begin{cases} a_1a + b_1b = 0 \\ a_2a + b_2b = 0 \end{cases}, \begin{cases} a_2a + b_2b = 0 \\ a_3a + b_3b = 0 \end{cases}, \text{ and: } \begin{cases} a_3a + b_3b = 0 \\ a_1a + b_1b = 0 \end{cases}$$

Each of these systems has (0, 0) as one solution. And, as we learned on page 175, the first of them has no other solution unless $a_1b_2 - a_2b_1 = 0$. Since if (***) has a nontrivial solution then each of the three

From the definition of the determinant of the pair, $((a_1, b_1), (a_2, b_2))$, and properties of real numbers;

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = a_1b_2 - b_1a_2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

It may be useful to spend a few minutes [but hardly any more] here to allow the students to state other forms for the determinant of the given pair, as such may occur to them after their having taken note of the above demonstration.

Answers to Part A

1. (a) linearly dependent
   (b) not linearly dependent
   (c) linearly dependent
   (d) not linearly dependent
   (e) linearly dependent

2. Letting $(\vec{u}_1, \vec{u}_2)$ be any linearly independent pair of vectors and, also, letting $\vec{a} = \vec{u}_1a_1 + \vec{u}_2a_2$ and $\vec{b} = \vec{u}_1b_1 + \vec{u}_2b_2$, then, by (2), $(\vec{u}_1a_1 + \vec{u}_2a_2, \vec{u}_1b_1 + \vec{u}_2b_2)$ is linearly dependent if and only if

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0.$$

Also, for a_1 and a_2 not both 0, $(\vec{u}_1a_1 + \vec{u}_2a_2, \vec{u}_1b_1 + \vec{u}_2b_2)$ is linearly dependent if and only if there is a real number x such that

$$\vec{u}_1b_1 + \vec{u}_2b_2 = (\vec{u}_1a_1 + \vec{u}_2a_2)x,$$

which is to say that $(\vec{u}_1a_1 + \vec{u}_2a_2, \vec{u}_1b_1 + \vec{u}_2b_2)$ is linearly dependent if and only if

$$\exists_x\, (b_1 = a_1x \text{ and } b_2 = a_2x)\ [a_1 \text{ and } a_2 \text{ not both } 0].$$

Hence,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0 \Longleftrightarrow \exists_x\, (b_1 = a_1x \text{ and } b_2 = a_2x)\ [a_1 \text{ and } a_2 \text{ not both } 0]$$

(a) Several such ordered pairs are; $(-2, 3)$, $(\frac{5}{3}, \frac{-5}{2})$, $(8, -12)$, $(a, -\frac{3}{2}a)$.

(b) $\{(x, y)\colon \exists_z\ (x = -3z \text{ and } y = 2z)\}$

Answers to Part B

1. $\vec{a}a + \vec{b}b = \vec{u}_1(a_1a + b_1b) + \vec{u}_2(a_2a + b_2b) + \vec{u}_3(a_3a + b_3b)$
2. $\begin{cases} a_1a + b_1b = 0 \\ a_2a + b_2b = 0 \\ a_3a + b_3b = 0 \end{cases}$
3. One obvious solution would be $a = b = 0$.
4. The ordered pair (a, b) must be such that a and b are not both zero, and if (a_1, b_1) is a common solution then, for each k, (a_1k, b_1k) is a solution.

systems of two equations has a nontrivial solution, it follows that if (***) has a nontrivial solution then

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0, \quad \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0, \text{ and } \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} = 0.$$

So, using (1), we see that *if* $(\vec{a}, \vec{b})$ is linearly dependent then

$$(4) \qquad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0, \quad \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = 0, \text{ and } \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} = 0.$$

To establish the converse it is enough to show that if (4) holds then (***) has a nontrivial solution. We shall consider two cases. First, in case $b_1 = b_2 = b_3 = 0$ it is easy to find solutions (a, b) of (***) in which not both a and b are 0. Give some. Second, suppose that at least one of the numbers b_1, b_2, and b_3 is not 0—for example, suppose that $b_3 \neq 0$. From the second two equations in (4) it follows that

$$a_1b_3 + b_1 \cdot -a_3 = 0 \text{ and } a_2b_3 + b_2 \cdot -a_3 = 0. \text{ [Explain.]}$$

What, then, is one nontrivial solution of the first two equations of (***)? Is this also a solution of the third equation?

We have seen that if (4) is satisfied then (***) has a solution other then (0,0) and, so, $(\vec{a}, \vec{b})$, as described in Part B, is linearly dependent. Combining this with the result obtained two paragraphs back we see that, *for* $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ linearly independent,

$$(\vec{u}_1a_1 + \vec{u}_2a_2 + \vec{u}_3a_3, \vec{u}_1b_1 + \vec{u}_2b_2 + \vec{u}_3b_3) \text{ is linearly dependent}$$

$$(5) \qquad \Longleftrightarrow$$

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0, \quad \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = 0, \text{ and } \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} = 0.$$

Part C

1. Suppose that $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is linearly independent. In each of the following cases, use (5) to determine whether or not $(\vec{a}, \vec{b})$ is linearly dependent.
   (a) $\vec{a} = \vec{u}_1 6 + \vec{u}_2 9 + \vec{u}_3 \cdot -4$, $\vec{b} = \vec{u}_1 \cdot -4 + \vec{u}_2 \cdot -6 + \vec{u}_3 3$
   (b) $\vec{a} = \vec{u}_1 6 + \vec{u}_2 \cdot -8 + \vec{u}_3 10$, $\vec{b} = \vec{u}_1 9 + \vec{u}_2 \cdot -12 + \vec{u}_3 15$
   (c) $\vec{a} = \vec{u}_1 8 + \vec{u}_2 0 + \vec{u}_3 4$, $\vec{b} = \vec{u}_1 12 + \vec{u}_2 0 + \vec{u}_3 5$
2. Use (5) to prove the real number theorem:
   For $(a_1, a_2, a_3) \neq (0, 0, 0)$,

$$(a_1b_2 - a_2b_1 = 0, a_2b_3 - a_3b_2 = 0, \text{ and } a_3b_1 - a_1b_3 = 0)$$

$$(6) \qquad \Longleftrightarrow$$

$$\exists_x (b_1 = a_1x, b_2 = a_2x, \text{ and } b_3 = a_3x).$$

Given that $b_1 = b_2 = b_3 = 0$, (***) is equivalent to the system:

$$\begin{cases} a_1a = 0 \\ a_2a = 0 \\ a_3a = 0 \end{cases}$$

So any ordered pair (a, b) having zero as a first component is a solution.

The first of the two equations is obtained from the third equation of (4) and by noting that '$a_1b_3 + b \cdot -a_3 = 0$' is equivalent to that equation. The second of these is obtained from the second equation of (4) in a like manner.

From $b_3 \neq 0$, $a_1b_3 + b_1 \cdot -a_3 = 0$, and $a_2b_3 + b_2 \cdot -a_3 = 0$, we have $a_1 = b_1(\frac{a_3}{b_3})$ and $a_2 = b_2(\frac{a_3}{b_3})$. Since not all of a_1, a_2, and a_3 are zero it follows, by (3), that the first two equations have a nontrivial solution because

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = b_1b_2(\frac{a_3}{b_3}) - b_1b_2(\frac{a_3}{b_3}) = 0.$$

An example of such a nontrivial solution would be any ordered pair $(a, -\frac{a_3}{b_3}a)$; such solutions do satisfy the third equation of (***).

Answers to Part C

1. (a) $\begin{vmatrix} 6 & 9 \\ -4 & -6 \end{vmatrix} = 0, \quad \begin{vmatrix} 9 & -4 \\ -6 & -3 \end{vmatrix} \neq 0, \quad \begin{vmatrix} -4 & 6 \\ 3 & -4 \end{vmatrix} \neq 0$

   Not linearly dependent because one of the determinants is not 0.

   (b) Linearly dependent

   (c) Not linearly dependent

2. In a manner analogous to that used to establish the theorem of Exercise 2, Part A, we first note that, according to (5), $(\vec{a}, \vec{b})$ is linearly dependent if and only if $a_1b_2 - a_2b_1 = 0$, $a_2b_3 - a_3b_2 = 0$, and $a_3b_1 - a_1b_3 = 0$. Further, $(\vec{a}, \vec{b})$ is linearly dependent if and only if, for some x and for at least one of a_1, a_2, and a_3 not equal to zero, $b_1 = a_1x$, $b_2 = a_2x$, and $b_3 = a_3x$. The theorem follows, as before, by an application of the replacement rule of biconditionals.

Chapter Ten
Dimension

10.01 Making Room – But Not Too Much

In our intuitive discussions about $\mathcal{E}$ and $\mathcal{T}$, we have tried to keep alive the notions that $\mathcal{E}$ is the set of points of what we commonly think of as "space" and that $\mathcal{T}$ is the set of all translations that act on these points. Our original intention was to construct a set of postulates which would in a formal way describe the sets $\mathcal{E}$ and $\mathcal{T}$. So, in our formal development of geometry, $\mathcal{E}$ and $\mathcal{T}$ are merely sets which satisfy our postulates. There are many ways of satisfying our current postulates.

Exercises

Part A

1. Suppose that $\mathcal{E}$ consists of exactly one point – say, O – and that $\mathcal{T}$ consists of exactly one translation.
   (a) What is $\mathcal{T}$?
   (b) Which of our current postulates are not satisfied?
2. Suppose that $\mathcal{E}$ contains two points.
   (a) Does it follow from our postulates that $\mathcal{E}$ contains other points? What points must $\mathcal{E}$ contain? Explain.
   (b) Does it follow that $\mathcal{E}$ contains a line? May $\mathcal{E}$ contain three noncollinear points? *Must* $\mathcal{E}$ contain three noncollinear points? Explain.
   (c) Is each point of $\mathcal{E}$ contained in a line? Need there be more than one line containing a given point of $\mathcal{E}$? Explain.
   (d) Does it follow that each line of $\mathcal{E}$ is contained in a plane? Explain.
   (e) Does it follow that $\mathcal{T}$ contains a non-$\vec{0}$ translation? May $\mathcal{T}$ contain two linearly independent translations? Must $\mathcal{T}$ contain two linearly independent translations?
3. Suppose that $\mathcal{E}$ contains three noncollinear points.
   (a) Does it follow that $\mathcal{E}$ contains a plane? Does it follow that each point of $\mathcal{E}$ is contained in a plane? That each line of $\mathcal{E}$ is contained in a plane? That $\mathcal{E}$ contains [at least] two planes?

There have been several indications — especially in Part C on page 391 that we need to have postulates which specify the dimension of $\mathcal{E}$. Even earlier [in Chapter 7], we discovered that without such postulates we could not prove that each point belongs to a line. Difficulties like this last are taken care of by Postulate 4_9 on page 400. Difficulties like those in Part C are taken care of by Postulate 4_{10} on page 403. With these new postulates we have a complete basis for 3-dimensional affine geometry — roughly, the geometry of incidence and ratio for 3-dimensional space. Postulates to be adopted in volume 2 will make it possible to introduce the notions of distance and perpendicularity and, so, to develop Euclidean metric geometry.

In Part A we investigate several sets which, according to our postulates, could be $\mathcal{E}$ or $\mathcal{T}$. Our purpose in this investigation is to demonstrate that there are features of the true sets $\mathcal{E}$ and $\mathcal{T}$ which have not been specified in our postulates. Then we investigate some alternate ways of specifying these features. We recommend that Part A be used for class discussion. In this way, students will more easily recognize the relationship between the exercises and our reasons for adopting Postulate 4_9. These exercises usually precipitate heated discussions. When used for homework, the opportunity for such spontaneous discussion is not present.

Answers for Part A

1. (a) $\mathcal{T} = \{\vec{0}\}$
   (b) None. [But, this should be checked for each postulate.]
2. (a) $\mathcal{E}$ must contain all points of the line determined by the two given points, but need contain no points other than those of this line.
   (b) Yes.; Yes.; No.; If $\mathcal{E}$ is a line and $\mathcal{T}$ is its direction then all postulates are satisfied.
   (c) Yes.; No.; If $\mathcal{E}$ contains two points then, given any point of $\mathcal{E}$, there is a point different from it, and the given point belongs to the line determined by it and any such other point. However, $\mathcal{E}$ may be merely a line, in which case each point of $\mathcal{E}$ is contained in exactly one line.
   (d) No.; As mentioned, $\mathcal{E}$ may be a line.
   (e) Yes.; Yes.; No.
3. (a) Yes.; Yes.; Yes.; No. [If $\mathcal{E}$ is a plane and $\mathcal{T}$ is its direction then all postulates are satisfied.]

(b) Does it follow that $\mathcal{T}$ contains two linearly independent translations? Three linearly independent translations?

4. Suppose that $\mathcal{E}$ contains four noncoplanar points.
   (a) In this context, would you give a different answer to any of the questions in Exercise 3(a)?
   (b) How about the questions in Exercise 3(b)?

5. (a) It is implicit in Postulates 1 and 2 that neither $\mathcal{E}$ nor $\mathcal{T}$ is empty. For example, from Postulates 1(a) and 2(a) we can infer '$A + (B - A) \in \mathcal{E}$'; from this and Postulate 2(a) we can infer '$B \in \mathcal{E}$'; from this, '$\exists_X X \in \mathcal{E}$'. In a similar manner, prove '$\exists_{\vec{x}} \vec{x} \in \mathcal{T}$'.
   (b) Can you infer the last more simply from another postulate?

6. (a) Suppose that $\mathcal{T}$ contains a non-$\vec{0}$ translation. Show that there are [at least] two points. [*Hint*: Use the contrapositive of '$A + \vec{a} = A \longrightarrow a = \vec{0}$'.]
   (b) Suppose that there are two points. Show that there is a non-$\vec{0}$ translation.

7. Show that there are two linearly independent translations if and only if there are three noncollinear points. [*Hint*: If A, B, and C are noncollinear points, what translations do you know to be linearly independent? Given linearly independent translations $\vec{a}$ and $\vec{b}$, how can you describe three noncollinear points?]

8. Show that there are three linearly independent translations if and only if there are four noncoplanar points.

*

The preceding exercises show that, as far as our present postulates are concerned, $\mathcal{E}$ need not be a very "roomy" space. $\mathcal{E}$ may consist of a single point, or it may be a single line or a single plane. None of these possibilities is agreeable with the aims in this course. As was suggested on page 27, these aims are somewhat vague; but, at least, we wish our postulates to describe as completely as possible some of the aspects of the space around us. So far we have been able to show, for example, that, *given* three noncollinear points of $\mathcal{E}$, there are subsets of $\mathcal{E}$ which satisfy at least some of our intuitive notions of what planes should be like. In particular, given three such points it is not difficult to show that each point of $\mathcal{E}$ is contained in some line, and that each line is a subset of some plane. Given only three noncollinear points there may, however, be only one plane. To be sure of enough space to move around in we need to be assured of the existence of at least four noncoplanar points. As we have seen, one way to ensure this is to postulate the existence of three linearly independent translations. We do so by adding to Postulate 4:

> **Postulate 4_9** There are three linearly independent members of $\mathcal{T}$.

Answers for Part A [cont.]

(b) Yes.; No.

4. (a), (b) In each case, the answer for the last question should be changed to 'Yes.'.

5. (a)

$$\dfrac{\dfrac{\vec{a} = (A + \vec{a}) - A \qquad \dfrac{B - A \in \mathcal{T} \qquad A + \vec{a} \in \mathcal{E}}{(A + \vec{a}) - A \in \mathcal{T}}}{\vec{a} \in \mathcal{T}}}{\exists_{\vec{x}}\, \vec{x} \in \mathcal{T}}$$

   (b) Yes, from the postulate '$\vec{0} \in \mathcal{T}$'. [Also, from '$B - A \in \mathcal{T}$'.]

6. (a) Suppose that $\vec{a} \neq \vec{0}$. Since $A + \vec{a} = A \Longrightarrow \vec{a} = \vec{0}$ it follows that $A + \vec{a} \neq A$. Since $A + \vec{a} \in \mathcal{E}$ it follows that $\exists_Y \exists_X\, Y \neq X$. Hence, $\exists_{\vec{x}}\, \vec{x} \neq \vec{0} \Longrightarrow \exists_Y \exists_X\, Y \neq X$.
   (b) Suppose that $A \neq B$. Since $B - A = \vec{0} \Longrightarrow A = B$ it follows that $B - A \neq \vec{0}$. Since $B - A \in \mathcal{T}$ it follows that $\exists_{\vec{x}}\, \vec{x} \neq \vec{0}$. Hence, $\exists_Y \exists_X\, Y \neq X \Longrightarrow \exists_{\vec{x}}\, \vec{x} \neq \vec{0}$.

7. Suppose that $\{A, B, C\}$ is noncollinear. It follows that $(B - A, C - A)$ is linearly independent and, so, that $\exists_{\vec{x}} \exists_{\vec{y}} (\vec{x}, \vec{y})$ is linearly independent. Hence, $\exists_X \exists_Y \exists_Z \{X, Y, Z\}$ is noncollinear $\Longrightarrow \exists_{\vec{x}} \exists_{\vec{y}} (\vec{x}, \vec{y})$ is linearly independent. Suppose that $(\vec{a}, \vec{b})$ is linearly independent and suppose that $A \in \mathcal{E}$. Since $\vec{a} = (A + \vec{a}) - A$ and $\vec{b} = (A + \vec{b}) - A$ it follows that $\{A, A + \vec{a}, A + \vec{b}\}$ is noncollinear and, so, that $\exists_X \exists_Y \exists_Z \{X, Y, Z\}$ is noncollinear. Since, as indicated in Exercise 5, $\exists_X X \in \mathcal{E}$ it follows that

$$\exists_{\vec{x}} \exists_{\vec{y}} (\vec{x}, \vec{y}) \text{ is linearly independent}$$
$$\Longrightarrow$$
$$\exists_X \exists_Y \exists_Z \{X, Y, Z\} \text{ is noncollinear.}$$

8. [Merely insert extra letters in answer for Exercise 7 and replace 'noncollinear' by 'noncoplanar'.]

TC 401 (1)

Parts B and C make a reasonable homework assignment to follow the introduction of Postulate 4_9. We recommend, however, that you give at least two illustrations of paragraph proofs appropriate for Exercise 1 of Part B.

Answers for Part B

1. (a) Since, by 4_9, $\mathcal{T}$ has three linearly independent members it certainly has two linearly independent members. So, [as in Exercise 7 of Part A], $\mathcal{E}$ has three noncollinear points.
   (b) Since $\mathcal{E}$ contains four points [by Theorem 10-1], given any point of $\mathcal{E}$, there is another. Since, given two points, there is a line containing them [Theorem 7-1], any point of $\mathcal{E}$ is contained in a line. [Note that we can now be certain that $\{A, B, C\}$ is collinear, by Definition 7-1, if and only if A, B, and C are contained in some line.]

As is shown in Exercise 8 it follows that

‖ Theorem 10–1 There are four noncoplanar points.

[As is also shown in Exercise 8, we might have chosen Theorem 10–1 as our new postulate, and called Postulate 4_g 'Theorem 10–1'.]

Part B

1. Give a short paragraph proof for each of these statements.
   - **(a)** There are three noncollinear points in $\mathcal{E}$.
   - **(b)** Each point is contained in a line.
   - **(c)** Each line is a subset of a plane.
   - **(d)** No plane contains all points of $\mathcal{E}$.
   - **(e)** Any plane contains at least three lines.
   - **(f)** There are at least six lines and four planes.
   - **(g)** Given any line, there is another line parallel to the given line.
   - **(h)** Given any plane, there is another plane parallel to the given plane.
   - **(i)** Any plane has a transversal.
2. **(a)** How many lines does any plane contain?
   - **(b)** How many planes does $\mathcal{E}$ contain?
   - **(c)** Given any line, how many lines in $\mathcal{E}$ are parallel to that line?
   - **(d)** Given any plane, how many planes are parallel to that plane?

Part C

1. Suppose that $\vec{a} \neq \vec{0}$. Draw a figure, and complete these sentences.
   - **(a)** Since $\vec{a} \neq \vec{0}$, $(\vec{a})$ is ________ ________.
   - **(b)** If $(\vec{a} \in [l]$ and $A \in l)$ then $A + \vec{a} \in$ ________.
   - **(c)** If $(\vec{a} \in [l]$ and $\vec{b} \in [l])$ then $(\vec{a}, \vec{b})$ is ________ ________.
   - **(d)** If $(\vec{a} \in [l]$ and $\vec{b} \notin [l])$ then $(\vec{a}, \vec{b})$ is ________ ________.
   - **(e)** Show a vector $\vec{b}$ such that $(\vec{a}, \vec{b})$ is linearly independent.
2. Suppose that $(\vec{a}, \vec{b})$ is linearly independent. Draw a figure, and complete these sentences.
   - **(a)** If $(\vec{a} \in [\pi]$ and $\vec{b} \in [\pi])$ then $[\pi] =$ ________.
   - **(b)** If $(\vec{a} \in [\pi.$ and $\vec{b} \in [\pi]$ and $A \in \pi)$ then $A + (\vec{a} + \vec{b}3) \in$ ________.
   - **(c)** If $\{\vec{a}, \vec{b}, \vec{c}\} \subseteq [\pi]$ then $(\vec{a}, \vec{b}, \vec{c})$ is ________ ________.
   - **(d)** Describe a translation $\vec{c}$ such that $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent.
3. The figure shows three linearly independent translations, $\vec{a}$, $\vec{b}$, and $\vec{c}$. Copy the figure on page 402.
   - **(a)** Given a plane π_1 such that $[\pi_1] = [\vec{a}, \vec{b}]$, is it the case that $\vec{c} \in [\pi_1]$?
   - **(b)** Suppose that R is a point of π_1 and that π_2 is a plane that contains R and has direction $[\vec{b}, \vec{c}]$. Then $R \in \pi_1 \cap \pi_2$. Describe three other points of $\pi_1 \cap \pi_2$.

(c) By part (a), $\mathcal{E}$ contains three noncollinear points. So, given any line in $\mathcal{E}$, there is a point of $\mathcal{E}$ not on this line. By Theorem 9-4, the given line and any such point are contained in a plane.

(d) By Theorem 10-1 there are four noncoplanar points. Since no plane can contain all of four such points, no plane contains all points of $\mathcal{E}$.

(e) Any plane contains three noncollinear points and, by Theorem 9-1, any two of three such points are contained in just one line, and no such line contains all three points. Since three pairs can be chosen from three objects, it follows that any plane contains at least three lines. [This theorem does not depend on 4_g.]

(f) Since, by Theorem 10-1, there are four noncoplanar points and any three of four such points are contained in a plane, and no plane contains all four points, there are at least as many planes as there are triples which can be chosen from four objects. So, there are at least four planes. Since no three of four noncoplanar points can be collinear, any two of four noncollinear points determine a line not containing the other two. Since six pairs can be chosen from four objects, there are at least six lines.

(g) Given any line ℓ, there is, by part (c), a plane containing ℓ. This plane contains a point not on ℓ. If A is any such point then $\overleftrightarrow{A[\ell]} \parallel \ell$ and $\overleftrightarrow{A[\ell]} \neq \ell$.

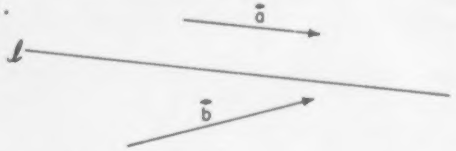
(h) [Like answer for part (g), but use part (d).]

(i) Let A be any point of the given plane. By part (d) there is a point not on this plane. Then $\overleftrightarrow{AB}$ is a transversal of the given plane.

2. [For each part, an acceptable answer is 'infinitely many'.]

Answers for Part C

1.



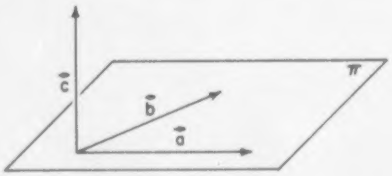
(a) linearly independent

(b) ℓ

(c) linearly dependent

(d) linearly independent

2.



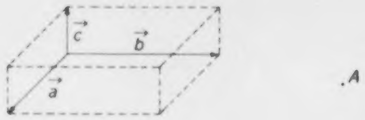
(a) $[\vec{a}, \vec{b}]$

(b) π

(c) linearly dependent

3. (a) No.

(b) $[\forall_x\ R + \vec{b}x \in \pi_1 \cap \pi_2]$



(c) Point A is shown in the figure. In your copy, show the position of the point $A + (\vec{a} + \vec{b} + \vec{c}3)$ and label it 'B'.

(d) What can you say about the sequence $(\vec{a}, \vec{b}, \vec{c}, B - A)$?

(e) Can you visualize the position of a point C such that $(\vec{a}, \vec{b}, \vec{c}, C - A)$ is linearly independent?

4. Given the linearly independent sequence $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ and a point A, consider the planes $\overline{A[\vec{a}, \vec{b}]}$ and $\overline{A[\vec{c}, \vec{d}]}$.

   (a) How do you know that $\overline{A[\vec{a}, \vec{b}]}$ is a plane?

   (b) Do $\overline{A[\vec{a}, \vec{b}]}$ and $\overline{A[\vec{c}, \vec{d}]}$ have a point in common?

   (c) How many points do $\overline{A[\vec{a}, \vec{b}]}$ and $\overline{A[\vec{c}, \vec{d}]}$ have in common?

5. Suppose that $A \in \pi_1 \cap \pi_2$, $[\pi_1] = [\vec{a}, \vec{b}]$, and $[\pi_2] = [\vec{c}, \vec{d}]$. Show that $\pi_1 \cap \pi_2 = \{A\}$ if and only if $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly independent.

6. Suppose, given the linearly independent sequence $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ and the point A, that $l = \overleftrightarrow{A[\vec{a}]}$, and $\pi = \overline{(A + \vec{d})[\vec{b}, \vec{c}]}$.

   (a) Is l parallel to π?

   (b) Do l and π have a common point?

7. Consider a line l and a plane π, where $l = \overleftrightarrow{A[\vec{a}]}$ and $\pi = \overline{B[\vec{b}, \vec{c}]}$. Show that if $l \not\parallel \pi$ and $l \cap \pi = \emptyset$ then there are four linearly independent translations. [*Hint*: You know that $(\vec{b}, \vec{c})$ is linearly independent, that $\vec{a} \notin [\vec{b}, \vec{c}]$ (Why?), and that $B - A \notin [\vec{a}, \vec{b}, \vec{c}]$ (Why?).]

8. Modify your work in solving Exercises 6 and 7 to show that there exist nonparallel planes which have no common point if and only if there exist four linearly independent translations. [*Hint*: In Exercise 7, consider planes $\overline{A[\vec{a}, \vec{b}]}$ and $\overline{B[\vec{c}, \vec{d}]}$ and show that some four of $\vec{a}, \vec{b}, \vec{c}, \vec{d}$, and $B - A$ must be linearly independent if the planes are nonparallel and have no common point.]

*

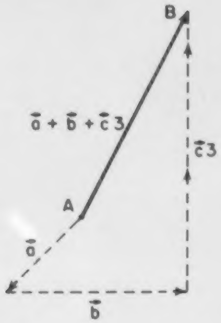
As is shown by Theorem 10-1 and—perhaps, better—by parts (g), (h), and (i) of Exercise 1 of Part B, Postulate 4_0 ensures that $\mathscr{E}$ is roomy enough for us to move around in. In short, Theorem 10-1 may be taken as asserting that

$\mathscr{E}$ is *at least* 3-dimensional.

It is customary, when dealing with vector spaces, to say that a vector space is at least *n-dimensional* if and only if it has at least n linearly

Answers for Part C [cont.]

3. (c)



(d) It is linearly dependent.

(e) No. [Unless you can visualize 4-dimensional space.]

4. (a) By Theorem 9-11(a). For, since $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly independent, so is $(\vec{a}, \vec{b})$.

   (b) They have the point A in common.

   (c) Only one, the point A. For, suppose that B belongs to both planes. Then there are numbers — say, a, b, c, and d — such that $B - A = \vec{a}a + \vec{b}b$ and $B - A = \vec{c}c + \vec{d}d$. It follows that $\vec{a}a + \vec{b}b = \vec{c}c + \vec{d}d$ and, since $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly independent, that $a = b = c = d = 0$. So, $B = A$.

5. The if-part is proved in the answer for Exercise 4(c). Suppose, then, that $\pi_1 \cap \pi_2 = \{A\}$. Suppose, now [to test for linear independence], that $\vec{a}a + \vec{b}b = \vec{c}c + \vec{d}d$. It follows that P, where $P = A + (\vec{a}a + \vec{b}b) = A + (\vec{c}c + \vec{d}d)$, belongs to both planes. So, $P = A$ and $\vec{a}a + \vec{b}b = \vec{0} = \vec{c}c + \vec{d}d$. Since $(\vec{a}, \vec{b})$ and $(\vec{c}, \vec{d})$ are linearly independent it follows that $a = b = 0$ and $c = d = 0$. Hence, $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly independent. [Note that it has been proved that there exist two planes which intersect in a single point if and only if there are four linearly independent translations. Combining this with the result of Exercise 2(a) of Part C on page 391, it follows that there exist two planes whose intersection is neither $\emptyset$ nor a line if and only if there exist four linearly independent translations.]

6. (a) Since $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly independent, so is $(\vec{a}, \vec{b}, \vec{c})$. So, $\vec{a} \notin [\vec{b}, \vec{c}]$ and $[\vec{a}] \not\subseteq [\vec{b}, \vec{c}]$. Hence, $\ell \not\parallel \pi$.

   (b) If $P \in \ell \cap \pi$ then there are numbers — say, a, b, and c — such that $P = A + \vec{a}a = (A + \vec{d}) + (\vec{b}b + \vec{c}c)$. From this it follows that $\vec{a}a = \vec{d}1 + \vec{b}b + \vec{c}c$ and, since $1 \neq 0$, that $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly dependent. Since the latter is not the case, $\ell \cap \pi = \emptyset$. The answer to the question is 'No.'.

Answers for Part C [cont.]

7. Suppose that $\ell \nparallel \pi$ and $\ell \cap \pi = \emptyset$. It follows that $[\ell] \not\subseteq [\pi]$ and, so, that $\vec{a} \notin [\vec{b}, \vec{c}]$. Since $[\vec{b}, \vec{c}]$ is a bidirection, $(\vec{b}, \vec{c})$ is linearly independent and it follows by Theorem 6-13 that $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent. If $B - A \in [\vec{a}, \vec{b}, \vec{c}]$ then B plus some linear combination of $\vec{b}$ and $\vec{c}$ would equal A plus some multiple of $\vec{a}$ and the point described in either of these equivalent ways would belong to both π and ℓ. Since there is no such point, $B - A \notin [\vec{a}, \vec{b}, \vec{c}]$. Since $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent it follows, by Theorem 6-13, that $(\vec{a}, \vec{b}, \vec{c}, B - A)$ is linearly independent. So, there are four linearly independent translations. [Note that it has been shown in Exercises 6 and 7 that there exist a line and a plane which are not parallel and do not intersect if and only if there are four linearly independent translations.]

8. Suppose that $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly independent. Since $(\vec{a}, \vec{b})$ and $(\vec{b}, \vec{c})$ are linearly independent, $A[\vec{a}, \vec{b}]$ and $(A + \vec{d})[\vec{b}, \vec{c}]$ are planes. Since $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent, $\vec{c} \notin [\vec{a}, \vec{b}]$ and, so, $[\vec{a}, \vec{b}] \neq [\vec{b}, \vec{c}]$ and the planes are not parallel. If the planes had a point in common then there would be numbers — say, a, b_1, b_2, and c — such that $A + (\vec{a}a + \vec{b}b_1) = (A + \vec{d}1) + (\vec{b}b_2 + \vec{c}c)$ and, since $1 \neq 0$, $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ would be linearly dependent. Since the latter is not the case, the planes have no common point. Hence, if there exist four linearly independent translations then there exist two planes which neither are parallel nor have a common point.

Suppose, now, that π and σ are planes such that $\pi \nparallel \sigma$ and $\pi \cap \sigma = \emptyset$. We may, of course, assume that $\pi = A[\vec{a}, \vec{b}]$ and $\sigma = B[\vec{c}, \vec{d}]$, where $(\vec{a}, \vec{b})$ and $(\vec{c}, \vec{d})$ are linearly independent. Since $\pi \cap \sigma = \emptyset$ it follows that $B - A \notin [\vec{a}, \vec{b}, \vec{c}, \vec{d}]$. Since $\pi \nparallel \sigma$ it follows that $[\vec{a}, \vec{b}] \neq [\vec{c}, \vec{d}]$. Since $(\vec{c}, \vec{d})$ is linearly independent it follows by the lemma to Theorem 9-1 that $\{\vec{c}, \vec{d}\} \not\subseteq [\vec{a}, \vec{b}]$. Suppose that $\vec{c} \notin [\vec{a}, \vec{b}]$. Since $(\vec{a}, \vec{b})$ is linearly independent it follows, by Theorem 6-13, that $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent. Since $B - A \notin [\vec{a}, \vec{b}, \vec{c}]$ it follows, by Theorem 6-13, that $(\vec{a}, \vec{b}, \vec{c}, B - A)$ is linearly independent. So, if $\vec{c} \notin [\vec{a}, \vec{b}]$ then $(\vec{a}, \vec{b}, \vec{c}, B - A)$ is linearly independent. Similarly, if $\vec{d} \notin [\vec{a}, \vec{b}]$ then $(\vec{a}, \vec{b}, \vec{d}, B - A)$ is linearly independent. Since, as shown previously, one of $\vec{c}$ and $\vec{d}$ must fail to belong to $[\vec{a}, \vec{b}]$, it follows that there exist four linearly independent translations. Hence, if there exist two planes which neither are parallel nor have a common point then there exist four linearly independent translations.

In discussing this page make sure that students are aware of the equivalence, as a postulate, of Postulate 4_{10} and any of:

> No two planes intersect in a single point.

[See note to the answer for Exercise 5 of Part C.]

> Any line and plane either are parallel or have [at least] a point in common.

[See note to the answer for Exercise 7.]

> Any two planes either are parallel or have [at least] a point in common.

Point out that, in particular, having adopted Postulate 4_{10}, each of these three sentences is, now, a theorem. [Their proofs are given in the answer for Exercise 5, in the answer for Exercise 7, and in the second part of the answer for Exercise 8, respectively.]

Next, point out that the first of these three theorems and the result of Exercise 2(a) of Part C on page 391 have, as an immediate consequence:

> The intersection of any two planes is either empty or a line.

or, equivalently:

> Two planes which have a point in common have a line as their intersection.

TC 404

Again, because of the discussions usually precipitated by the exercises, we recommend Part D as class discussion exercises. Parts E and F provide a homework assignment to follow such discussions.

Answers for Part D

1. (a) 2 (b) most; 2; dimensional
   (c) 2; dimensional; vector; space
2. Yes.; Such a subset is called a proper bidirection.

independent members. So, by definition, Postulate 4_9 can be abbreviated to:

$\mathcal{T}$ is at least 3-dimensional.

On the other hand, Postulate 4_9 puts no upper limit on the number of linearly independent translations. As a consequence of this our present postulates allow for the existence of planes which intersect at a single point, of a line and a plane which have no common point but are not parallel, and of skew planes. As far as our intuitive notions of $\mathcal{E}$ go, there should not be room enough in $\mathcal{E}$ for such situations to occur. According to Exercises 4 – 8 of Part C, ruling out any of these possibilities amounts to adopting:

> **Postulate 4_{10}** There are not four linearly independent members of $\mathcal{T}$.

By the definition mentioned in the preceding paragraph, Postulate 4_{10} amounts to saying that it is not the case that $\mathcal{T}$ is at least 4-dimensional. More simply:

$\mathcal{T}$ is *at most* 3-dimensional.

As was shown by Exercise 5 we might have adopted the following theorem as a postulate in place of Postulate 4_{10}.

No two planes intersect in a single point.

Other alternatives to Postulate 4_{10} were suggested by Exercises 6 and 7:

A line which is not parallel to a given
plane contains a point of that plane.

[or: A line and a plane which have no common point are parallel.] and by Exercise 8:

Two nonparallel planes have a nonempty intersection.

[or: Two planes which have no point in common are parallel.]
Any of these alternatives to Postulate 4_{10} may be taken as asserting that

$\mathcal{E}$ is at most 3-dimensional.

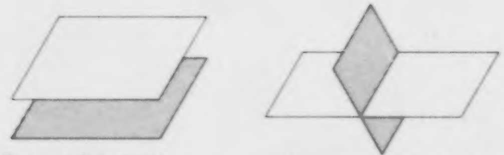


Fig. 10 – 1

The first and third of these may, in view of earlier theorems, be combined into:

> **Theorem 10 – 2** The intersection of two nonparallel planes is a line.

Note that Postulate 4_{10} is equivalent to:

Any four members of $\mathcal{T}$ are linearly dependent.

Since Postulates 4_9 and 4_{10} amount to saying that $\mathcal{T}$ is exactly 3-dimensional we may summarize Postulates 4_9 – 4_{10} in:

> **Postulate 4′** $\mathcal{T}$, under function composition, is a 3-dimensional vector space over $\mathcal{R}$.

[Postulates 4_0 – 4_8 are given on page 191 and are summarized succinctly in Postulate 4″ on page 191.] As the ″′ suggests, we shall have another bunch of parts to include in our final Postulate 4. These will be introduced in the second volume.

Part D

Suppose that, instead of adopting Postulates 4_9 and 4_{10}, we choose to adopt the following statements as postulates:

(i) There are two linearly independent members of $\mathcal{T}$.
(ii) Any three members of $\mathcal{T}$ are linearly dependent.

1. **(a)** Statement (i) tells us that our space is at least ____-dimensional.
   (b) Statement (ii) tells us that our space is at ________ ____-____________.
   (c) Statements (i) and (ii) can be summarized as follows:

   $\mathcal{T}$ is a ____-____________ __________ __________.

2. Do you suppose that our 3-dimensional vector space $\mathcal{T}$ has a subset which satisfies (i) and (ii)? If so, do we have a name for this kind of subset?

3. Under the assumptions made for this part, how might you describe $\mathscr{E}$? Which of the following would you expect to be theorems?
   (a) There are four noncoplanar points.
   (b) There are three noncollinear points.
   (c) Each two points are on a line.
   (d) Given any line, there is another line parallel to it.
   (e) Two lines have at most one point in common.
   (f) Two lines can have different directions and still not intersect.
   (g) Two lines are either parallel or they intersect.
4. Give a reason for rejecting any of those statements in Exercise 3 which you did not select as possible theorems.

Part E

1. Repeat Exercises 1 and 2 of Part D with (i) and (ii) replaced by:

   There is a non-$\vec{0}$ member of $\mathscr{T}$.
   Any two members of $\mathscr{T}$ are linearly dependent.

2. (a) Write statements similar to (i) and (ii) of Part D which, together with Postulates $4_0 - 4_8$, can be summarized as:

   (*) $\mathscr{T}$, under function composition,
   is a 4-dimensional vector space.

   (b) Describe some of the "peculiar" situations which would occur in $\mathscr{E}$ if Postulates 1 – 3 and (*) were satisfied.
   (c) Give some of the numbered theorems in this and earlier chapters which would *not* be theorems if Postulate 4′ were replaced by (*).
3. (a) Show that the direction of a line is a 1-dimensional vector space.
   (b) Show that the direction of a plane is a 2-dimensional vector space.

Part F

1. (a) Consider the plane $\overleftrightarrow{OUV}$ [so, $\{O, U, V\}$ is noncollinear]. Show that

$$O + (U - O)p + (V - O)q = O + (U - O)r + (V - O)s$$

   if and only if $p = r$ and $q = s$.
   (b) Consider the plane $O\overleftrightarrow{[\vec{u}, \vec{v}]}$ [so, $(\vec{u}, \vec{v})$ is linearly independent]. State and prove a result similar to that of part (a).
   (c) What do the results of parts (a) and (b) tell you about a plane and the set $\mathscr{R} \times \mathscr{R}$ of all ordered pairs of real numbers?
   (d) What analogous result have you previously established concerning lines?

Answers for Part D [cont.]

3. Under these assumptions $\mathcal{E}$ is a plane. Statements (b), (c), (d), (e) and (g) are theorems. [In fact, (c) and (e) are, as we have shown in Chapter 7, independent of any postulate of dimensionality.]

4. By Exercise 8 of Part A on page 400, (a) of Exercise 3 contradicts assumption (ii). Statement (f) contradicts (g) which, by (ii) and Theorem 9-7, would be a theorem under the present assumptions. Moreover, by the method used in the second part of the answer for Exercise 8 of Part C on page 402, it could be proved that (f) implies the existence of three linearly independent translations.

Answers for Part E

1. The assumptions considered in this exercise tell us that $\mathcal{E}$ is at least and at most 1-dimensional. They can be summarized as follows:

   $\mathcal{T}$ is a 1-dimensional vector space.

   Our 3-dimensional vector space $\mathcal{T}$ has many 1-dimensional subspaces. We have called them proper directions.

2. (a) There are four linearly independent members of $\mathcal{T}$.; Any five members of $\mathcal{T}$ are linearly dependent.

   (b) If ($\star$) were added to Postulate 1-3 [and 5] then, as proved in the exercises for Part C, there would exist pairs of planes which have exactly one common point; there would exist lines and planes which are not parallel but do not intersect; and there would exist skew planes. $\mathcal{E}$ would contain 3-dimensional "hyperplanes" [defined in analogy with Definitions 7-2 and 9-2] which would have many of the properties which planes have in our 3-dimensional space. For example, two such hyperplanes would be parallel or have a plane as their intersection. A hyperplane would be determined by four noncoplanar points, by a plane and a line transverse to it. Each hyperplane would be "just like" the 3-dimensional space we are studying and, for example, in analogy with Theorem 9-7, we could prove that two nonparallel planes contained in the same hyperplane intersect in a line. [In fact, the proof would be precisely that of Theorem 10-2.]

   (c) Of course all the theorems of earlier chapters would remain theorems. Theorem 10-1 would also be a theorem, but Theorem 10-2 would not. [What would take the place of Theorem 10-2 has been indicated above.]

3. That, for any $\vec{a}$ and $\vec{b}$, $[\vec{a}]$ and $[\vec{a}, \vec{b}]$ are vector spaces has been proved in Parts A and B on pages 192 and 193. It remains only to be shown that if $\vec{a} \neq \vec{0}$ then $[\vec{a}]$ is 1-dimensional, and that if $(\vec{a}, \vec{b})$ is linearly independent then $[\vec{a}, \vec{b}]$ is 2-dimensional.

   (a) Suppose that $\vec{a} \neq \vec{0}$. Then $(\vec{a})$ is linearly independent and $\vec{a} \in [\vec{a}]$. So, $[\vec{a}]$ is at least 1-dimensional. Supposing, now, that $\{\vec{b}, \vec{c}\} \subseteq [\vec{a}]$, we need to show that $(\vec{b}, \vec{c})$ is linearly dependent. This is obviously the case if $\vec{b} = \vec{0}$. On the other hand, if $\vec{b} \neq \vec{0}$ then $[\vec{a}] = [\vec{b}]$ and since $\vec{c} \in [\vec{a}] = [\vec{b}]$, $(\vec{b}, \vec{c})$ is linearly dependent. So, $[\vec{a}]$ is at most 1-dimensional.

Answers for Part E [cont.]

(b) Suppose that $(\vec{a}, \vec{b})$ is linearly independent. Since $\{\vec{a}, \vec{b}\} \subseteq [\vec{a}, \vec{b}]$ it follows that $[\vec{a}, \vec{b}]$ is at least 2-dimensional. On the other hand, by Exercise 3(b) of Part D on page 384, $[\vec{a}, \vec{b}]$ is at most 2-dimensional.

*

Students whose interest in 4-dimensional space is aroused by, say, Exercise 2, may consult H. P. Manning, Geometry of Four Dimensions, Dover, 1956. [But, they will find it tough going.] See, also, D. M. Y. Sommerville, An Introduction to the Geometry of N Dimensions, Dover, 1958.

Answers for Part F

1. (a) [Compare with Theorem 7-15.] Suppose that

$$(*)\quad O + (U - O)p + (V - O)q = O + (U - O)r + (V - O)s.$$

It follows that $(U - O)(p - r) + (V - O)(q - s) = \vec{0}$. Since $\{O, U, V\}$ is noncollinear, $(U - O, V - O)$ is linearly independent. It follows that $p - r = 0 = q - s$. Hence, if $(*)$ then $p = r$ and $q = s$. On the other hand, $(*)$ holds trivially if $p = r$ and $q = s$.

(b) [Merely replace '$(U - O)$' in the preceding by '$\vec{u}$', and '$(V - O)$' by '$\vec{v}$', and delete the sentence beginning with 'Since'.

(c) Corresponding to any 3-termed sequence of noncollinear points of a plane there is a one-to-one correspondence between the points of the plane and the members of $\mathcal{R} \times \mathcal{R}$. [By (b), instead of three points, one may start with one point of the plane and two linearly independent members of the direction of the plane.]

(d) Theorem 7-15.

2. Suppose that $\{O, U, V, W\}$ is noncoplanar. [Draw a figure.]
   - (a) Is there a point – say, A – such that $A - O$ is not a linear combination of $U - O$, $V - O$, and $W - O$? Explain.
   - (b) State and prove a result analogous to that of Exercise 1(a).
   - (c) Formulate and answer a question like that of Exercise 1(c).

10.02 Intersections

Now that we have postulated that $\mathscr{T}$ – and, hence, $\mathscr{E}$ – is 3-dimensional we are in a position to tidy up some of the theorems on lines and planes which were proved in Chapter 9. [You have already done much of the necessary work in Part C of the preceding exercises.]

Before doing this, let's recall a problem which arose in Chapter 7 in connection with collinearity. Having defined 'collinear', by referring to collinear points, we wished to make sure that

if $\{A, B, C\}$ is collinear then there
is a line which contains A, B and C.

This was easy to do in case $A \neq B$. [In that case, $\overleftrightarrow{AB}$ is a line which contains A and B and, if $\{A, B, C\}$ is collinear, also contains C.] The cases in which $B \neq C$ and $C \neq A$ are, of course, treated in the same way; but that in which $A = B = C$ could not be settled at that time. You can now see why. Use the notion of dimension to do so. [*Hint*: Recall Exercise 1 of Part A on page 399. In the context of this exercise, what is the dimension of $\mathscr{E}$?] We can now settle the matter by proving that each point belongs to some line. [Which of Postulates 4_9 and 4_{10} is useful in doing this?]

You may have thought of a similar problem involving 'coplanar' and 'plane'. Suppose that $\{A, B, C, D\}$ is coplanar. Does it follow that there is a plane which contains A, B, C, and D? As in the previous case, it is easy to see that there is such a plane in case $\{A, B, C\}$ [or $\{B, C, D\}$, or $\{C, D, A\}$, or $\{D, A, B\}$] is noncollinear. One way to complete the argument is to show that in the remaining case there is a line which contains A, B, C, and D, and then to show that each line is a subset of some plane. Do this.

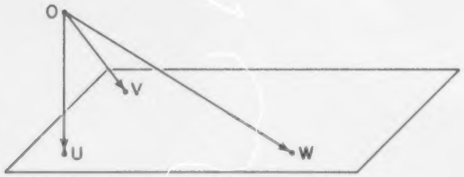
The last theorem – 'Each line is a subset of a plane.' – has another application. Recall that we were able to show that *two* parallel lines are coplanar. But, up to this point we could not have proved:

$$l_1 \parallel l_2 \longrightarrow l_1 \text{ and } l_2 \text{ are coplanar}$$

Explain.

The preceding problems are not very exciting. The more interesting problems are those whose solution requires us to assume that there is not too much room in $\mathscr{E}$. The question as to whether $\mathscr{E}$ contains skew

2.



(a) No. Since $\{O, U, V, W\}$ is noncoplanar, $(U - O, V - O, W - O)$ is linearly independent. By Postulate 4_{10}, $(U - O, V - O, W - O, A - O)$ is linearly dependent. So, by Theorem 6-13, $A - O \in [U - O, V - O, W - O]$.

(b) For $\{O, U, V, W\}$ noncoplanar,

$$O + (U - O)p + (V - O)q + (W - O)r$$
$$= O + (U - O)s + (V - O)t + (W - O)u$$

if and only if $p = s$, $q = t$, and $r = u$.

[The proof is too like that given in answer to Exercise 1(a) to bear repetition here.]

(c) Given any 4-termed sequence of noncoplanar points, there is a one-to-one correspondence between the points of $\mathscr{E}$ and the members [ordered triples of real numbers] of $\mathscr{R} \times \mathscr{R} \times \mathscr{R}$.

As shown in Exercise 1 on page 399, in the absence of dimension postulates $\mathscr{E}$ might consist of a single point — and the dimension of $\mathscr{E}$ would then be 0. If so, it would not be the case that each point of $\mathscr{E}$ belongs to a line. Postulate 4_9, however, makes it possible to show that each point of $\mathscr{E}$ belongs to a line. A weaker postulate — $\exists_{\vec{x}}\ \vec{x} \neq \vec{0}$ — would do as well for this purpose.

The proof that if $\{A, B, C, D\}$ is coplanar and $\{A, B, C\}$ is noncollinear then there is a plane which contains A, B, C, and D can be carried out by using definitions and Theorem 6-13. In case each of the sets $\{A, B, C\}$, $\{B, C, D\}$, $\{C, D, A\}$ and $\{D, A, B\}$ is collinear we prove that $\{A, B, C, D\}$ is a subset of a line by considering two cases — that in which two of A, B, C, and D are different and that [treated above] in which $A = B = C = D$. If, for example, $A \neq B$ then, by definition, C and D [as well as A and B] belong to the line $\overleftrightarrow{AB}$ because $\{A, B, C\}$ and $\{D, A, B\}$ are collinear. That each line is a subset of a plane is Exercise 1(c) of Part B on page 401.

The proof of:

$$\ell_1 \parallel \ell_2 \Longrightarrow \ell_1 \text{ and } \ell_2 \text{ are coplanar}$$

is treated in two cases. In case $\ell_1 \neq \ell_2$ use Theorem 9-6. In case $\ell_1 = \ell_2$ use Exercise 1(c).

planes is an example. As we have seen, this is settled by Theorem 10–2 which is, essentially, equivalent to Postulate 4_{10}. In view of this theorem we can restate the theorem you proved in Exercise 4 of Part C on page 392 as follows:

> **Theorem 10–3** A line which is parallel to each of two nonparallel planes is parallel to their intersection.

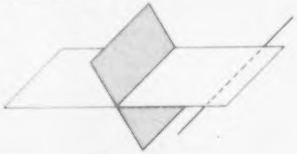


Fig. 10–2

Here is another theorem which depends essentially on the assumption that $\mathcal{E}$ is at most 3-dimensional:

> **Theorem 10–4** A line and a plane which are not parallel intersect at a single point.

This theorem has two useful corollaries:

> **Corollary 1** A line which is a transversal of one plane is a transversal of any parallel plane.

> **Corollary 2** Parallel lines are transversals of the same planes.

[Notice in what way these corollaries are stronger than the results used in proving Theorem 9–15.] We can also beef up Theorem 9–14:

> **Theorem 10–5** A plane which intersects one of two parallel planes intersects the other, and the intersections are parallel lines.

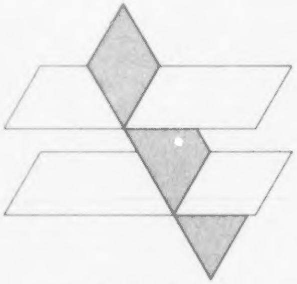


Fig. 10–3

In proving Theorem 9-17, we assumed that a given line was a transversal of each of certain parallel planes. By Corollary 1 it would now be sufficient to assume that the line is a transversal of one of the planes. We also had to assume that a line parallel to the given transversal intersected one of the planes in order to show that it was a transversal of each of them. Now that Corollary 2 is available this assumption is no longer necessary.

That neither of the corollaries to Theorem 10-4 holds if, contrary to Postulate 4_{10}, there are four linearly independent translations is easy to prove. For, if $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly independent and $A \in \mathcal{E}$ then $A[\vec{a}]$ is a transversal of $\overline{A[\vec{b}, \vec{c}]}$ which does not intersect the parallel plane $(A + \vec{d})[\vec{b}, \vec{c}]$ [if it did, some multiple of $\vec{a}$ would be the sum of $\vec{d}$ and some linear combination of $\vec{b}$ and $\vec{c}$, contrary to the linear independence of $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$]; and, while $\overline{A[\vec{a}]}$ is a transversal of $\overline{A[\vec{b}, \vec{c}]}$, the parallel line $\overline{(A + \vec{d})[\vec{a}]}$ is not [if it were, the sum of $\vec{d}$ and some multiple of $\vec{a}$ would be a linear combination of $\vec{b}$ and $\vec{c}$].

Theorem 10-5 is not quite correct, since it leaves open the possibility that the first-mentioned plane might be the first of the two parallel planes.

Finally, we can now settle two questions brought up on page 390:

> **Theorem 10-6** (a) $\sigma \parallel \pi \longleftrightarrow (\sigma = \pi \text{ or } \sigma \cap \pi = \emptyset)$
> (b) $l \parallel \pi \longleftrightarrow (l \subseteq \pi \text{ or } l \cap \pi = \emptyset)$

[Compare Theorem 10-6(a) with Theorem 9-8 – which we could now rewrite as:

$$l \parallel m \longleftrightarrow (l \text{ and } m \text{ are coplanar and } (l = m \text{ or } l \cap m = \emptyset))$$

Do you see the effect, on Theorem 10-6(a), of Postulate 4_{10}?]

Proofs of Theorems 10-2 through 10-6 can be obtained by putting together results you have already proved. As indicated by the hints for some of the following exercises, it will be a good idea to repeat some of the work you have already done.

Exercises

1. **(a)** Show that the intersection of two planes which have two points in common is a line.
   (b) Show that two planes which have one point in common have another point in common.
   (c) Show that two planes which are not parallel have a common point.
   (d) Prove Theorem 10-2.

   [*Hint*: For part (b), show first that planes $\overleftrightarrow{A[\vec{a}, \vec{b}]}$ and $\overleftrightarrow{A[\vec{c}, \vec{d}]}$ have a point other than A in common if and only if $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly dependent. For part (c), show that $\overleftrightarrow{A[\vec{a}, \vec{b}]}$ and $\overleftrightarrow{B[\vec{c}, \vec{d}]}$ have a point in common if and only if $B - A \in [\vec{a}, \vec{b}, \vec{c}, \vec{d}]$. Next, show that if $(\vec{c}, \vec{d})$ is linearly independent and $[\vec{c}, \vec{d}] \neq [\vec{a}, \vec{b}]$ then $\vec{c}$ and $\vec{d}$ do not both belong to $[\vec{a}, \vec{b}]$. Conclude that if $[\vec{a}, \vec{b}]$ and $[\vec{c}, \vec{d}]$ are two proper bidirections then either $(\vec{a}, \vec{b}, \vec{c})$ or $(\vec{a}, \vec{b}, \vec{d})$ is linearly independent. Show that, in the first case $B - A \in [\vec{a}, \vec{b}, \vec{c}] = [\vec{a}, \vec{b}, \vec{c}, \vec{d}]$ and, in the second, $B - A \in [\vec{a}, \vec{b}, \vec{d}] = [\vec{a}, \vec{b}, \vec{c}, \vec{d}]$.]
2. Prove Theorem 10-3. [*Hint*: Here a reference to Theorem 10-2 and an earlier exercise may be sufficient. If you have doubts, repeat the work of the earlier exercise.]
3. Prove Theorem 10-4. [As in Exercise 1(c), your principal tool is the fact that if $\vec{a} \notin [\vec{b}, \vec{c}]$, where $(\vec{b}, \vec{c})$ is linearly independent, then $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent. What theorem tells you this?]
4. Prove Corollary 1 of Theorem 10-4. [*Hint*: It is helpful to realize that a line is a transversal of a plane if and only if the line and plane are not parallel. One part of this is Theorem 10-4. Where did you prove the other part?]
5. Prove Corollary 2 of Theorem 10-4.

Taking into account results obtained in Exercise 1 of Part B on page 390, what remains to the proof of Theorem 10-6 are the proofs of:

(a) $\sigma \cap \pi = \emptyset \Longrightarrow \sigma \parallel \pi$

(b) $\ell \cap \pi = \emptyset \Longrightarrow \ell \parallel \pi$

As noted on page 403, each of these can [and, in fact, has been] proved by the use of Postulate 4_{10}.

Theorem 10-6(a) is analogous to the displayed form of Theorem 9-8. But, because of Postulate 4_{10} there is no need in Theorem 10-6(a) for a clause analogous to 'ℓ and m are coplanar'. There would be no need for this phrase in Theorem 9-8 if, in place of Postulate 4_{10}, we had assumed that there are no three linearly independent members of $\mathcal{T}$. On the other hand, the analogous clause, 'σ and π are cohyperplanar', would be required in Theorem 10-6 if Postulate 4_{10} were not assumed to hold.

Answers for Exercises

1. (a) By Theorem 9-3, two planes which have two points in common have the line determined by these points as a subset of their intersection. By the corollary to Theorem 9-1, the intersection cannot contain any other points.

   (b) We may assume that the planes in question are $\overleftrightarrow{A[\vec{a}, \vec{b}]}$ and $\overleftrightarrow{A[\vec{c}, \vec{d}]}$. By Postulate 4_{10}, there are numbers — say, a, b, c, and d — which are not all zero and are such that $\vec{a}a + \vec{b}b = \vec{c}c + \vec{d}d$. Since $(\vec{a}, \vec{b})$ and $(\vec{c}, \vec{d})$ are linearly independent it follows that $\vec{a}a + \vec{b}b \neq \vec{0}$. [If it were $\vec{0}$ then a, b, c, and d would all be 0.] Hence, $A + \vec{a}a + \vec{b}b$ is a point other than A which is common to the two planes.

   (c) We may assume that the planes are $\overleftrightarrow{A[\vec{a}, \vec{b}]}$ and $\overleftrightarrow{B[\vec{c}, \vec{d}]}$, where $[\vec{a}, \vec{b}] \neq [\vec{c}, \vec{d}]$. Since $(\vec{c}, \vec{d})$ is linearly independent it follows by the lemma to Theorem 9-1 that either $\vec{c} \notin [\vec{a}, \vec{b}]$ or $\vec{d} \notin [\vec{a}, \vec{b}]$. Since $(\vec{a}, \vec{b})$ is linearly independent it follows, by Theorem 6-13, that either $(\vec{a}, \vec{b}, \vec{c})$ or $(\vec{a}, \vec{b}, \vec{d})$ is linearly independent. In the first case, since it follows by Postulate 4_{10} that $(\vec{a}, \vec{b}, \vec{c}, B - A)$ is linearly dependent, it follows that $B - A \in [\vec{a}, \vec{b}, \vec{c}]$. So, B plus some multiple of $\vec{c}$ is the sum of A and some linear combination of $\vec{a}$ and $\vec{b}$. In other words, the planes have a common point. The second case — that in which $(\vec{a}, \vec{b}, \vec{d})$ is linearly independent — leads to the same conclusion. Hence, two nonparallel planes have a common point.

   (d) Given two nonparallel planes, these planes have a common point, by (c), and so, by (b), have two common points. Hence, by (a), their intersection is a line.

2. By Theorem 10-2, two nonparallel planes intersect in a line. By Exercise 4 of Part C on page 392, a line parallel to each of two such planes is parallel to their line of intersection. [Rather than Theorem 10-2, it is sufficient to refer to Exercise 1(c), above. Doing so will better bring out the relation of Theorem 10-3 to that of Exercise 4 on page 392.]

Answers for Exercises [cont.]

3. We may suppose the line and plane in question to be $\overleftrightarrow{A[\vec{a}]}$ and $B[\vec{b}, \vec{c}]$, where $[\vec{a}] \not\subseteq [\vec{b}, \vec{c}]$ and, so, $\vec{a} \notin [\vec{b}, \vec{c}]$. Since, also, $(\vec{b}, \vec{c})$ is linearly independent it follows, by Theorem 6-13, that $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent. Since, by Postulate 4_{10}, $(\vec{a}, \vec{b}, \vec{c}, B - A)$ is linearly dependent it follows, again by Theorem 6-13, that $B - A \in [\vec{a}, \vec{b}, \vec{c}]$. Hence, the sum of B and some member of $[\vec{b}, \vec{c}]$ is the sum of A and some member of $[\vec{a}]$. In other words, $\overleftrightarrow{A[\vec{a}]}$ and $B[\vec{b}, \vec{c}]$ have a common point. If they had two points in common then, by Theorems 9-3 and 9-1, the line would be a subset of — and, so, parallel to — the plane. Since the latter is not the case, the line and plane intersect at a single point.

4. By Theorem 10-4 a line which is not parallel to a plane is a transversal of that plane. On the other hand, a line which is a transversal to a plane intersects the plane but is not a subset of it and so, by Exercise 3(c) of Part B on page 390, is not parallel to the plane. So, to prove Corollary 1 we must show that a line which is not parallel to a plane is, also, not parallel to any parallel plane. But this is equivalent to the result of Exercise 1(f) of Part B on page 390. [Corresponding sentences of the forms '(p and q) $\Longrightarrow$ r' and '(not r and q) $\Longrightarrow$ not p' are logically equivalent.]

5. $\ell \parallel m$ and $\ell \not\parallel \pi$ then $m \not\parallel \pi$. [See Exercise 1(e) of Part B on page 390.]

6. Suppose that $\sigma \cap \pi_1 \neq \emptyset$ [and that $\sigma \neq \pi_1$] and that $\pi_2 \not\parallel \pi_1$. It follows that $\sigma \not\parallel \pi_1$ and, so that $\sigma \not\parallel \pi_2$. So, by Theorem 10-2, $\sigma \cap \pi_1$ and $\sigma \cap \pi_2$ are lines and, by Theorem 9-16, are parallel.

7. Using results of Exercise 1 of Part B on page 390 [specifically, parts (g), (h), (i), and (j)] all that remains is to prove:

   (a) $\sigma \cap \pi = \emptyset \Longrightarrow \sigma \parallel \pi$

   (b) $\ell \cap \pi = \emptyset \Longrightarrow \ell \parallel \pi$

   It will be simpler [and sufficient] to prove the contrapositives of (a) and (b). The contrapositive of (a) is an immediate consequence of Theorem 10-2. That of (b) is a consequence of Theorem 10-4.

A sequence $(\vec{a}_1, \ldots, \vec{a}_n)$ of members of a vector space is said to span the space if and only if each member of the space is a linear combination of terms of the sequence. The sequence is called a basis for the space if and only if it spans the space and is linearly independent. Intuitively, then [and also formally], a basis is a spanning sequence without redundant terms.

Part A and the text leading to Definition 10-1 can be used to promote class discussion of the idea of a basis. Parts B, C, and D can be used as a single homework assignment, but it is unreasonable to have each student do each of the exercises in Part D. Instead, each student can work all exercises of Parts B and C and then students can team up for Part D, with each team preparing and presenting the discussion for just one of the exercises.

Answers for Part A

1. Yes. Any sequence with repeated terms is linearly dependent. [Theorem 6-3(b)]

2. No. Any sequence with $\vec{0}$ as a term is linearly dependent. [Theorem 6-3(a)]

3. By Postulate 4_{10}, $(\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{b})$ is linearly dependent.

4. Yes, by the definition of linear dependence.

5. If d were 0 then, since $\vec{b}0 = \vec{0}$ and $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is linearly independent, it would follow that a, b, and c are, also, 0.

6. Since, by Exercise 4, $\vec{a}_1 a + \vec{a}_2 b + \vec{a}_3 c + \vec{b} d = \vec{0}$ and, by Exercise 5, $d \neq 0$, it follows that $\vec{b} = \vec{a}_1(-a/d) + \vec{a}_2(-b/d) + \vec{a}_3(-c/d)$. So, there exist numbers x, y, and z such that $\vec{b} = \vec{a}_1 x + \vec{a}_2 y + \vec{a}_3 z$.

7. Yes, for the conclusion in Exercise 6 made no use of any special properties of the translation $\vec{b}$.

8. T

9. Yes.; $\vec{a}_1 a + \vec{a}_2 b + \vec{a}_3 c = (\vec{a}_1 2)(a/2) + (\vec{a}_2 3)(b/3) + (\vec{a}_3 \cdot -1) \cdot -c$

6. Prove Theorem 10–5.
7. Prove Theorem 10–6. [*Hint*: Most of this has been done on page 390.]

10.03 Bases for $\mathcal{T}$

In this chapter, we have postulated that there are three linearly independent translations, and that any four translations are linearly dependent. This and the postulates that tell us that $\mathcal{T}$ is a vector space combine to give us:

$$\mathcal{T} \text{ is a 3-dimensional vector space}$$

We shall now investigate some important properties of the 3-dimensional vector space $\mathcal{T}$.

Exercises

Part A

Suppose that $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is linearly independent.

1. Does it follow that $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$ are three vectors? Explain.
2. Can any of the vectors $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$ be $\vec{0}$? Explain your answer.
3. Suppose that $\vec{b} \in \mathcal{T}$. What can be said of $(\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{b})$?
4. Based on your answer in Exercise 3, are we justified in stating that there are numbers—say, a, b, c, and d—not all zero, such that $\vec{a}_1 a + \vec{a}_2 b + \vec{a}_3 c + \vec{b} d = \vec{0}$? Explain your answer.
5. Show that the number d of Exercise 4 is not zero.
6. Using the results of Exercises 4 and 5, show that there are numbers x y, and z such that
$$\vec{b} = \vec{a}_1 x + \vec{a}_2 y + \vec{a}_3 z.$$
7. Does it follow from Exercises 4, 5, and 6 that each vector in $\mathcal{T}$ is a linear combination of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$? Explain your answer.
8. What is another name for $[\vec{a}_1, \vec{a}_2, \vec{a}_3]$?
9. Is each vector in $\mathcal{T}$ a linear combination of $\vec{a}_1 2$, $\vec{a}_2 3$, and $\vec{a}_3 \cdot -1$? Explain your answer.
10. Suppose that $a \neq 0$, $b \neq 0$, and $c \neq 0$. Prove that $[\vec{a}_1 a, \vec{a}_2 b, \vec{a}_3 c] = \mathcal{T}$.
11. Prove that each translation is a linear combination of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3 - \vec{a}_1$.

*

In the preceding exercises we have found that each translation is a linear combination of any three linearly independent translations.

Answers for Part A [cont.]

10. Any linear combination of $\vec{a}_1 a$, $\vec{a}_2 b$, and $\vec{a}_3 c$ is a member of T. For the rest, it is sufficient [by Exercise 7] to prove that if $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is linearly independent and $a \neq 0$, $b \neq 0$, and $c \neq 0$ then $(\vec{a}_1 a, \vec{a}_2 b, \vec{a}_3 c)$ is linearly independent. Suppose, then, that $(\vec{a}_1 a)d + (\vec{a}_2 b)e + (\vec{a}_3 c)f = \vec{0}$. It follows that $\vec{a}_1(ad) + \vec{a}_2(be) + \vec{a}_3(cf) = \vec{0}$ and, supposing that $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is linearly independent, that $ad = 0$, $be = 0$, and $cf = 0$. Assuming that a, b, and c are nonzero, it follows that $d = e = f = 0$. So, under the indicated assumptions, $(\vec{a}_1 a, \vec{a}_2 b, \vec{a}_3 c)$ is linearly independent.
11. As in Exercise 10, all that needs to be shown is that if $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is linearly independent then so is $(\vec{a}_1, \vec{a}_2, \vec{a}_3 - \vec{a}_1)$. Suppose, then, that $\vec{a}_1 a + \vec{a}_2 b + (\vec{a}_3 - \vec{a}_1)c = \vec{0}$. It follows that $\vec{a}_1(a - c) + \vec{a}_2 b + \vec{a}_3 c = \vec{0}$ and, assuming $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ linearly independent, $a - c = b = c = 0$. It follows at once that $a = b = c = 0$.

*

[This result is a consequence of Postulate 4_{10} and Theorem 6–13.] Introducing a word which is often used in discussing vector spaces, we may state this result as:

(*) Each 3-termed linearly independent sequence of translations *spans* $\mathcal{T}$.

[Similarly, as we noted long ago, any bidirection is a vector space and, as we proved more recently, such a vector space is spanned by any linearly independent 2-termed sequence of its members. (Locate the references referred to by 'long ago' and 'more recently'.)]

There are many sequences which span $\mathcal{T}$ but which are not linearly independent. For example, if $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, and $\vec{a}_4$ are translations such that $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ spans $\mathcal{T}$ then so does the sequence $(\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4)$. In fact, it follows from (*) that

a sequence of translations spans $\mathcal{T}$ if it has a 3-termed linearly independent subsequence.

[Explain.]

Having a sequence of translations which spans $\mathcal{T}$ we know that each translation is a linear combination of terms of $\mathcal{T}$ and, of course, each such linear combination is a translation. It is helpful if we can also be sure that each translation is a linear combination of members of the sequence "in only one way". By Theorem 6–8 this is the case if [and only if] the sequence is linearly independent. It is customary to call a linearly independent sequence which spans a vector space a *basis* for the space. [The plural of 'basis' is 'bases', pronounced as 'bas'eez'.] In particular:

Definition 10–1 $(\vec{a}_1, \ldots, \vec{a}_n)$ is a basis for $\mathcal{T}$ if and only if
(i) $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly independent, and
(ii) $[\vec{a}_1, \ldots, \vec{a}_n] = \mathcal{T}$.

So, we can restate (*) as:

Theorem 10–7 Each 3-termed linearly independent sequence of translations is a basis for $\mathcal{T}$.

Up to now, Postulate 4_9 has not been referred to in our discussion of bases. Together with Theorem 10–7 it tells us that there is a basis for $\mathcal{T}$. [As Exercises 10 and 11 of Part A indicate, it follows that there are many bases for $\mathcal{T}$.] In addition to this, Postulate 4_9 makes it possible to show that the bases specified in Theorem 10–7 are the only ones there are.

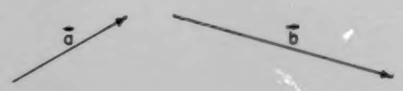
By Postulate 4_{10}, $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is linearly dependent. So, by Theorem 6-13, if $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent then $\vec{d} \in [\vec{a}, \vec{b}, \vec{c}]$.

'long ago' refers to Part B on page 193; 'more recently' refers to the lemma to Theorem 9-1.

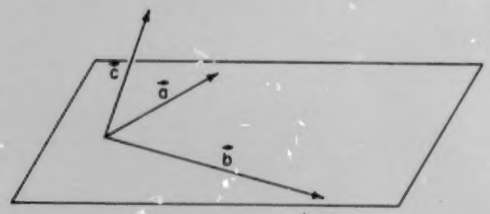
Given any sequence of vectors, a vector which is a linear combination of the terms of some subsequence is, also, a linear combination of terms of the given sequence.

Answers for Part B

1. (a)



(b) Yes. [By Postulate 4_9, there are three linearly independent translations and, by Exercise 3(b) of Part D on page 384, three linearly independent translations cannot all belong to $[\vec{a}, \vec{b}]$.] Here is a picture which describes such a translation.



Any translation $\vec{c}$ which maps points of a plane with direction $[\vec{a}, \vec{b}]$ out of that plane is a translation which does not belong to $[\vec{a}, \vec{b}]$.

(c) No.

2. If $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is linearly independent then so is $(\vec{a}_1, \vec{a}_2)$ and, if $\{\vec{a}_1, \vec{a}_2\} \subseteq [\vec{a}, \vec{b}]$ then, by the lemma to Theorem 9-1, $[\vec{a}_1, \vec{a}_2] = [\vec{a}, \vec{b}]$. If, in addition, $\vec{a}_3 \in [\vec{a}, \vec{b}]$ it follows that $\vec{a}_3 \in [\vec{a}_1, \vec{a}_2]$, thus contradicting the linear independence of $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$. Since, by Postulate 4_9, there exist three linearly independent translations it follows that there is a translation which does not belong to $[\vec{a}, \vec{b}]$.

3. By Exercise 2, no sequence with fewer than three terms spans $\mathcal{T}$. By Postulate 4_{10}, no sequence with more than three terms is linearly independent. So, each basis for $\mathcal{T}$ must be a 3-termed sequence and, by definition, be linearly independent.

Part B

1. **(a)** Draw arrows to represent a translation $\vec{a}$ and a translation $\vec{b}$.
   (b) Is there a translation which does not belong to $[\vec{a}, \vec{b}]$? [If your answer is 'Yes.', describe such a translation; if 'No.', tell why.]
   (c) Might your answer for part (b) have been different if you had chosen different translations in part (a)?
2. Prove that, for any $\vec{a}$ and $\vec{b}$, there is a translation which does not belong to $[\vec{a}, \vec{b}]$. [*Hint*: Suppose that $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is linearly independent and that $\{\vec{a}_1, \vec{a}_2\} \subseteq [\vec{a}, \vec{b}]$. What follows about $[\vec{a}_1, \vec{a}_2]$? About $\vec{a}_3$ and $[\vec{a}, \vec{b}]$?]
3. Prove:

> **Theorem 10–8** Each basis for $\mathscr{T}$ is a 3-termed linearly independent sequence of translations.

[*Hint*: By part (i) of Definition 10–1 and Postulate 4_{10}, each basis for $\mathscr{T}$ has at most 3 terms (and is linearly independent). Can a basis for $\mathscr{T}$ have fewer than 3 terms?]

4. Prove:

> **Theorem 10–9**
> $[\vec{a}_1, \vec{a}_2, \vec{a}_3] = \mathscr{T} \longleftrightarrow (\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is linearly independent

Part C

1. Suppose that $(\vec{a}, \vec{b})$ is linearly independent.
   (a) Do you believe that there is a vector—say, $\vec{c}$—such that $(\vec{a}, \vec{b}, \vec{c})$ is a basis for $\mathscr{T}$?
   (b) Try to use Exercise 2 of Part B [and a well-worn theorem] to show that there is such a vector $\vec{c}$.
2. Suppose that $\vec{a} \neq \vec{0}$ [that is, suppose that $(\vec{a})$ is linearly independent].
   (a) Do you believe that there is a vector—say, $\vec{b}$—such that $(\vec{a}, \vec{b})$ is linearly independent?
   (b) Try to prove that there is such a vector $\vec{b}$.
   (c) Are there vectors—say, $\vec{b}$ and $\vec{c}$—such that $(\vec{a}, \vec{b}, \vec{c})$ is a basis for $\mathscr{T}$?
3. Does $\vec{0}$ belong to any basis for $\mathscr{T}$? Explain.
4. Suppose, as in Exercise 1, that $(\vec{a}, \vec{b})$ is linearly independent. Let A be any point.
   (a) What can you say about $\{A, A + \vec{a}, A + \vec{b}\}$?
   (b) Is there a point—say, B—such that $B \notin \overleftrightarrow{A[\vec{a}, \vec{b}]}$? Explain.
   (c) If there is such a point as B, what can you say about $(\vec{a}, \vec{b}, B - A)$?
5. Suppose that, as in Exercise 2, $\vec{a} \neq \vec{0}$. Give an argument which, like that of Exercise 4, uses what you know about points, lines and planes, to show how to find vectors $\vec{b}$ and $\vec{c}$ such that $(\vec{a}, \vec{b}, \vec{c})$ is a basis for $\mathscr{T}$.

Answers for Part B [cont.]

4. As pointed out on page 410, it follows from Postulate 4_{10} and Theorem 6-13 that each 3-termed linearly independent sequence spans T. This establishes the if-part of Theorem 10-9. On the other hand, if $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ spans T it cannot be linearly dependent. For, if $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is linearly dependent then, by Theorem 6-2, $[\vec{a}_1, \vec{a}_2, \vec{a}_3]$ is a bidirection and, by Exercise 2, no bidirection contains all members of T. This establishes the only if-part of Theorem 10-9.

   [As a result of Theorem 10-9, we can show that a 3-termed sequence of members of T is a basis for T either by showing that it is linearly independent or by showing that it spans T. We need not go to the effort of showing both.]

Answers for Part C

1. (a) Yes.

   (b) By Exercise 2 of Part B it follows that there is a translation — say, $\vec{c}$ — such that $\vec{c} \notin [\vec{a}, \vec{b}]$. Since $(\vec{a}, \vec{b})$ is linearly independent it follows, by Theorem 6-13, that $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent. So, by Theorem 10-9, $(\vec{a}, \vec{b}, \vec{c})$ is a basis for T.

2. (a) Yes.

   (b) By Exercise 2 of Part B, there is a vector — say, $\vec{b}$ — such that $\vec{b} \notin [\vec{a}, \vec{a}] = [\vec{a}]$. Since $\vec{a} \neq \vec{0}$ it follows, by Theorem 6-13, that $(\vec{a}, \vec{b})$ is linearly independent.

   (c) Yes. This follows from part (b) and Exercise 1(b).

3. No. A basis is, by definition, linearly independent and no sequence one of whose terms is $\vec{0}$ is linearly independent.

4. (a) This set is noncollinear.

   (b) Yes. For by Postulate 4_9, not all points belong to one plane.

   (c) This sequence is linearly independent.

5. Let A be any point. Since $\vec{a} \neq \vec{0}$, $\overleftrightarrow{A(A + \vec{a})}$ is a line. It has previously been shown that there is a plane containing this line. Let B be a point of such a plane which does not belong to $\overleftrightarrow{A(A + \vec{a})}$. [Or, let $\vec{b}$ be a translation in the direction of such a plane which does not belong to $[\vec{a}]$.] Then $(\vec{a}, B - A)$ [or $(\vec{a}, \vec{b})$] is linearly independent and we can continue as in Exercise 4.

The exercises of Part C suggest two proofs for:

Theorem 10–10
(a) Each non-$\vec{0}$ translation is a term of some basis for $\mathcal{T}$.
(b) Each two linearly independent translations are terms of some basis for $\mathcal{T}$.

Part D

Tell which of the following are theorems and justify your answer.

1. If A, B, C, and D are noncoplanar points of $\mathcal{E}$ then $(B - A, C - A, D - A)$ is a basis for $\mathcal{T}$.
2. $(\vec{a}, \vec{b}, \vec{c})$ is a basis for $\mathcal{T}$ if and only if $\mathcal{T} = [\vec{a}, \vec{b}, \vec{c}]$.
3. If $(\vec{a}, \vec{b}, \vec{c})$ is a basis for $\mathcal{T}$ and $O \in \mathcal{E}$, then

$$\mathcal{E} = \{X: \exists_x \exists_y \exists_z\, X = O + \vec{a}x + \vec{b}y + \vec{c}z\}.$$

4. If $\vec{a} \in \mathcal{T}$, there is a basis $(\vec{a}, \vec{b}, \vec{c})$ for $\mathcal{T}$.
5. If $\vec{a} \neq \vec{0}$, there is a basis $(\vec{a}, \vec{b}, \vec{c})$ for $\mathcal{T}$.
6. If $(\vec{a}, \vec{b}, \vec{c})$ is a basis for $\mathcal{T}$ then $\vec{a} \notin [\vec{b}, \vec{c}]$.
7. If $A \neq B$ then $(B - A)$ is a basis for $[\overleftrightarrow{AB}]$.
8. If $\{A, B, C\}$ is noncollinear then $(B - A, C - A)$ is a basis for $[\overline{ABC}]$.

10.04 Components of Vectors

Suppose that $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is a basis for $\mathcal{T}$. Then, each vector belongs to $[\vec{a}_1, \vec{a}_2, \vec{a}_3]$, the set of linear combinations of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$. [Explain this.] It follows that, for each $\vec{x}$, there are numbers x_1, x_2, and x_3 such that

$$\vec{x} = \vec{a}_1x_1 + \vec{a}_2x_2 + \vec{a}_3x_3.$$

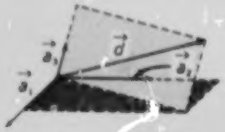


Fig. 10–4

Exercises

1. Given that $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is a basis for $\mathcal{T}$, show that, for each $\vec{x}$, there is *exactly one* triple (x_1, x_2, x_3) such that $\vec{x} = \vec{a}_1x_1 + \vec{a}_2x_2 + \vec{a}_3x_3$.

Answers for Part D

1. Theorem, by Definition 9-1 and Theorem 10-7.
2. Theorem, by Theorems 10-7 and 10-9.
3. Theorem, by definition of 'basis' and Postulates 1(a) and 2(a).
4. Not a theorem.
5. Theorem, by Theorem 10-10(a).
6. Theorem, by definition of 'basis' and Theorem 6-13.
7. Theorem, by Theorem 3-2(b) and Theorem 7-1
8. Theorem, by Definition 7-1 and Theorem 9-3.

*

Explanation asked for in line 2, is merely that by definition, a basis for $\mathcal{T}$ spans $\mathcal{T}$.

Sample Quiz

1. Tell what is means to say that a given sequence of vectors is a basis for $\mathcal{T}$.
2. Suppose that $(\vec{a}, \vec{b}, \vec{c})$ is a basis for $\mathcal{T}$. Tell which of the following are also bases for $\mathcal{T}$ and which are not. Justify your answers.
   (a) $(\vec{a},\ \vec{a} + \vec{b},\ \vec{a} + \vec{b} + \vec{c})$
   (b) $(\vec{a} - \vec{b},\ \vec{b},\ \vec{a} + \vec{b})$
   (c) $(\vec{a} + \vec{b} - \vec{c},\ \vec{a} - \vec{b} + \vec{c},\ -\vec{a} + \vec{b} + \vec{c})$

Key to Sample Quiz

1. The sequence has as its terms three linearly independent vectors.
2. (a) is a basis for $\mathcal{T}$, for the sequence consists of three linearly independent vectors. This is easy to show, for to say that $\vec{a}a + (\vec{a} + \vec{b})b + (\vec{a} + \vec{b} + \vec{c})c = \vec{0}$ implies that $\vec{a}(a + b + c) + \vec{b}(b + c) + \vec{c}c = \vec{0}$, and the latter together with the linear independence of $(\vec{a}, \vec{b}, \vec{c})$ implies that $a = b = c = 0$.

   (b) is not a basis for $\mathcal{T}$, for $\vec{c} \notin [\vec{a} - \vec{b}, \vec{b}, \vec{a} + \vec{b}]$. Alternately $(\vec{a} - \vec{b}, \vec{b}, \vec{a} + \vec{b})$ is linearly dependent for $\vec{a} + \vec{b} = (\vec{a} - \vec{b})1 + \vec{b}2$

   (c) is a basis for $\mathcal{T}$, for the sequence consists of three linearly independent vectors. This is easy to show, for to say that $(\vec{a} + \vec{b} - \vec{c})a + (\vec{a} - \vec{b} + \vec{c})b + (-\vec{a} + \vec{b} + \vec{c})c = \vec{0}$ implies that $\vec{a}(a + b - c) + \vec{b}(a - b + c) + \vec{c}(-a + b + c) = \vec{0}$, and the latter together with the linear independence of $(\vec{a}, \vec{b}, \vec{c})$ implies that $a = b = c = 0$.

2. Is the converse of Exercise 1 a theorem? That is, is it the case that if $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is a sequence of vectors such that for each $\vec{x}$ there is exactly one triple (x_1, x_2, x_3) such that $\vec{x} = \vec{a}_1x_1 + \vec{a}_2x_2 + \vec{a}_3x_3$, then $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is a basis for $\mathcal{T}$? If you think that it is a theorem, prove it. If not, give a counter-example.

As a result of the above exercises, we are assured that, for a given basis for $\mathcal{T}$, there is exactly one triple of numbers associated with a given vector, and there is exactly one vector associated with a given triple of numbers. Put another way, given a basis for $\mathcal{T}$, there is a *one-to-one correspondence* between the members of $\mathcal{T}$ and the triples of real numbers.

One consequence of the one-to-one correspondence between translations and triples of real numbers is the following. Suppose that, with respect to the basis $(\vec{a}, \vec{b}, \vec{c})$, $\vec{d}$ corresponds to the triple (d_1, d_2, d_3). Then,

$$\vec{d} = \vec{a}d_1 + \vec{b}d_2 + \vec{c}d_3.$$

So, in a sense, "part" of $\vec{d}$ results from $\vec{a}$, "part" from $\vec{b}$, and "part" from $\vec{c}$. This suggests our next definition.

> **Definition 10-2** If $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is a basis for $\mathcal{T}$
> and $\quad \vec{a} = \vec{u}_1a_1 + \vec{u}_2a_2 + \vec{u}_3a_3$
> then a_1, a_2, and a_3 are, respectively, the *first*, *second*, and *third components* of $\vec{a}$ with respect to the given basis. Also, (a_1, a_2, a_3) is the *component-triple* of $\vec{a}$ with respect to this basis.

[Instead of saying that the components of $\vec{a}$ are a_1, a_2, and a_3, respectively, or that the component-triple of $\vec{a}$ is (a_1, a_2, a_3), we shall often – but improperly – say that the components of $\vec{a}$ are (a_1, a_2, a_3).]

Exercises

Part A

1. Suppose that $(\vec{a}, \vec{b}, \vec{c})$ is a basis for $\mathcal{T}$.
   (a) If $\vec{d} = \vec{a}5 + \vec{b} \cdot -2 + \vec{c}3$ then, with respect to the given basis, the triple ________ corresponds to $\vec{d}$, and the first, second and third components of $\vec{d}$ are ____, ____, and ____.
   (b) If $\vec{e} = \vec{0}$, then the first, second and third components of $\vec{e}$ are ____, ____, and ____.
   (c) If $\vec{f}$ has components $(3, 2, -1)$ then $\vec{f} =$ ________.
   (d) If the components of $\vec{g}$ are $(0, 1, 0)$ then $\vec{g} =$ ________.

Answers for Exercises

1. Since a basis for $\mathcal{T}$ spans $\mathcal{T}$, there is at least one such triple. Since a basis is linearly independent it follows by Theorem 6-8 that there is at most one such triple.

2. The converse is a theorem. If, for each $\vec{x}$, there is at least one such triple then $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ spans $\mathcal{T}$. If, for each $\vec{x}$, there is at most one such triple then there is at most one such triple for $\vec{0}$ and, by Theorem 6-7, $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is linearly independent.

   Consider the correspondence

$$\vec{a} \longleftrightarrow (a_1, a_2, a_3)$$

between members of $\mathcal{T}$ and members of $\mathcal{R} \times \mathcal{R} \times \mathcal{R}$ which holds when

$$\vec{a} = \vec{a}_1a_1 + \vec{a}_2a_2 + \vec{a}_3a_3.$$

Since $\mathcal{T}$ is a vector space over the reals, each member (a_1, a_2, a_3) of $\mathcal{R} \times \mathcal{R} \times \mathcal{R}$ corresponds with exactly one member $\vec{a}$ of $\mathcal{T}$. Assuming that $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ spans $\mathcal{T}$ amounts, by definition, to assuming that each member of $\mathcal{T}$ corresponds with at least one member of $\mathcal{R} \times \mathcal{R} \times \mathcal{R}$. Assuming that $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is linearly independent amounts, by Theorem 6-8, to assuming that no member of $\mathcal{T}$ corresponds with two members of $\mathcal{R} \times \mathcal{R} \times \mathcal{R}$. So, if $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is a basis for $\mathcal{T}$ then the correspondence in question is one-to-one between all of $\mathcal{T}$ and all of $\mathcal{R} \times \mathcal{R} \times \mathcal{R}$.

Exercise 2 shows that the correspondence in question has this property only if $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is a basis for $\mathcal{T}$.

The exercises on page 413 together with Parts A and B following provide a medium for introducing the language of components. The exercises of Parts A and B also help to point out the intimate relationship between addition of vectors and multiplication of a vector by a real number, and addition of the components [with respect to a particular basis] and multiplication of the components by a real number. If properly planned, Parts A and B and the discussion on pages 415 - 417 can be presented in one class period, with a homework assignment of Part A on pages 418 - 419.

Answers for Part A

1. (a) $(5, -2, 3)$; 5; -2; 3 (b) 0; 0; 0

TC 414 (1)

   (c) $\vec{a}3 + \vec{b}2 + \vec{c} \cdot -1$ (d) $\vec{b}$
   (e) $\vec{a}3 + \vec{b}4 + \vec{c}2$; $\vec{a}1 + \vec{b} \cdot -5 + \vec{c}3$; $\vec{a}4 + \vec{b} \cdot -1 + \vec{c}5$; $(4, -1, 5)$
   (f) $\vec{a}2 + \vec{b} \cdot -1 + \vec{c}3$; $\vec{a}4 + \vec{b} \cdot -2 + \vec{c}6$; $(4, -2, 6)$
   (g) $\vec{a}(a + d) + \vec{b}(b + e) + \vec{c}(e + f)$; $(a + d,\ b + e,\ c + f)$
   (h) $\vec{a}_1(a_1b) + \vec{a}_2(a_2b) + \vec{a}_3(a_3b)$; a_1b; a_2b; a_3b

2. (a) By Exercise 1(b).
   (b) Suppose that the components [with respect to the given basis $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$] are (a_1, a_2, a_3). Then $\vec{a} = \vec{a}_1a_1 + \vec{a}_2a_2 + \vec{a}_3a_3$ and, so,

$$\begin{aligned} -\vec{a} &= -(\vec{a}_1a_1 + \vec{a}_2a_2 + \vec{a}_3a_3) = -(\vec{a}_1a_1) + -(\vec{a}_2a_2) + -(\vec{a}_3a_3) \\ &= \vec{a}_1 \cdot -a_1 + \vec{a}_2 \cdot -a_2 + \vec{a}_3 \cdot -a_3. \end{aligned}$$

   So, each component of $-\vec{a}$ is the opposite of the corresponding component of $\vec{a}$.
   (c) By Exercise 1(g). (d) By Exercise 1(h).

(e) If (3, 4, 2) are the components of $\vec{h}$ and (1, −5, 3) are the components of $\vec{i}$ then

$\vec{h}$ = ____________ $\vec{i}$ = ____________
$\vec{h} + \vec{i}$ = ____________
and the components of $\vec{h} + \vec{i}$ are ____________.

(f) If $\vec{j}$ has components (2, −1, 3) then

$\vec{j}$ = ____________ $\vec{j}2$ = ____________
and $\vec{j}2$ has the components ________.

(g) If (a, b, c) are the components of $\vec{k}$, and (d, e, f) are the components of $\vec{m}$ then

$\vec{k} + \vec{m}$ = ____________,

and the components of $\vec{k} + \vec{m}$ are ____________.

(h) If a_1, a_2, and a_3 are the first, second and third components of $\vec{n}$ then

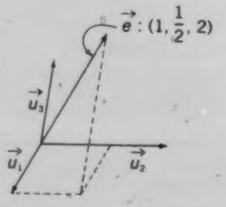
$\vec{n}b$ = ____________

and the first, second and third components of $\vec{n}b$ are ____, ____, and ____.

2. Prove:

> **Theorem 10-11** For any basis for $\mathscr{T}$,
> (a) each component of $\vec{0}$ is 0,
> (b) each component of $-\vec{a}$ is the opposite of the corresponding component of $\vec{a}$,
> (c) each component of $\vec{a} + \vec{b}$ is the sum of the corresponding components of $\vec{a}$ and $\vec{b}$,
> (d) each component of $\vec{a}a$ is the product of the corresponding component of $\vec{a}$ by a.

3. Here is a figure representing a linearly independent triple of vectors $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$. Copy the figure on your paper and draw arrows for the vectors corresponding to each of the triples given below. [An arrow for the vector—say $\vec{e}$—corresponding to (1, $\frac{1}{2}$, 2) is shown in the figure.]



(a) $\vec{a}$: (1, 0, 0)
(b) $\vec{b}$: (2, 0, −1)
(c) $\vec{c}$: (−1, −2, 2)
(d) $\vec{d}$: (−3, 1, −2)

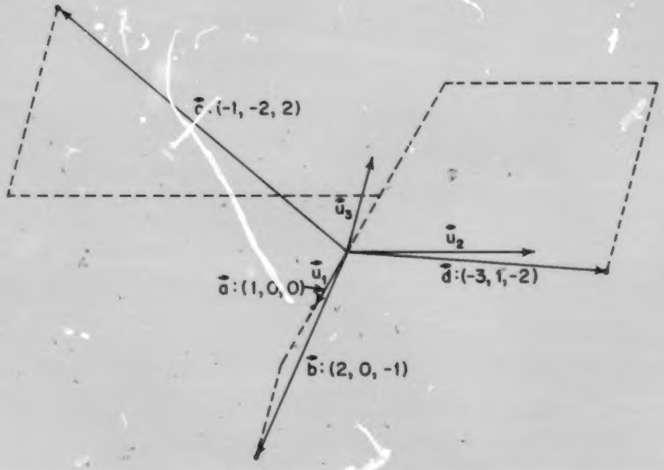
Theorem 10-11 shows that the correspondence

$$\vec{a} \longleftrightarrow (a_1, a_2, a_3)$$

determined, as in TC 413 by a given basis $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is an isomorphism between $\mathcal{T}$ and the vector space of 3-dimensional measure vectors discussed in section 5.06. [See, in particular, Exercise 2 on page 207.] As vector spaces, these two "have the same structure". As a consequence, any two 3-dimensional vector spaces are isomorphic and, more generally, two vector spaces of the same finite dimension are isomorphic. [For example, any two proper bidirections are isomorphic, for each is isomorphic with the space of 2-dimensional measure vectors. Derivatively from this, any plane, considered in isolation, is "just like" any other plane.]

Answers for Part A [cont.]

3.



4. Suppose that $(\vec{a}, \vec{b}, \vec{c})$ is a basis for $\mathcal{T}$, and that

$$\vec{d} = \vec{a}2 + \vec{b}3 + \vec{c} \cdot -1.$$

(a) Show that $(\vec{a}5, \vec{b} \cdot -10, \vec{c}3)$ is a basis for $\mathcal{T}$.
(b) $\vec{d} = (\vec{a}5)$ ____ $+ (\vec{b} \cdot -10)$ ____ $+ (\vec{c}3)$ ____, so the first, second, and third components of $\vec{a}$ with respect to the basis $(\vec{a}5, \vec{b} \cdot -10, \vec{c}3)$ are ________, ________, and ________.
(c) The first, second, and third components of $\vec{d}$ with respect to the basis $(\vec{a}, \vec{b}2, \vec{c} \cdot -1)$ are ________, ________, and ________, because

$$\vec{d} = \vec{a}\ \_\_\_\_ + (\vec{b}2)\ \_\_\_\_ + \vec{c} \cdot -1\ \_\_\_\_ .$$

Part B

Suppose that $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is a basis for $\mathcal{T}$ and that $O \in \mathcal{E}$. Complete the following:

1. If $(2, 1, 4)$ are the components of $\vec{b}$ with respect to $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ then

$$\vec{b} = \underline{\qquad\qquad\qquad}\,.$$

and

$$O + \vec{b} = \underline{\qquad\qquad\qquad}\,.$$

2. If P is a point of $\mathcal{E}$ such that

$$P = O + (\vec{u}_1 4 + \vec{u}_2 7 + \vec{u}_3 \cdot -1)$$

then

$$P - O = \vec{u}_1\ \_\_\_\_ + \vec{u}_2\ \_\_\_\_ + \vec{u}_3\ \_\_\_\_$$

and the components of $P - O$ with respect to $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ are ____

3. If (a_1, a_2, a_3) are the components of $A - O$ with respect to $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ then

$$A = \underline{\qquad\qquad\qquad}$$

4. State the converse of the conditional sentence in Exercise 3.

10.05 Coordinate Systems for $\mathcal{E}$

In Section 3.08 you learned that, given any point O in $\mathcal{E}$, the mapping of $\mathcal{T}$ into $\mathcal{E}$ which maps $\vec{a}$ on $O + \vec{a}$ and the mapping of $\mathcal{E}$ into $\mathcal{T}$ which maps A on $A - O$ are inverses of each other. [Explain. By what

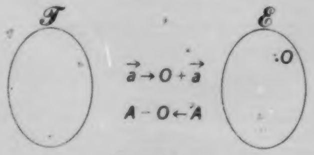


Fig. 10–5

4. (a) By Theorem 10-9 it is sufficient to show that $(\vec{a}5, \vec{b} \cdot -10, \vec{c}3)$ spans $\mathcal{T}$. But, this is obvious. For, given any $\vec{d}$, there are numbers — say, a, b, and c — such that $\vec{d} = \vec{a}a + \vec{b}b + \vec{c}c = (\vec{a}5)(a/5) + (\vec{b} \cdot -10)(b/-10) + (\vec{c}3)(c/3)$.

   (b) 2/5; −3/10; −1/3; 2/5; −3/10; −1/3.

   (c) 2; 3/2; 1; 2; 3/2; 1.

Answers for Part B

1. $\vec{u}_1 2 + \vec{u}_2 + \vec{u}_3 4$; $O + (\vec{u}_1 2 + \vec{u}_2 + \vec{u}_3 4)$
2. 4; 7; −1; (4, 7, −1)
3. $O + (\vec{u}_1 a_1 + \vec{u}_2 a_2 + \vec{u}_3 a_3)$
4. If $A \in \mathcal{E}$ such that $A = O + \vec{u}_1 a_1 + \vec{u}_2 a_2 + \vec{u}_3 a_3$ that the components of $A - O$ with respect to $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ are (a_1, a_2, a_3).

TC 415-417

The first mapping maps $\mathcal{T}$ into $\mathcal{E}$ by Postulate 1(b); the second is a mapping of $\mathcal{E}$ into $\mathcal{T}$ by Postulate 1(a). That each is the inverse of the other results from Postulate 2. [For more details, see section 3.08 and, especially, the accompanying commentary.]

A − O is the position vector of A with respect to O.

postulate is the first mapping a mapping of $\mathscr{T}$ into $\mathscr{E}$? By what postulate is the second mapping a mapping of $\mathscr{E}$ into $\mathscr{T}$? Why is each of the two mappings the inverse of the other?] It follows that these two mappings establish a one-to-one correspondence between the points in $\mathscr{E}$ and the translations in $\mathscr{T}$. [In Chapter 8 we made use of this one-to-one correspondence. How did we, there, refer to the translation $A - O$?]

In the preceding section you have seen that, given any basis $(\overrightarrow{u}_1, \overrightarrow{u}_2, \overrightarrow{u}_3)$ for $\mathscr{T}$, there is a one-to-one correspondence between triples of real numbers – that is, the members of $\mathscr{R} \times \mathscr{R} \times \mathscr{R}$ – and the members of $\mathscr{T}$.

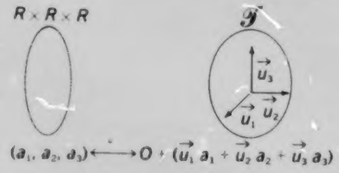
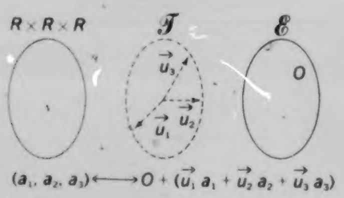

R × R × R
𝒯
$\overrightarrow{u}_3$
$\overrightarrow{u}_1$
$\overrightarrow{u}_2$
$(a_1, a_2, a_3) \longleftrightarrow O + (\overrightarrow{u}_1 a_1 + \overrightarrow{u}_2 a_2 + \overrightarrow{u}_3 a_3)$

Fig. 10–6

Composing these two one-to-one correspondences we see that, given any point O in $\mathscr{E}$ *and* any basis $(\overrightarrow{u}_1, \overrightarrow{u}_2, \overrightarrow{u}_3)$ for $\mathscr{T}$, there is a one-to-one correspondence between the triples of real numbers which are the members of $\mathscr{R} \times \mathscr{R} \times \mathscr{R}$ and the points which are the members of $\mathscr{E}$.


R × R × R
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O
$\overrightarrow{u}_3$
$\overrightarrow{u}_2$
$\overrightarrow{u}_1$
$(a_1, a_2, a_3) \longleftrightarrow O + (\overrightarrow{u}_1 a_1 + \overrightarrow{u}_2 a_2 + \overrightarrow{u}_3 a_3)$

Fig. 10–7

This correspondence can be described by saying that to any triple (a_1, a_2, a_3) there corresponds the point $O + (\overrightarrow{u}_1 a_1 + \overrightarrow{u}_2 a_2 + \overrightarrow{u}_3 a_3)$ and, to any point A there corresponds the component-triple of $A - O$, with respect to the basis $(\overrightarrow{u}_1, \overrightarrow{u}_2, \overrightarrow{u}_3)$. This suggests the following definition.

Definition 10–3 If $O \in \mathscr{E}$, $(\overrightarrow{u}_1, \overrightarrow{u}_2, \overrightarrow{u}_3)$ is a basis for $\mathscr{T}$, and

$$A = O + (\overrightarrow{u}_1 a_1 + \overrightarrow{u}_2 a_2 + \overrightarrow{u}_3 a_3)$$

then a_1, a_2, and a_3 are, respectively, the first, second, and third coordinates of A with respect to the given point and the given basis. Also, (a_1, a_2, a_3) is the *coordinate-triple* of A with respect to this point and basis.

[As in the case of components of translations, we shall sometimes say – improperly – that the coordinates of A are (a_1, a_2, a_3) when – properly – (a_1, a_2, a_3) is the coordinate-triple of A.]

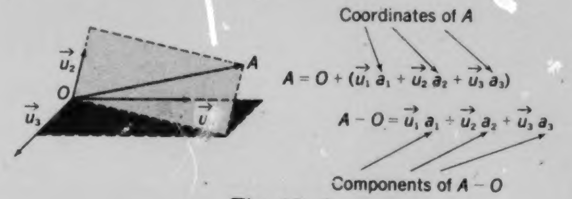

$\overrightarrow{u}_2$
O
$\overrightarrow{u}_3$
$\overrightarrow{u}$
A
Coordinates of A
$A = O + (\overrightarrow{u}_1 a_1 + \overrightarrow{u}_2 a_2 + \overrightarrow{u}_3 a_3)$
$A - O = \overrightarrow{u}_1 a_1 + \overrightarrow{u}_2 a_2 + \overrightarrow{u}_3 a_3$
Components of A − O

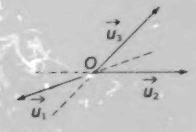
Fig. 10–8

A correspondence between triples of real numbers and points of $\mathscr{E}$ of the kind described in Definition 10–3 is called [after the French mathematician René Descartes] *a cartesian coordinate system for* $\mathscr{E}$. Other correspondences between triples of real numbers and points are also called *coordinate systems for* $\mathscr{E}$, but we shall have little to do with other than cartesian coordinate systems.

You have seen that many results concerning points, lines, planes, and other geometric figures, such as triangles and quadrilaterals, can be established by using translations. In Section 8.05 you saw that establishing such results is sometimes simplified by using position vectors from an arbitrarily chosen point O. The use of a coordinate system, based on a point O and a basis $(\overrightarrow{u}_1, \overrightarrow{u}_2, \overrightarrow{u}_3)$ for $\mathscr{T}$, is another way of studying geometric figures. This procedure makes it possible to solve a problem in geometry by, first, translating it into a related problem in the algebra of real numbers, solving this problem, and, then, interpreting its solution to obtain the solution of the original problem. Often it is simpler to work directly with translations, as we have done in the past; but sometimes this "coordinate method" of solving geometric problems has advantages. In later sections we shall give some illustrations of the use of this method.

Exercises

Part A



1. Copy the figure, showing a point O, and a basis, $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$, for $\mathcal{T}$. On your picture, graph and label the points whose coordinates are given below.

(a) P: $(1, 2, 0)$	**(b)** Q: $(\frac{1}{2}, 0, 1)$	**(c)** R: $(0, 2, 1)$
(d) S: $(0, 0, -1)$	**(e)** T: $(\frac{1}{2}, -1, 1)$	**(f)** U: $(\frac{3}{2}, \frac{3}{2}, \frac{1}{2})$
(g) U_1: $(1, 0, 0)$	**(h)** U_2: $(0, 1, 0)$	**(i)** U_3: $(0, 0, 1)$

2. Referring to the points in Exercise 1, complete the following:
   (a) Since $P - O = \vec{u}_1\_\_\_\_ + \vec{u}_2\_\_\_\_ + \vec{u}_3\_\_\_\_$, the components of $P - O$ are ____ ____ and ____.
   (b) Give the components of $Q - O, R - O, S - O, T - O, U - O, U_1 - O, U_2 - O$, and $U_3 - O$.
3. Referring to the points in Exercise 1, complete the following:
   (a) $P - R = [O + \vec{u}_1\_\_\_\_ + \vec{u}_2\_\_\_\_ + \vec{u}_3\_\_\_\_] - [O + \vec{u}_1\_\_\_\_ + \vec{u}_2\_\_\_\_ + \vec{u}_3\_\_\_\_]$. Therefore, $P - R = \vec{u}_1\_\_\_\_ + \vec{u}_2\_\_\_\_ + \vec{u}_3\_\_\_\_$ and the components of $P - R$ are ________.
   (b) Give the components of the vectors $P - T, Q - S, S - Q$ and $U_1 - U_2$.
4. **(a)** Suppose that A is a point of $\overleftrightarrow{O[\vec{u}_1]}$. Can you say anything about the first coordinate of A? About its second coordinate? About its third coordinate?
   (b) Suppose that $A \in \overleftrightarrow{O[\vec{u}_2]}$. Answer the questions in part (a).
   (c) Complete the following:

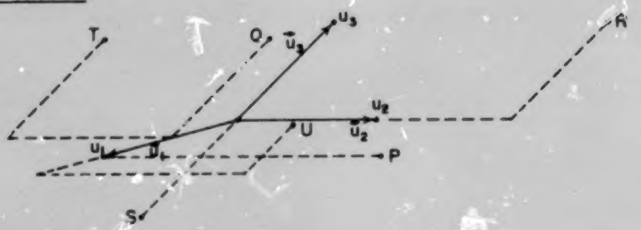
   $A \in \overleftrightarrow{O[\vec{u}_3]} \Longleftrightarrow$ the first coordinate of A is ____ and the second coordinate of A is ____.
5. In each of the following exercises you are given the coordinates for a pair of points A and B. Your job is to describe the coordinates of the points of $\overleftrightarrow{AB}$.
   (a) A: $(2, 0, 0)$, B: $(-3, 0, 0)$
   (b) A: $(0, 2, 0)$, B: $(0, 4, 0)$
   (c) A: $(0, 0, 0)$, B: $(0, 0, 3)$

*

In discussing the cartesian coordinate system determined by a point O and a basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$, O is called *the origin* [of the coordinate system]; the points $O + \vec{u}_1$, $O + \vec{u}_2$, and $O + \vec{u}_3$ are called *the first, second,* and *third unit points*, respectively; and the lines $\overleftrightarrow{O[\vec{u}_1]}$, $\overleftrightarrow{O[\vec{u}_2]}$, and $\overleftrightarrow{O[\vec{u}_3]}$ are called *the first, second,* and *third coordinate axes.*

Answers for Part A

1.



2. (a) 1; 2; 0; 1, 2, 0

(b)	Q - O: $\frac{1}{2}$, 0, 1	R - O: 0, 2, 1
	S - O: 0, 0, −1	T - O: $\frac{1}{2}$, −1, 1
	U - O: $\frac{3}{2}$, $\frac{3}{2}$, $\frac{1}{2}$	U_1 - O: 1, 0, 0
	U_2 - O: 0, 1, 0	U_3 - O: 0, 0, 1

3. (a) 1, 2, 0, 0, 2, 1;
   1, 0, −1; 1, 0, −1

(b)	P - T: $\frac{1}{2}$, 3, −1	Q - S: $\frac{1}{2}$, 0, 2
	S - Q: $-\frac{1}{2}$, 0, −2	U_1 - U_2: 1, −1, 0

4. (a) Some real number; zero; zero
   (b) Zero; some real number; zero
   (c) Zero; zero
5. (a) (a, 0, 0), for any a.
   (b) (0, a, 0), for any a.
   (c) (0, 0, a), for any a.

Note that a point belongs to, say, the *second* coordinate axis if and only if its *first* and *third* coordinates are both 0. Make similar remarks concerning the other two coordinate axes.

6. (a) Is $\overline{O[\vec{u}_2, \vec{u}_3]}$ a plane? Explain your answer.
   (b) Since $\overline{O[\vec{u}_2, \vec{u}_3]} = \{X:$ ________$\}$, $A \in \overline{O[\vec{u}_2, \vec{u}_3]}$ if and only if the first coordinate of A is ________.
   (c) Make a statement about $\overline{O[\vec{u}_3, \vec{u}_1]}$ like that which you obtained by completing part (b).
   (d) Repeat part (c) for $\overline{O[\vec{u}_1, \vec{u}_2]}$.
7. For each of the following exercises, describe a plane which contains the three points whose coordinates are given.
   (a) (0, 1, 3), (0, 3, 6), (0, 6, −7)
   (b) (1, 0, 5), (−2, 0, 5), (7, 0, 0)
   (c) (3, 0, 0), (7, −2, 0), (−3, 3, 0)
8. (a) Show that there is only one plane which contains the points whose coordinates are given in Exercise 7(a). [*Hint*: Suppose that the points in question are A, B, and C, respectively. You need, of course, to show that $(B - A, C - A)$ is linearly independent—that is, you need to show that if $(B - A)a + (C - A)b = \vec{0}$ then $a = b = 0$. Do this by, first, expressing $B - A$ and $C - A$ as linear combinations of the basis vectors $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$. Knowledge of determinants (pages 273, 274) will be helpful but is not necessary.]
   (b) Is there more than one plane which contains the points whose coordinates are given in Exercise 7(b)?
   (c) Repeat part (b), but for Exercise 7(c).

*

The planes $\overline{O[\vec{u}_2, \vec{u}_3]}$, $\overline{O[\vec{u}_3, \vec{u}_1]}$, and $\overline{O[\vec{u}_1, \vec{u}_2]}$ are the *first, second*, and *third coordinate planes*, respectively [with respect, of course, to the cartesian coordinate system determined by the point O and the basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$]. Note that a point belongs to, say, the *first* coordinate plane if and only if its *first* coordinate is 0. What can you say about the coordinates of a point which belongs to the first coordinate plane and, also, to the first coordinate axis? Is there such a point? Is there more than one?

*

9. Suppose that, with respect to a coordinate system with origin O, the first, second, and third coordinate axes are the lines l_1, l_2, and l_3, respectively, and the first, second, and third coordinate planes are π_1, π_2, and π_3, respectively. Use what you know about the coordinates of points on the coordinate axes and planes to simplify each of the following:
   (a) $l_1 \cap l_2$ (b) $\pi_2 \cap \pi_3$ (c) $l_2 \cap \pi_3$ (d) $l_2 \cap \pi_2$

Answers for Part A [cont.]

6. (a) Yes.; Since $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is a basis for $\mathcal{T}$, $(\vec{u}_2, \vec{u}_3)$ is linearly independent. So, by Theorem 9-11(a), $\overline{O[\vec{u}_2, \vec{u}_3]}$ is a plane.
   (b) $X - O \in [\vec{u}_2, \vec{u}_3]$; 0
   (c) $A \in \overline{O[\vec{u}_3, \vec{u}_1]}$ if and only if the second coordinate of A is 0.
   (d) $A \in \overline{O[\vec{u}_1, \vec{u}_2]}$ if and only if the third coordinate of A is 0.
7. (a) $\overline{O[\vec{u}_2, \vec{u}_3]}$ (b) $\overline{O[\vec{u}_3, \vec{u}_1]}$ (c) $\overline{O[\vec{u}_1, \vec{u}_2]}$
8. (a) Using the notation of the hint, $B - A = \vec{u}_2 2 + \vec{u}_3 3$ and $C - A = \vec{u}_2 5 + \vec{u}_3 \cdot -10$. Suppose that $(B - A)a + (C - A)b = \vec{0}$. It follows that $\vec{u}_2(2a + 5b) + \vec{u}_3(3a - 10b) = \vec{0}$ and, since $(\vec{u}_2, \vec{u}_3)$ is linearly independent, that $2a + 5b = 0$ and $3a - 10b = 0$. Solving for 'a' and 'b' [by, for example, multiplying the first equation with '2' and adding the result with the second equation] shows that $a = 0$ and $b = 0$. So, $(B - A, C - A)$ is linearly independent and, hence, $\{A, B, C\}$ is a subset of just one plane. [The solution is much simplified by using the results mentioned on determinants. It is sufficient to add, after the first sentence, above: Since $(\vec{u}_2, \vec{u}_3)$ is linearly independent and $2 \cdot -10 - 5 \cdot 3 = -35 \neq 0$ it follows that $(B - A, C - A)$ is linearly independent. Hence $\{A, B, C\}$ is a subset of just one plane.]
   (b) As in part (a) [but with new values for 'A', 'B', and 'C'], $B - A = \vec{u}_1 \cdot -3$ and $C - A = \vec{u}_1 \cdot 6 + \vec{u}_3 \cdot -5$. So, if $(B - A)a + (C - A)b = \vec{0}$ then $\vec{u}_1(-3a + 6b) + \vec{u}_3(-5b) = \vec{0}$ and, since $(\vec{u}_1, \vec{u}_3)$ is linearly independent, $-3a + 6b = 0$ and $-5b = 0$ — that is, $a = 0$ and $b = 0$. So, ...
   (c) In this case, $B - A = \vec{u}_1 4 + \vec{u}_2 \cdot -2$ and $C - A = \vec{u}_1 \cdot -6 + \vec{u}_2 3$. By inspection, $(B - A)3 + (C - A)2 = \vec{0}$. So, $\{A, B, C\}$ is collinear and is a subset of infinitely many planes. [Using determinants, it is sufficient to note that $4 \cdot 3 - -6 \cdot -2 = 0$ in order to conclude that $(B - A, C - A)$ is linearly dependent.]

*

The intersection of the first coordinate plane and the first coordinate axis consists of the origin.

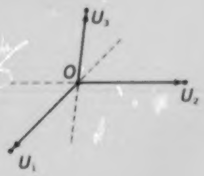
9. (a) {O} (b) l_1 (c) l_2 (d) {O}

If, as suggested earlier, Part A is used as a homework assignment, we do not recommend Part B as part of this same assignment. Rather Part B can be used as in-class exercises to reinforce the work of Part A and to identify possible misconceptions. Part C serves as a good class summary of the work on coordinate systems. This is difficult for students to do alone, however. Because of the way we name the points in our noncoplanar quadruple, students frequently do not realize that a choice of origin, first coordinate axis, etc. has been made. Part D can then be used as a homework assignment.

Part B

Suppose that (O, U_1, U_2, U_3) is a quadruple of noncoplanar points. [That is, (O, U_1, U_2, U_3) is a 4-termed sequence whose terms are points, and the set $\{O, U_1, U_2, U_3\}$ is noncoplanar.]

1. (a) Show that $(U_1 - O, U_2 - O, U_3 - O)$ is a basis for $\mathscr{T}$.



(b) What are the coordinate axes of the coordinate system determined by the basis $(U_1 - O, U_2 - O, U_3 - O)$ and O? What are the unit points? What are the coordinate planes?

(c) Give the coordinates of O, U_1, U_2, and U_3.

2. If A has coordinates $(3, -2, 1)$ then

$$A = O + (U_1 - O)\_\_\_\_ + (U_2 - O)\_\_\_\_ + (U_3 - O)\_\_\_\_.$$

Therefore, $A - O =$ __________ and the components of $A - O$ are __________.

3. (a) Describe how one can use the basis $(U_1 - O, U_2 - O, U_3 - O)$ to assign a triple of real numbers (a_1, a_2, a_3) as coordinates of any given point A.

(b) Describe how one can use the given basis to assign a triple (b_1, b_2, b_3) as components of $A - O$.

(c) Is there any relationship between the triples (a_1, a_2, a_3) and (b_1, b_2, b_3) determined in parts (a) and (b)? Explain your answer.

4. If A and B have coordinates (a_1, a_2, a_3) and (b_1, b_2, b_3), respectively, with respect to the basis $(U_1 - O, U_2 - O, U_3 - O)$ then

$A = O +$ __________
$B = O +$ __________
$B - A =$ __________

and $B - A$ has the components (__________).

Part C

You have seen in Part B how a quadruple of noncoplanar points can be used to determine a unique cartesian coordinate system for $\mathscr{E}$. Discuss the possibility of using a noncoplanar set $\{O, U_1, U_2, U_3\}$ to determine a cartesian coordinate system.

Answers for Part B

1. (a) Since $\{O, U_1, U_2, U_3\}$ is noncoplanar it follows by Definition 9-1 that $(U_1 - O, U_2 - O, U_3 - O)$ is linearly independent. Hence, by Theorem 10-7 [or, Theorem 10-9 and the definition of 'basis'], $(U_1 - O, U_2 - O, U_3 - O)$ is a basis for T.

(b) The first, second, and third coordinate axes are, respectively, $\overleftrightarrow{OU_1}$, $\overleftrightarrow{OU_2}$, and $\overleftrightarrow{OU_3}$. The first, second, and third coordinate planes are, respectively, $\overline{OU_2U_3}$, $\overline{OU_3U_1}$, and $\overline{OU_1U_2}$.

(b) (0, 0, 0), (1, 0, 0), (0, 1, 0), and (0, 0, 1), respectively.

2. 3; −2; 1; $(U_1 - O)3 + (U_2 - O)\cdot -2 + (U_3 - O)1$; $(3, -2, 1)$

3. (a) The coordinates to be assigned to A should be the components of A - O. So, in order to assign an ordered triple of real numbers to any given point one must know the components of the translation A - O with respect to the given basis.

(b) The components of A - O are precisely the numbers b_1, b_2, and b_3 such that $A - O = (U_1 - O)b_1 + (U_2 - O)b_2 + (U_3 - O)b_3$.

(c) The coordinates of A with respect to O and $(U_1 - O, U_2 - O, U_3 - O)$ and the components of A - O with respect to $(U_1 - O, U_2 - O, U_3 - O)$ are precisely the same real numbers. This follows from Theorem 2-1 and the fact that $(U_1 - O, U_2 - O, U_3 - O)$ is a basis for T.

4. $(U_1 - O)a_1 + (U_2 - O)a_2 + (U_3 - O)a_3$; $(U_1 - O)b_1 + (U_2 - O)b_2 + (U_3 - O)b_3$; $(U_1 - O)(b_1 - a_1) + (U_2 - O)(b_2 - a_2) + (U_3 - O)(b_3 - a_3)$; $(b_1 - a_1, b_2 - a_2, b_3 - a_3)$

*

The points U_1, U_2, and U_3 are often called the unit points (first, second, and third, respectively) of the coordinate system discussed in Part B. Given any four noncoplanar points, one may define a coordinate system by choosing one point as origin and the other three — in any order — as unit points. So, one has a choice of 24 coordinate systems. This is the reason for starting Part B with a 4-termed sequence of points rather than with a set of four points.

Answers for Part C

To determine a cartesian coordinate system, as in Part B, based on the points O, U_1, U_2, and U_3, one must decide which of these four points is to be taken as the origin. Then, one must decide which of the remaining points is to determine [together with the chosen origin] the first coordinate axis, and which is to determine the second coordinate axis. Given only the set of four noncoplanar points, there are 24 coordinate systems which may be obtained by using them in the way described.

Part D

Suppose that we have a coordinate system for ℰ with origin O and basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$. Complete the following.

1. (a) O has the coordinates ____________.
   (b) $\vec{u}_1$ has the components ____________.
   (c) $O + \vec{u}_1 = O + \vec{u}_1$ ____ $+ \vec{u}_2$ ____ $+ \vec{u}_3$ ____, and so $O + \vec{u}_1$ has the coordinates ____________.
2. (a) $\vec{u}_2 + \vec{u}_3$ has the components ____________, and $O + (\vec{u}_2 + \vec{u}_3)$ has the coordinates ____________.
   (b) $\vec{u}_1 + \vec{u}_3 4$ has the components ____________, and $O + (\vec{u}_1 + \vec{u}_3 4)$ has the coordinates ____________.
   (c) If $\vec{a}$ has the components (a, b, c) then $\vec{a} = \vec{u}_1$ ____ $+ \vec{u}_2$ ____ $+ \vec{u}_3$ ____ and $O + \vec{a}$ has coordinates ____________.
3. Suppose that A has coordinates $(2, -1, 3)$. Then $A = O + \vec{u}_1$ ____ $+ \vec{u}_2$ ____ $+ \vec{u}_3$ ____. If $\vec{a}$ has components $(5, -3, 2)$ then $\vec{a} = \vec{u}_1$ ____ $+ \vec{u}_2$ ____ $+ \vec{u}_3$ ____. Also,

$$\begin{aligned} A + \vec{a} &= [O + (\vec{u}_1\_\_\_\_ + \vec{u}_2\_\_\_\_ + \vec{u}_3\_\_\_\_)] \\ &\quad + (\vec{u}_1\_\_\_\_ + \vec{u}_2\_\_\_\_ + \vec{u}_3\_\_\_\_) \\ &= O + \vec{u}_1\_\_\_\_ + \vec{u}_2\_\_\_\_ + \vec{u}_3\_\_\_\_. \end{aligned}$$

   Thus, $A + \vec{a}$ has the coordinates ____________.
4. In each of the following exercises you are given a point A and its coordinates, and a vector $\vec{a}$ and its components. You are to find the coordinates of $A + \vec{a}$.
   (a) A: $(3, 5, 7)$
   $\vec{a}$: $(1, 2, 3)$
   $A + \vec{a}$: ________
   (b) A: $(1, 2, 5)$
   $\vec{a}$: $(-2, 1, -7)$
   $A + \vec{a}$: ________
   (c) A: $(-1, -2, -3)$
   $\vec{a}$: $(1, 2, 3)$
   $A + \vec{a}$: ________
   (d) A: (a, b, c)
   $\vec{a}$: (d, e, f)
   $A + \vec{a}$: ________
5. Suppose that $\vec{a}$ has components $(1, 2, 3)$ and $\vec{b}$ has components $(3, -2, 4)$. Then

$$\vec{a} = \vec{u}_1\_\_\_\_ + \vec{u}_2\_\_\_\_ + \vec{u}_3\_\_\_\_$$

   and

$$\vec{b} = \vec{u}_1\_\_\_\_ + \vec{u}_2\_\_\_\_ + \vec{u}_3\_\_\_\_.$$

   Therefore,

$$\vec{a} + \vec{b} = \vec{u}_1\_\_\_\_ + \vec{u}_2\_\_\_\_ + \vec{u}_3\_\_\_\_$$

   and so $\vec{a} + \vec{b}$ has the components ____________.
6. Suppose that $\vec{a}$, $\vec{b}$, and $\vec{c}$ have the components $(1, 2, -3)$, $(-4, 1, 3)$ and $(1, 4, -2)$ respectively. In each exercise you are given a linear combination of these vectors. You are to find its components.
   (a) $\vec{a} + \vec{c}$
   (b) $\vec{a} + \vec{b} + \vec{c}$
   (c) $\vec{a}2$
   (d) $\vec{a}3 + \vec{b} \cdot -1$
   (e) $\vec{a} - \vec{b}$
   (f) $\vec{a} - \vec{b}2 + \vec{c}$

Answers for Part D

1. (a) (0, 0, 0) (b) (1, 0, 0) (c) 1; 0; 0; (1, 0, 0)
2. (a) (0, 1, 1); (0, 1, 1)
   (b) (1, 0, 4); (1, 0, 4)
   (c) a; b; c; (a, b, c)
3. 2, −1, 3
   5, −3, 2
   2, −1, 3; 5, −3, 2
   7, −4, 5
   (7, −4, 5)
4. (a) (4, 7, 10) (b) (−1, 3, −2)
   (c) (0, 0, 0) (d) (a + d, b + e, c + f)
5. 1; 2; 3; 3; −2; 4; 4; 0; 7; (4, 0, 7)
6. (a) (2, 6, −5) (b) (−2, 7, −2) (c) (2, 4, −6)
   (d) (7, 5, −12) (e) (5, 1, −6) (f) (10, 4, −11)

7. Suppose that A has coordinates (1, 2, 3) and that B has coordinates (5, 3, 1). Then
   (a) $B - A$ has components ____________.
   (b) $(B - A)t$ has components ____________.
   (c) $A + (B - A)t$ has coordinates ____________.
   (d) Recall that

$$\overleftrightarrow{AB} = \{X: \exists_t\, X = A + (B - A)t\}.$$

Therefore,

$X \in \overleftrightarrow{AB}$ if and only if X has the coordinates (____, ____, ____).

   (e) $X \in \overleftrightarrow{AB}$ if and only if

$X = \overrightarrow{u}_1 x_1 + \overrightarrow{u}_2 x_2 + \overrightarrow{u}_3 x_3$ where, for some real number t,

$$(*) \qquad \begin{cases} x_1 = 1 + (5 - 1)t = \_\_\_\_\_\_\_\_ \\ x_2 = \_\_\_\_\_\_\_\_ \\ x_3 = \_\_\_\_\_\_\_\_ \end{cases}$$

   (f) Since for $t = \frac{1}{2}$,

$$\begin{cases} x_1 = 1 + 4 \cdot \frac{1}{2} = 3 \\ x_2 = 2 + 1 \cdot \frac{1}{2} = \frac{5}{2} \\ x_3 = 3 + -2 \cdot \frac{1}{2} = 2 \end{cases}$$

it follows that (____________) are the coordinates of a point on $\overleftrightarrow{AB}$.
   (g) What value of 't' in the equations (*) will yield coordinates of the point A? Of the point B?
   (h) What value of 't' in the equations (*) will yield the coordinates of the midpoint of $\overline{AB}$?
   (i) Give the coordinates of three other points of $\overleftrightarrow{AB}$.
8. Suppose that A has coordinates (a_1, a_2, a_3) and B has coordinates (b_1, b_2, b_3). Then
   (a) $(B - A)t$ has components ____________.
   (b) $A + (B - A)t$ has coordinates ____________.
   (c) $C \in \overleftrightarrow{AB} \longleftrightarrow$ for some t, C has coordinates (____, ____, ____).
   (d) $C \in \overleftrightarrow{AB} \longleftrightarrow C = O + \overrightarrow{u}_1 c_1 + \overrightarrow{u}_2 c_2 + \overrightarrow{u}_3 c_3$ where, for some t,

$$\begin{cases} c_1 = \_\_\_\_\_\_\_\_\_\_ \\ c_2 = \_\_\_\_\_\_\_\_\_\_ \\ c_3 = \_\_\_\_\_\_\_\_\_\_ \end{cases}$$

   (e) If the coordinates of the point C and the real number t are related as in the last three equations in part (d), in what ratio does C divide the interval from A to B?

Answers for Part D [cont.]

7. (a) (4, 1, −2) (b) (4t, t, −2t)
   (c) (1 + 4t, 2 + t, 3 − 2t)
   (d) (1 + 4t, 2 + t, 3 − 2t) [for some t]
   (e) 1 + 4t; 2 + t; 3 − 2t
   (f) (3, 5/2, 2)
   (g) 0; 1
   (h) 1/2
   (i) [Various answers, each to be obtained by substituting for 't' in the answer for part (d).]
8. (a) $((b_1 - a_1)t,\ (b_2 - a_2)t,\ (b_3 - a_3)t)$
   (b) $(a_1 + (b_1 - a_1)t,\ a_2 + (b_2 - a_2)t,\ a_3 + (b_3 - a_3)t)$
   (c) $a_1 + (b_1 - a_1)t;\ a_2 + (b_2 - a_2)t;\ a_3 + (b_3 - a_3)t$
   (d) $a_1 + (b_1 - a_1)t;\ a_2 + (b_2 - a_2)t;\ a_3 + (b_3 - a_3)t$
   (e) t : 1 − t [Assuming, of course that A ≠ C ≠ B. Compare with Theorem 8-14.]

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Note that interchanging the 'a's and 'b's in (☆) yields another set of parametric coordinate equations for $\overleftrightarrow{AB}$. Of course, the same value of 't' [other than 1/2] will yield the coordinates of different points of $\overleftrightarrow{AB}$ when used in the two sets of equations. Each line has as many sets of parametric coordinate equations as there are choices of an ordered pair of points belonging to the line. One can, for example, given the coordinates of A and B, use (☆) to find the coordinates of two points of $\overleftrightarrow{AB}$ and, using these coordinates write new equations of the form (☆) for $\overleftrightarrow{AB}$. In particular, when, as in Exercise 1 of Part A, students are given coordinates of two points and asked for parametric equations, each student might give a different answer, all of which answers might be correct. But, this is not likely to occur.

The fact that two types of parametric equations [coordinate and vector] are introduced in the exercises of this section sometimes causes confusion. This is aggravated by the inclusion of two-point coordinate equations. To help avoid confusion we recommend using Exercises 1 - 3 of Part A and Exercises 1 - 3 of Part B as class discussion exercises. The Exercises 4 - 8 of Part A and Exercises 4 - 9 of Part B can be one homework assignment. Part C makes a good class activity for individual work. Part D can be used as a homework assignment but Part E can be very complicated for some students. We recommend that Part E be used as the basis of a class discussion and demonstration.

10.06 Equations of Lines

In the last exercises you noted that if the coordinates of A are (a_1, a_2, a_3) and the coordinates of B are (b_1, b_2, b_3) with respect to a coordinate system whose basis is $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ and whose origin is O then, for each $X \in \mathscr{E}$,

$X \in \overleftrightarrow{AB}$ if and only if $X = O + \vec{u}_1 x_1 + \vec{u}_2 x_2 + \vec{u}_3 x_3$
where, for some t,

$$(*) \qquad \begin{cases} x_1 = a_1 + (b_1 - a_1)t, \\ x_2 = a_2 + (b_2 - a_2)t, \\ x_3 = a_3 + (b_3 - a_3)t. \end{cases}$$

The equations (*) are called *parametric coordinate equations* of $\overleftrightarrow{AB}$ with respect to the given coordinate system. [The variable 't' whose values serve to single out points of $\overleftrightarrow{AB}$ is called a *parameter*.]

Exercises

In each of the following exercises, assume that a cartesian coordinate system with origin O has been established.

Part A

1. Write parametric coordinate equations for $\overleftrightarrow{AB}$, given that A and B have coordinates:
   (a) (12, 10, 8) and (−5, −5, 0) **(b)** (6, 3, 0) and (0, 2, 6)
   (c) (−4, −4, −4) and (4, 4, 4) **(d)** (1, −1, −1) and (−2, −2, 2)
   (e) (0, 0, 0) and (1, 2, 3) **(f)** (1, 0, 0) and (1, 1, 0)
2. For each part of Exercise 1, solve each of the parametric equations (*) for 't'. In which part can you not do this? Draw a figure for this part.

*

If $b_1 \neq a_1$, $b_2 \neq a_2$, and $b_3 \neq a_3$, you may solve the parametric equations (*) for 't' to obtain:

$$t = \frac{x_1 - a_1}{b_1 - a_1},\ t = \frac{x_2 - a_2}{b_2 - a_2},\ t = \frac{x_3 - a_3}{b_3 - a_3}$$

Since (x_1, x_2, x_3) are coordinates of a point of $\overleftrightarrow{AB}$ if and only if there is a value of 't' such that all three of these equations are satisfied it follows that the equations of $\overleftrightarrow{AB}$ can be written as:

$$(**) \qquad \frac{x_1 - a_1}{b_1 - a_1} = \frac{x_2 - a_2}{b_2 - a_2} = \frac{x_3 - a_3}{b_3 - a_3}$$

Answers for Part A

1. (a) $x_1 = 12 - 17t$
$x_2 = 10 - 15t$
$x_3 = 8 - 8t$

(b) $x_1 = 6 - 6t$
$x_2 = 3 - t$
$x_3 = 6t$

(c) $x_1 = -4 + 8t$
$x_2 = -4 + 8t$
$x_3 = -4 + 8t$

(d) $x_1 = 1 - 3t$
$x_2 = -1 - 3t$
$x_3 = -1 + 3t$

(e) $x_1 = t$
$x_2 = 2t$
$x_3 = 3t$

(f) $x_1 = 1$
$x_2 = t$
$x_3 = 0$

2. (a) $t = \frac{x_1 - 12}{-17} = \frac{x_2 - 10}{-15} = \frac{x_3 - 8}{-8}$

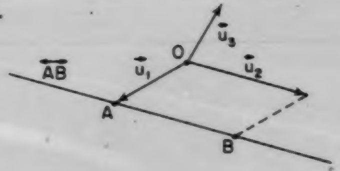
(b) $t = \frac{x_1 - 6}{-6} = \frac{x_2 - 3}{-1} = \frac{x_3}{6}$

(c) $t = \frac{x_1 + 4}{8} = \frac{x_2 + 4}{8} = \frac{x_3 + 4}{8}$

(d) $t = \frac{x_1 - 1}{-3} = \frac{x_2 + 1}{-3} = \frac{x_3 + 1}{3}$

(e) $t = \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}$

(f) $t = \frac{x_2}{1}$; but 't' cannot be expressed in terms of either 'x_1' or 'x_3'.



It will turn out that each of the three equations summarized in (☆☆) is the coordinate equation of a plane — the plane which contains $\overleftrightarrow{AB}$ and is parallel to one of the coordinate axes. These planes are called projecting planes for $\overleftrightarrow{AB}$, with respect to the given coordinate system. Note that, mostly by Theorem 9-15, there is a unique plane which contains a given line ℓ and is parallel to a given line $u \nparallel \ell$.

Of course, any two of the three equations summarized in (☆) imply the third.

These equations are called *two-point coordinate equations* for $\overleftrightarrow{AB}$. [Notice that (**) summarized three equations.]

*

3. Write two-point coordinate equations for the line $\overleftrightarrow{AB}$, given that A and B have coordinates:
   - (a) (12, 10, 8) and (−5, −5, 0)
   - (b) (1, 2, 3) and (2, −1, 4)
   - (c) (−4, −4, −4) and (4, 4, 4)
   - (d) (1, −1, −1) and (−2, −2, 2)
   - (e) (0, 0, 0) and (1, 2, 3)
   - (f) (0, 0, 0) and (1, 1, 1)
4. Given P with coordinates (−7, 4, 0) and R with coordinates (1, 0, 2), decide which of the points whose coordinates are given below belong to $\overleftrightarrow{PR}$.
   - (a) (−3, 2, 1)
   - (b) (1, 8, −2)
   - (c) $(2, \frac{1}{2}, \frac{7}{4})$
   - (d) $(-\frac{13}{3}, \frac{8}{3}, \frac{2}{3})$
   - (e) (9, −4, 4)
   - (f) (−11, 6, −1)
   - (g) (8, −3, 4)
   - (h) (2, −7, 6)
   - (i) (−23, 12, −4)
5. (a) For each of the points of Exercise 4 which belongs to $\overleftrightarrow{PR}$, tell the ratio in which the point divides the interval from P to R.
   (b) Is there a point of $\overleftrightarrow{PR}$ in the first coordinate plane?
6. Given the lines

   $\overleftrightarrow{AB}$ with parametric equations: $\begin{cases} x_1 = 1 - t \\ x_2 = 3 + t \\ x_3 = 1 - 2t \end{cases}$

   and $\overleftrightarrow{CD}$ with parametric equations: $\begin{cases} x_1 = 2 + s \\ x_2 = 5 - 2s \\ x_3 = 3s \end{cases}$

   Determine whether the lines intersect and, if they do, give the coordinates of the point of intersection.
7. Given the lines

   $\overleftrightarrow{AB}$ with parametric equations: $\begin{cases} x_1 = 1 - t \\ x_2 = 3 + t \\ x_3 = 1 - 2t \end{cases}$

   and $\overleftrightarrow{CD}$ with parametric equations: $\begin{cases} x_1 = 2 + s \\ x_2 = 5 - 2s \\ x_3 = 1 + 3s \end{cases}$

   Determine whether the lines intersect and, if they do, give the coordinates of the point of intersection.
8. Suppose that A, B C, and D have coordinates (1, 2, 1), (0, 5, 2), (6, 5, 4), and (−1, 8, 3) respectively. Determine whether $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect, and if they do, give the coordinates of the point of intersection.

Part B

Suppose that A and B are two points of a line l. Let O be any point. Recall that the position vector $\vec{a}$, with respect to O, of the point A is, by definition, $A - O$.

Answers for Part A [cont.]

3. (a) $\frac{x_1 - 12}{-17} = \frac{x_2 - 10}{-15} = \frac{x_3 - 8}{-8}$ [Note that equivalent equations are obtained by replacing the denominators '−17', '−15' and '−8' by numerals for multiples of the corresponding numbers by any nonzero number.]

   (b) $\frac{x_1 - 1}{1} = \frac{x_2 - 2}{-3} = \frac{x_3 - 3}{1}$

   (c) $\frac{x_1 + 4}{-8} = \frac{x_2 + 4}{-8} = \frac{x_3 + 4}{-8}$ [or: $x_1 + 4 = x_2 + 4 = x_3 + 4$, or: $x_1 = x_2 = x_3$]

   (d) $\frac{x_1 - 1}{-3} = \frac{x_2 + 1}{-1} = \frac{x_3 + 1}{3}$

   (e) $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}$

   (f) $\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$ [Compare with part (c).]

4. [We may use either (☆) or (☆☆) to describe $\overleftrightarrow{PR}$:

   $x_1 = -7 + 8t$
   $x_2 = 4 - 4t$ or: $\frac{x_1 + 7}{8} = \frac{x_2 - 4}{-4} = \frac{x_3}{2}$
   $x_3 = 2t$

   To use the former, we would substitute for 'x_1', 'x_2', and 'x_3' and see if the resulting three equations were satisfied by the same value of 't'. The two-point equations are, perhaps, a bit more convenient for the purpose at hand.]

   (a) $\frac{-3 + 7}{8} = \frac{2 - 4}{-4} = \frac{1}{2}$ So, the point with coordinates (−3, 2, 1) belongs to $\overleftrightarrow{PR}$.

   (b) $\frac{1 + 7}{8} \neq \frac{8 - 4}{-4} = \frac{-2}{2}$ So, not in $\overleftrightarrow{PR}$.

   (c) $\frac{2 + 7}{8} \neq \frac{\frac{1}{2} - 4}{-4} \neq \frac{\frac{1}{2}}{2}$ So, not in $\overleftrightarrow{PR}$.

   (d) $\frac{-(13/3) + 7}{8} = \frac{(8/3) - 4}{-4} = \frac{2/3}{2}$ So, in $\overleftrightarrow{PR}$.

   (e) $\frac{9 + 7}{8} = \frac{-4 - 4}{-4} = \frac{4}{2}$ So, in $\overleftrightarrow{PR}$.

   (f) $\frac{-11 + 7}{8} = \frac{6 - 4}{-4} = \frac{-1}{2}$ So, in $\overleftrightarrow{PR}$.

   (g) $\frac{8 + 7}{8} \neq \frac{-3 - 4}{-4} \neq \frac{4}{2}$ So, not in $\overleftrightarrow{PR}$.

   (h) $\frac{2 + 7}{8} \neq \frac{-7 - 4}{-4} \neq \frac{6}{2}$ So, not in $\overleftrightarrow{PR}$.

   (i) $\frac{-23 + 7}{8} = \frac{12 - 4}{-4} = \frac{-4}{2}$ So, in $\overleftrightarrow{PR}$.

Answers for Part A [cont.]

5. (a) [The points in question are those of parts (a), (d), (e), (f), and (i). Had we used parametric equations, the desired ratios could have been found by substituting into '$t/(1 - t)$'. However, since, for $\overleftrightarrow{PR}$, $t = x_3/2$ we can still do the equivalent.]

(a) 1 (d) 1/2 (e) -2 (f) $-1/3$ (i) $-2/3$

(b) [To find such a point, we must find a point of $\overleftrightarrow{PR}$ such that $x_1 = 0$. We may, again, use either the parametric or the two-point equations. We give both methods.]

Using parametric equations:

For such a point, $0 = -7 + 8t$, so $t = 7/8$. Hence, $x_2 = 4 - 4\cdot\frac{7}{8} = \frac{1}{2}$ and $x_3 = 2\cdot\frac{7}{8} = \frac{7}{4}$. So, there is a point of $\overleftrightarrow{PR}$ in the first coordinate plane, and its coordinates are $(0, 1/2, 7/4)$.

Using two-point equations:

For such a point, $\frac{0 + 7}{8} = \frac{x_2 - 4}{-4} = \frac{x_3}{2}$. Solving any two of these equations we find that $x_2 = 1/2$ and $x_3 = 7/4$. So, ...

6. To find a point common to the two lines, we must find values of 's' and 't' which satisfy the three equations:

$$1 - t = 2 + s$$
$$3 + t = 5 - 2s$$
$$1 - 2t = 3s$$

Solving the first two yields $s = 3$, $t = -4$. Since these values satisfy the third equation $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at the point whose coordinates are $(5, -1, 9)$.

7. ($1 - t = 2 + s$ and $3 + t = 5 - 2s$) if and only if ($s = 3$ and $t = -4$). Since $1 - 2\cdot -4 \neq 1 + 3\cdot 3$, the lines have no common point.

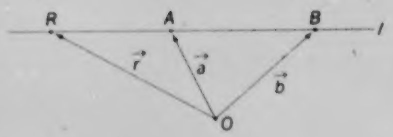
8. Parametric coordinate equations for $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are:

$$\begin{array}{lcl} x_1 = 1 - t & & x_1 = 6 - 7t \\ x_2 = 2 + 3t & \text{and:} & x_2 = 5 + 3t \\ x_3 = 1 + t & & x_3 = 4 - t \end{array}$$

For there to be a point common to the lines there must be a first value of 't' and a second value of 't' such that the first and second sets of equations yield the same coordinates for the respective values of 't'. For simplicity, then, we replace the parameter in the second set of equations by, say, 's' and look for solutions of:

$$1 - t = 6 - 7s$$
$$2 + 3t = 5 + 3s$$
$$1 + t = 4 - s$$

Solving two of these and checking in the third yields $s = 1$, $t = 2$. So, the lines do intersect, and the point of intersection has coordinates $(-1, 8, 3)$.



1. Since $A \neq B$ and $\{A, B\} \subseteq l$ it follows that $l = \overleftrightarrow{AB}$. Complete:

$$R \in l \longleftrightarrow \exists_t\, R = \underline{\hspace{3cm}}$$

2. Complete the following without using '*A*' and '*B*':

$$R \in l \longleftrightarrow \exists_t\, \vec{r} = \underline{\hspace{3cm}}$$

[*Hint*: Recall, for one thing, that $B - A = (B - O) - (A - O)$.]

*

The equation:

$$(1) \qquad \vec{r} = \vec{a} + (\vec{b} - \vec{a})t$$

is, for $\vec{a} \neq \vec{b}$, a *parametric vector equation* for the line determined by the points whose position vectors are $\vec{a}$ and $\vec{b}$. What this means is that the position vectors of points of this line are just the values, for the various values of 't', of the right side of (1). Equation (1) is, of course, equivalent to:

$$(2) \qquad \vec{r} = \vec{a}(1 - t) + \vec{b}t$$

This suggests a more symmetrical form of equation which serves the same purpose as (1) and (2) is:

$$(3) \qquad \vec{r} = \vec{a}a + \vec{b}b \quad [a + b = 1]$$

Given the position vectors $\vec{a}$ and $\vec{b}$ of two points, a given vector $\vec{r}$ is the position vector of a point on the line containing these points if and only if there are numbers—say, a and b—whose sum is 1 and which satisfy (3).

3. (a) Assume as given a basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ for $\mathscr{T}$ and, taking the point O as origin both for coordinates and for position vectors, explain the relation of the parametric vector equation (1) to the parametric equations (*) on page 423.
   (b) For what value of 't' do the equations (*) give the coordinates of the point which divides the interval from A to B in $a : b$? [*Hint*: Recall Theorem 8-14.]

Answers for Part B

1. $A + (B - A)t$
2. $\vec{a} + (\vec{b} - \vec{a})t$ [For, $R = A + (B - A)t$ if and only if $R - O = (A - O) + (B - A)t$; and $R - O = \vec{r}$, $A - O = \vec{a}$, and $B - A = \vec{b} - \vec{a}$, where $\vec{b} = B - O$.]

*

In "coordinate geometry" it is customary to use 'x_1', 'x_2', and 'x_3', or 'x', 'y', and 'z' as variables whose values are interpreted as coordinates of points. In "vector geometry" it is customary to use '$\vec{r}$' as a variable whose values are interpreted as position vectors [or "radius vectors"]. These conventions motivate our somewhat oscillatory choices of notation.

Students may recall equation (2) as equation (1) on page 346.

3. (a) Since $\vec{r}$, $\vec{a}$, and $\vec{b}$, are $R - O$, $A - O$, and $B - O$, the components of $\vec{r}$, $\vec{a}$, and $\vec{b}$ are the coordinates of R, A, and B. Furthermore, by Theorem 10-11, the vectors in question satisfy (1) if and only if their components satisfy (☆).
   (b) By Theorem 8-14, a point R whose position vector is given by equation (2) divides the interval from A to B in the ratio $t/(1 - t)$. The same holds when the coordinates of R are given by (☆). In order for $t/(1 - t)$ to be $a : b$ we must take $t = a/(a + b)$. [Obtained by solving '$t/(1 - t) = a/b$' for 't'.]

4. The vector equation (1) is appropriate for use in situations where it would be natural to use '$\overleftrightarrow{AB}$' in referring to a given line. What form of vector equation would it be natural to use in describing a line $\overleftrightarrow{A[\vec{c}]}$?

5. Suppose that A has coordinates (a_1, a_2, a_3) and that the non-$\vec{0}$ vector $\vec{c}$ has components (c_1, c_2, c_3). Write parametric equations for the coordinates (x_1, x_2, x_3) of a point of $\overleftrightarrow{A[\vec{c}]}$. [*Hint*: If you are not clear as to what is wanted here, see equations (*) on page 423.]

*

The equation:

$$\vec{r} = \vec{a} + \vec{c}t \tag{4}$$

is, like (1), a parametric equation for the position vectors of points of a line. In the case of (1), the line in question is the line which contains the points whose position vectors are $\vec{a}$ and $\vec{b}$. In the case of (4), the line in question is the line whose direction is $[\vec{c}]$ and which contains the point whose position vector is $\vec{a}$. Corresponding with (1) we have equations (*) on page 423 which are parametric equations for the coordinates of points of the line through, say, A and B. Corresponding with (4) we have similar equations:

$$(**) \qquad \begin{cases} x_1 = a_1 + c_1t \\ x_2 = a_2 + c_2t \\ x_3 = a_3 + c_3t \end{cases}$$

In these equations, (a_1, a_2, a_3) are the coordinates of a point A of the line and (c_1, c_2, c_3) are the components of some non-$\vec{0}$ vector in the direction of the line. [The components of such a vector are sometimes called *direction numbers* of the line—of course, with respect to the given coordinate system.]

*

6. In each of the following exercises you are given the coordinates of a point A and the components of a vector $\vec{c}$. Using (**) as a model, write parametric equations for the coordinates of points on $\overleftrightarrow{A[\vec{c}]}$.

(a) A: $(0, 1, 2)$ — $\vec{c}$: $(2, 4, -1)$

(b) A: $(2, 5, 1)$ — $\vec{c}$: $(2, 4, -1)$

(c) A: $(3, -2, 1)$ — $\vec{c}$: $(2, 0, -4)$

(d) A: $(4, -2, -3)$ — $\vec{c}$: $(-1, 0, 2)$

(e) A: $(5, -6, 3)$ — $\vec{c}$: $(0, 2, 0)$

(f) A: $(4, 7, -1)$ — $\vec{c}$: $(0, 0, 0)$

Answers for Part B [cont.]

4. $\vec{r} = \vec{a} + \vec{c}t$ [Some students will have already recalled that, for $\{A, B\} \subseteq \ell$, $B - A \in [\ell]$ and that, for $A \in \ell$ and $\vec{a} \in [\ell]$, $A + \vec{a} \in \ell$.]

5. $x_1 = a_1 + c_1t$, $x_2 = a_2 + c_2t$, $x_3 = a_3 + c_3t$

In volume 2, where we can use mutually perpendicular coordinate axes, it will turn out that, for such a coordinate system, the direction numbers of a line are proportional to the cosines of angles determined by the line and the coordinate axes. [Which angles may be specified by choosing a sense on the line and on each of the axes.] the cosines of these angles are, then, called direction cosines of the line. Of course, students are not yet prepared to discuss these matters.

6. (a) $x_1 = 2t$, $x_2 = 1 + 4t$, $x_3 = 2 - t$

(b) $x_1 = 2 + 2t$, $x_2 = 5 + 4t$, $x_3 = 1 - t$

(c) $x_1 = 3 + 2t$, $x_2 = -2$, $x_3 = 1 - 4t$

(d) $x_1 = 4 - t$, $x_2 = -2$, $x_3 = -3 + 2t$

(e) $x_1 = 5$, $x_2 = -6 + 2t$, $x_3 = 3$

(f) $x_1 = 4$, $x_2 = 7$, $x_3 = -1$

7. Consider the various sets $\overleftrightarrow{A[\vec{c}]}$ described in Exercise 6.
   (a) Are there any of these sets which are not lines?
   (b) Are two of the lines described in Exercise 6 parallel?
   (c) Is the same line described twice in Exercise 6?
   (d) Are any of the lines of Exercise 6 parallel to any of the coordinate axes?
   (e) Are any of the lines parallel to any of the coordinate planes?
8. Suppose that A has coordinates (1, 2, 3) and that B has coordinates (2, 4, −3).
   (a) Use (*) to write parametric equations for $\overleftrightarrow{AB}$.
   (b) Use (**) to write parametric equations for $\overleftrightarrow{A[\vec{c}]}$, where $\vec{c} = B - A$.
   (c) Compare your answers for parts (a) and (b).
9. Suppose that A has coordinates (0, 1, 2) and that $\vec{c}$ has components (3, 5, 7).
   (a) Use (**) to write parametric equations for $\overleftrightarrow{A[\vec{c}]}$.
   (b) Use (*) to write parametric equations for $\overleftrightarrow{AB}$, where $B = A + \vec{c}$.
   (c) Compare your answers for parts (a) and (b).

Part C

1. If l is the line which contains two given points, A and B, then, as you know, $[l] = [B - A]$. You also know that, for $a \neq 0$, $[\vec{a}a] = [\vec{a}]$. Suppose, now that A, B, C, and D have coordinates A: (2, 3, 5), B: (4, 2, 6), C: (6, 3, 9), and D: (8, 2, 10).
   (a) What are the components of $B - A$?
   (b) What are the components of $D - C$?
   (c) What can you say about the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$?
2. Repeat Exercise 1 when the coordinates of the given points are A: (3, 5, 1), B: (2, 1, 4), C: (4, 6, 3), and D: (2, −2, 5).
3. In each of the following exercises, determine whether or not $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.
   (a) A: (3, 2, −1)
   B: (4, 7, 4)
   C: (3, 9, 1)
   D: (6, 24, 16)
   (b) A: (2, 4, 0)
   B: (−1, −2, −3)
   C: (4, 8, 2)
   D: (9, 3, 7)
   (c) A: (4, 7, 0)
   B: (3, 6, 0)
   C: (−2, 4, 0)
   D: (−4, 2, 0)
   (d) A: (2, 0, 4)
   B: (4, 0, −1)
   C: (1, 0, 4)
   D: (2, 0, 2)
4. In each of the following exercises you are to find coordinates of a point D such that $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$.
   (a) A: (2, 4, 0)
   B: (1, 2, 5)
   C: (8, 4, 0)
   (b) A: (1, 4, 7)
   B: (2, 9, 1)
   C: (3, −2, 4)
   (c) A: (2, 4, 0)
   B: (1, 2, 0)
   C: (4, 1, 0)
   (d) A: (3, 5, 0)
   B: (4, 3, 0)
   C: (2, 6, 0)

Answers for Part B [cont.]

7. (a) The set described in part (f) is not a line. [As should be the case, since $\overleftrightarrow{A[\vec{0}]} = \{A\}$, the parametric coordinate equations for $\overleftrightarrow{A[\vec{0}]}$ yield only the coordinates of A.]

   (b) The lines described in (a) and (b) are parallel since the direction of each is described by the same vector. The lines described in (c) and (d) are also parallel since $[\vec{u}_1 2 + \vec{u}_3 \cdot -4] = [\vec{u}_1 \cdot -1 + \vec{u}_3 2]$.

   (c) The same line is described in (a) and (b). This can be discovered in several ways: (i) Use the equations for one line to find coordinates of a point on that line, and use the equations of the other to show that the same point belongs to it. (ii) Note that replacing 't' in the equations of (b) by 't - 1' yields the equations of (a). This means that any coordinates obtainable from either set of equations is obtainable from the other. [That different lines are described in (c) and (d) can be shown by showing that one coordinate triple — for example, (3, −2, 1) — which satisfies (c) does not satisfy (d).]

   (d) The line described in part (e) is parallel to the second coordinate axis, since its direction is $[\vec{u}_2 2]$.

   (e) The lines described in (c) and (d) are parallel to the second coordinate plane since their direction, $[\vec{u}_1 - \vec{u}_3 2]$, is a subset of the direction $[\vec{u}_3, \vec{u}_1]$ of this plane. And, by part (d), the line described in part (e) is parallel to both the first and the third coordinate plane.

8. (a)-(c) The equations obtained in answer to (a) and (b) are, of course, the same:

$$x_1 = 1 + t$$
$$x_2 = 2 + 2t$$
$$x_3 = 3 - 6t$$

9. (a)-(c) The equations obtained in answers to (a) and (b) are:

$$x_1 = 3t$$
$$x_2 = 1 + 5t$$
$$x_3 = 2 + 7t$$

Answers for Part C

1. (a) (2, −1, 1)   (b) (2, −1, 1)   (c) $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$
2. (a) (−1, −4, 3)   (b) (−2, −8, 2)   (c) $\overleftrightarrow{AB} \not\parallel \overleftrightarrow{CD}$
3. In parts (a) and (c), $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$; in parts (b) and (d), $\overleftrightarrow{AB} \not\parallel \overleftrightarrow{CD}$.
4. [In each part, one needs to find a point D belonging to $\overleftrightarrow{C[B - A]}$. The simplest choice is $C + (B - A)$ but, of course, for any $t \neq 0$, $C + (B - A)t$ will do as well. An easy check is $D - C \in [B - A]$.]

   (a) (7, 2, 5)   (b) (4, 3, −2)   (c) (3, −1, 0)   (d) (3, 4, 0)

Part D

1. Consider the points A, B, C, and D whose coordinates are A: $(1, -2, 3)$, B: $(2, -3, 5)$, C: $(-1, 4, -3)$, and D: $(-2, 5, -5)$.
   - (a) Show that $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect.
   - (b) In what ratio does the point found in part (a) divide the interval from A to C? The interval from B to D?
   - (c) What follows about $ABCD$ from the result obtained in part (b)?
   - (d) Obtain the conclusion you reached in part (c) in another way.
2. Suppose that $\{A, B, C, D\}$ is noncoplanar.
   - (a) Draw a figure and mark on it the midpoints P, Q, R, and S of $\overline{AB}$, $\overline{CD}$, $\overline{BD}$, and $\overline{AC}$, respectively.
   - (b) Using A as origin and $(B - A, C - A, D - A)$ as a basis for $\mathscr{T}$, find the coordinates of P, Q, R, and S.
   - (c) Show that $\overline{PQ}$ and $\overline{RS}$ intersect.
   - (d) What does your work in part (c) tell you about $\overline{PQ}$ and $\overline{RS}$?
   - (e) Formulate a theorem concerning the segments joining the midpoints of opposite sides of a tetrahedron. ['tetrahedron' is a synonym for 'triangular pyramid'.]
3. Suppose that $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ are position vectors of four noncoplanar points.
   - (a) What can you say about $(\vec{b} - \vec{a}, \vec{c} - \vec{a}, \vec{d} - \vec{a})$?
   - (b) Suppose that $a + b + c + d = 0$ and $\vec{a}a + \vec{b}b + \vec{c}c + \vec{d}d = \vec{0}$. What can you say about $\vec{a}(a + b + c + d)$? About $(\vec{b} - \vec{a})b + (\vec{c} - \vec{a})c + (\vec{d} - \vec{a})d$? About b, c, and d?
   - (c) Prove:

> **Theorem 10–12** If $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ are position vectors of noncoplanar points and $a + b + c + d = 0$ then
> $\vec{a}a + \vec{b}b + \vec{c}c + \vec{d}d = \vec{0}$ if and only if
> $a = 0, b = 0, c = 0$, and $d = 0$.

4. Reconsider Exercise 2. Suppose that $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ are the position vectors of A, B, C, and D, respectively.
   - (a) What are the position vectors $\vec{p}, \vec{q}, \vec{r}$, and $\vec{s}$ of P, Q, R, and S?
   - (b) The equation:

$$\vec{x} = \vec{p}(1 - t) + \vec{q}t$$

is a parametric vector equation of $\overline{PQ}$. Use your results from part (a) to rewrite this equation in terms of the position vectors $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$, and write a similar equation for $\overline{RS}$, using 's' as parameter.
   - (c) Use Theorem 10–12 and your equations from part (b) to show that $\overline{PQ}$ and $\overline{RS}$ bisect each other.

Answers for Part D

1. (a) $\overleftrightarrow{AC}$: $x_1 = 1 - 2t$, $x_2 = -2 + 6t$, $x_3 = 3 - 6t$; $\overleftrightarrow{BD}$: $x_1 = 2 - 4s$, $x_2 = -3 + 8s$, $x_3 = 5 - 10s$

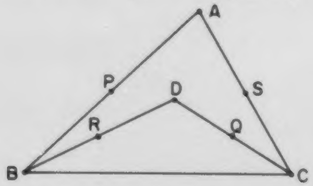
$\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at the point whose coordinates are given by $t = 1/2$ and $s = 1/2$.

   (b) Since, for the point of intersection, $t:(1 - t) = s:(1 - s) = 1$, this point is the midpoint of both intervals and, so, divides each in $1:1$.

   (c) ABCD is a parallelogram.

   (d) The components of $B - A$ and $C - D$ are [for both] $(1, -1, 2)$. So, $B - A = C - D$. $C \notin \overleftrightarrow{AB}$ since the components of $C - B$ are $(-3, 7, -8)$ and, so, $C - B \notin [B - A]$. Hence [by Theorem 8-16], ABCD is a parallelogram.

2. (a)



   (b) Since $P - A = (B - A)\frac{1}{2}$, the coordinates of P are $(\frac{1}{2}, 0, 0)$. Since $Q - A = (C - A)(1 - \frac{1}{2}) + (D - A)\frac{1}{2}$, the coordinates of Q are $(0, \frac{1}{2}, \frac{1}{2})$. Similarly, the coordinates of R are $(\frac{1}{2}, 0, \frac{1}{2})$ and those of S are $(0, \frac{1}{2}, 0)$.

   (c) It is a reasonable guess that $\overline{PQ}$ and $\overline{RS}$ have the same midpoint. This is quickly checked by noting that, for each, the coordinates of its midpoint are $(1/4, 1/4, 1/4)$. Since $\{A, B, C, D\}$ is noncoplanar, $\{P, Q, R\}$ is noncollinear and, so, $\overline{PQ}$ and $\overline{RS}$ have no other common point. [A more pedestrian approach is to find parametric equations of $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$, and proceed as in Exercise 1(a). A more inspired approach is to note that ABDC is a simple quadrilateral and use Theorem 8-20(b) — thus avoiding the somewhat cumbersome use of coordinates.]

   (d) $\overline{PQ}$ and $\overline{RS}$ bisect each other.

   (e) The three segments which join the midpoints of opposite sides of a tetrahedron intersect at their common midpoint.

Answers for Part D [cont.]

3. (a) Since $\vec{b} - \vec{a} = B - A$, etc., $(\vec{b} - \vec{a}, \vec{c} - \vec{a}, \vec{d} - \vec{a})$ is linearly independent.

(b) Suppose that $a + b + c + d = 0$ and $\vec{a}a + \vec{b}b + \vec{c}c + \vec{d}d = \vec{0}$. It follows that $\vec{a}(a + b + c + d) = \vec{0}$ and, so, that $(\vec{a}a + \vec{b}b + \vec{c}c + \vec{d}d) - (\vec{a}a + \vec{a}b + \vec{a}c + \vec{a}d) = \vec{0}$. Hence, $(\vec{b} - \vec{a})b + (\vec{c} - \vec{a})c + (\vec{d} - \vec{a})d = \vec{0}$ and, since [by part (a)] $(\vec{b} - \vec{a}, \vec{c} - \vec{a}, \vec{d} - \vec{a})$ is linearly independent, that $b = c = d = 0$. Since $a + b + c + d = 0$, $a = 0$ also.

(c) The proof is contained in the preceding arguments except for noting that if $a = b = c = d = 0$ then $\vec{a}a + \vec{b}b + \vec{c}c + \vec{d}d = \vec{0}$.

[Theorem 10-12 is, of course, analogous to the corollary to Theorem 8-15. Theorem 8-15 has a similar analogue of which Theorem 10-12 could be considered a corollary.]

4. (a) $\vec{p} = \vec{a}\frac{1}{2} + \vec{b}\frac{1}{2}$, $\vec{q} = \vec{c}\frac{1}{2} + \vec{d}\frac{1}{2}$, $\vec{r} = \vec{b}\frac{1}{2} + \vec{d}\frac{1}{2}$, $\vec{s} = \vec{a}\frac{1}{2} + \vec{c}\frac{1}{2}$ [If, as in Exercise 2, position vectors are taken with respect to A then $\vec{a} = \vec{0}$. The preceding formulas, however, hold for any origin. The symmetry which results when the origin is unspecified is often of more value than the simplicity which sometimes results from a special choice of origin. This is one reason why vector methods are often more efficient than coordinate methods.]

(b) $\overleftrightarrow{PQ}$: $\vec{x} = (\vec{a} + \vec{b})\frac{1 - t}{2} + (\vec{c} + \vec{d})\frac{t}{2}$

$\overleftrightarrow{RS}$: $\vec{x} = (\vec{b} + \vec{d})\frac{1 - s}{2} + (\vec{a} + \vec{c})\frac{s}{2}$

(c) [To find where, if at all, $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect we look for values of 's' and 't' which give the same value for '$\vec{x}$'.]

$$\vec{a}\frac{1 - t}{2} + \vec{b}\frac{1 - t}{2} + \vec{c}\frac{t}{2} + \vec{d}\frac{t}{2} = \vec{a}\frac{s}{2} + \vec{b}\frac{1 - s}{2} + \vec{c}\frac{s}{2} + \vec{d}\frac{1 - s}{2}$$

$$(*) \quad \vec{a}(1 - t - s) + \vec{b}(s - t) + \vec{c}(t - s) + \vec{d}(s + t - 1) = \vec{0}$$

Since $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are position vectors of noncollinear points and since

$$(1 - t - s) + (s - t) + (t - s) + (s + t - 1) = 0$$

it follows that (*) is satisfied if and only if $s + t = 1$ and $s = t$ — that is, if and only if $s = 1/2 = t$. So, $\overline{PQ}$ and $\overline{RS}$ bisect each other.

Sample Quiz

1. Suppose that $(\vec{a}, \vec{b}, \vec{c})$ is a basis for $\mathcal{T}$, that O is the origin, and that A, B, and C have coordinates $(2, -3, 1)$, $(-1, 2, 4)$, and $(2, -7, 1)$, respectively, with respect to the coordinate system determined by O and the given basis.

(a) Find the coordinates of all points X such that A, B, C, and X are the vertices of a parallelogram.

(b) Find the coordinates of the point of intersection of the medians of $\triangle ABC$. [This point is called the centroid of $\triangle ABC$.]

(c) Given that the points you found in (a) are X_1, X_2, and X_3, show that the centroid of $\triangle X_1X_2X_3$ is the centroid of $\triangle ABC$.

2. Suppose that A has coordinates $(6, -3, 5)$, that B has coordinates $(-9, 7, 7)$ and that $\vec{a}$ has components $(2, -4, 6)$ with respect to a given coordinate system.

(a) Write parametric equations for $\overleftrightarrow{A[\vec{a}]}$.

(b) Compute the coordinates of M such that B is the midpoint of $\overline{AM}$.

(c) Show that A, $A + \vec{a}$, B, and $B - \vec{a}$ are the vertices of a parallelogram.

Key to Sample Quiz

1. (a) $(5, -12, -2)$, $(-1, 6, 4)$, $(-1, -2, 4)$

(b) $(1, -8/3, 2)$

(c) The centroid of $\triangle X_1X_2X_3$ has coordinates $(1, -8/3, 2)$. Compare with (b).

2. (a) $x_1 = 6 + 2t$, $x_2 = -3 - 4t$, $x_3 = 5 + 6t$

(b) $(-24, 17, 9)$

(c) [Many solutions.]

(d) Show that, even if $\{A, B, C, D\}$ is coplanar, it shall follow that $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ have the same midpoint. Conclude that if a simple quadrilateral is not a parallelogram then the midpoints of its diagonals and of a pair of opposite sides are the vertices of a parallelogram.

Part E

Recall the two-point coordinate equations of lines which were introduced in Part A on page 423. If A and B have coordinates (a_1, a_2, a_3) and (b_1, b_2, b_3) then the parametric equations:

$$\begin{cases} x_1 = a_1 + (b_1 - a_1)t \\ x_2 = a_2 + (b_2 - a_2)t \\ x_3 = a_3 + (b_3 - a_3)t \end{cases}$$

can, each, be solved for 't'—assuming that $b_1 \neq a_1$, $b_2 \neq a_2$, and $b_3 \neq a_3$. On doing so, we note that (x_1, x_2, x_3) are coordinates of a point of $\overleftrightarrow{AB}$ if and only if

$$\frac{x_1 - a_1}{b_1 - a_1} = \frac{x_2 - a_2}{b_2 - a_2} = \frac{x_3 - a_3}{b_3 - a_3}.$$

Something similar to this can be done if the restrictions that $b_1 \neq a_1$, $b_2 \neq a_2$, and $b_3 \neq a_3$ are not all satisfied.

1. What is $\overleftrightarrow{AB}$ in case *none* of the three restrictions just mentioned are satisfied?
2. Suppose that A and B have coordinates $(2, 6, -1)$ and $(5, 4, -1)$.
   - **(a)** Write parametric coordinate equations for $\overleftrightarrow{AB}$.
   - **(b)** Use your answer for part (a) to find two equations, neither of which contains the parameter, and which are satisfied by the coordinates of just those points which belong to $\overleftrightarrow{AB}$.
   - **(c)** What do you know about the third coordinate of any point of $\overleftrightarrow{AB}$?
   - **(d)** Describe the set of all points of $\mathscr{E}$ whose third coordinate is -1. [What kind of set is it, and how is it related to the coordinate planes or axes?]
3. In solving Exercise 2 you should have found that the line $\overleftrightarrow{AB}$ referred to there can be described by the two equations:

$$\frac{x_1 - 2}{3} = \frac{x_2 - 6}{-2}, \quad x_3 = -1$$

You also should have noted that the second of these equations is satisfied by the coordinates (y_1, y_2, y_3) of just those points which belong to a certain plane which is parallel to the third coordinate plane.

(d) In any case, (*) is satisfied if $s = 1/2 = t$. So, in any case, $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ have the same midpoint [but, they may have other points in common]. Consider the quadrilateral to be ABCD and assume that $\overline{AD} \nparallel \overline{BC}$. In the notation of this exercise, P and Q are the midpoints of the opposite sides $\overline{AB}$ and $\overline{CD}$, and R and S are the midpoints of the diagonals $\overline{BD}$ and $\overline{AC}$. We have seen that $\overline{PQ}$ and $\overline{RS}$ have the same midpoint. By earlier results [Part B on page 355] we know that, since $\overline{AC} \nparallel \overline{BC}$, {P, Q, R} is not collinear. Hence, the intersection of $\overline{PQ}$ and $\overline{RS}$ must consist of their common midpoint — in short, they bisect each other. Hence, PRQS is a parallelogram.

Answers for Part E

1. If $b_1 = a_1$, $b_2 = a_2$, and $b_3 = a_3$ then $B = A$ and, so, $\overleftrightarrow{AB} = \overleftrightarrow{AA} = \{A\}$.

2. (a)
$$x_1 = 2 + 3t$$
$$x_2 = 6 - 2t$$
$$x_3 = -1$$

(b) $\dfrac{x_1 - 2}{3} = \dfrac{x_2 - 6}{-2}$, $x_3 = -1$

(c) It is -1.

(d) It is the plane which is parallel to the third coordinate plane and contains the point $O - \vec{u}_3$ whose coordinates are $(0, 0, -1)$. For, D belongs to this set if and only if $D = O + \vec{u}_1 s + \vec{u}_2 t + \vec{u}_3 \cdot -1$, for some choice of values of 's' and 't'. So, the set in question is $\{X: \exists_x \exists_y\ X = (O - \vec{u}_3) + \vec{u}_1 x + \vec{u}_2 y\}$ and this is $\overline{(O - \vec{u}_3)[\vec{u}_1, \vec{u}_2]}$.

3. (a) One such translation is $-\vec{u}_3$; but the sum of this and any member of $[\vec{u}_1, \vec{u}_2]$ will do as well.

(b) It is reasonable to guess that the equation:

$$(*) \qquad \frac{x_1 - 2}{3} = \frac{x_2 - 6}{-2}$$

represents another plane containing the line $\overleftrightarrow{AB}$. Since (*) places no restriction on 'x_3', (x_1, x_2, x_3) satisfies (*) if and only if there are numbers s and t such that

$$x_1 = 2 + 3s$$
$$x_2 = 6 - 2s$$
$$x_3 = t.$$

Equivalently,
$$x_1 = 2 + 3s + 0t$$
$$x_2 = 6 - 2s + 0t$$
$$x_3 = 0 + 0s + 1t$$

So, if C has coordinates $(2, 6, 0)$ and $\vec{a}$ has components $(3, -2, 0)$ then the coordinates of a point P are given by (*) if and only if $P \in C[a, u_3]$. Hence, (*) represents a plane which is parallel to the third coordinate axis and whose direction contains $\vec{u}_1 3 + \vec{u}_2 \cdot -2$.

(a) Describe a translation $\vec{c}$ which maps the third coordinate plane onto the plane whose equation is '$x_3 = -1$'. [*Hint*: $\vec{c}$ must be a linear combination of the terms of the basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ of the coordinate system.]

(b) Theorem 10–2 might now suggest something about the first of the two equations. Make a guess and think of ways of checking it.

4. Consider the line through points A and B which have coordinates (3, 2, 1) and (4, 2, 1).

(a) Repeat parts (a) and (b) of Exercise 2.

(b) Describe two planes whose intersection is $\overleftrightarrow{AB}$.

(c) How is $\overleftrightarrow{AB}$ related to the coordinate planes and axes?

10.07 Equations of Planes

If you recall the similarities between Theorems 7–1 and 9–1 and between Theorems 7–5 and 9–11, it should be clear that we can expect that vector equations and coordinate equations of planes will be similar to those of lines. For example, if $\{A, B, C\}$ is noncollinear then there is a unique plane, $\overleftrightarrow{ABC}$ which contains A, B, and C, and a point R belongs to this plane if and only if, for some numbers s and t,

$$R = A + (B - A)s + (C - A)t.$$

As in the case of the line $\overleftrightarrow{AB}$, if $\vec{a}, \vec{b}, \vec{c}$, and $\vec{r}$ are the position vectors of A, B, C, and R, with respect to some point O, it follows that $R \in \overleftrightarrow{ABC}$ if and only if, for some s and t,

$$\vec{r} = \vec{a} + (\vec{b} - \vec{a})s + (\vec{c} - \vec{a})t.$$

[Explain.] If, now, we choose a basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ for $\mathscr{T}$ then the coordinates – say (x_1, x_2, x_3) – of a point $R \in \overleftrightarrow{ABC}$ must satisfy parametric coordinate equations:

$$(\star) \qquad \begin{cases} x_1 = a_1 + (b_1 - a_1)s + (c_1 - a_1)t \\ x_2 = a_2 + (b_2 - a_2)s + (c_2 - a_2)t \\ x_3 = a_3 + (b_3 - a_3)s + (c_3 - a_3)t \end{cases}$$

where (a_1, a_2, a_3) are the coordinates of A [or, equivalently, are the components of the position vector $A - O$], etc. [Explain.]

In the following exercises, we suppose that a coordinate system based in some point O and some basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is chosen.

This result can, of course, be generalized to apply to any equation of the form of (*). For example, the equation:

$$\frac{x_2 + 1}{4} = \frac{x_3 - 5}{7}$$

is satisfied by the coordinates of just those points which belong to the plane whose direction is $[\vec{u}_1, \vec{u}_2 4 + \vec{u}_3 7]$ and which contains the point whose coordinates are (0, −1, 5) [or, for that matter (−217, −1, 5)].

As mentioned earlier, the plane represented by (*) is the projecting plane, parallel to the third coordinate axes, for $\overleftrightarrow{AB}$. [In case the axes were mutually perpendicular, this plane would customarily be described as the projecting plane, perpendicular to the third coordinate plane, for $\overleftrightarrow{AB}$, or, as the x_1x_2-projecting plane for $\overleftrightarrow{AB}$.]

Since, for each point of $\overleftrightarrow{AB}$, $x_3 = -1$ [see Exercise 2], this last equation represents the projecting plane for $\overleftrightarrow{AB}$ parallel to the second coordinate axis and [the same] projecting plane for $\overleftrightarrow{AB}$ parallel to the first coordinate axis.

In general, the three equations summarized in the two-point coordinate equations of a line represent three planes projecting the line parallel to the three coordinate axes. In case one of the denominators represents 0, two of these planes coincide in a plane parallel to the other two axes.

The subject of projecting planes deserves to be discussed in class in connection with the present exercise. See also, the following exercise.

4. (a) $x_1 = 3 + t$

$x_2 = 2$

$x_3 = 1$

(b) $x_2 = 2$, $x_3 = 1$ [Evidently, the first of the parametric equations puts no restriction on 'x_1'. It merely serves to specify the parameter-value, associated with a point on the line, in terms of the first coordinate of the point.]

(b) The plane which is parallel to the second coordinate plane and contains, say, the point whose coordinates are (0, 2, 0); the plane which is parallel to the third coordinate plane and contains, say, the point whose coordinates are (0, 0, 1).

(c) $\overleftrightarrow{AB}$ is parallel to the first coordinate axis and, so, to each of the second and third coordinate planes.

A line like $\overleftrightarrow{AB}$ is contained in each of infinitely many planes parallel to the first coordinate axis. The two of these each of which is parallel to one of the other axes are called the projecting planes of the line.

In discussing Exercise 3(b) of Part E we came upon an example of (☆):

$$\begin{aligned} x_1 &= 2 + 3s + 0t \\ x_2 &= 6 - 2s + 0t \\ x_3 &= 0 + 0s + 1t \end{aligned}$$

Exercises

Part A

1. Consider the points A, B, and C whose coordinates are $(1, 2, 3)$, $(2, 5, 7)$, $(3, 1, 1)$.
   - **(a)** Show that $\{A, B, C\}$ is noncollinear.
   - **(b)** What are the components of $B - A$? Of $C - A$? Of $(B - A)s + (C - A)t$?
   - **(c)** What are the coordinates of $A + (B - A)s + (C - A)t$?
   - **(d)** For each point X, $X \in \overleftrightarrow{ABC}$ if and only if, for some s and t, the coordinates (x_1, x_2, x_3) of X are given by the equations:

$$\begin{cases} x_1 = \underline{\hspace{3cm}} \\ x_2 = \underline{\hspace{3cm}} \\ x_3 = \underline{\hspace{3cm}} \end{cases}$$

[Complete.]

   - **(e)** Give, as linear combinations of the terms of $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$, two translations which are terms of a basis for $[\overleftrightarrow{ABC}]$.
2. Write parametric coordinate equations for the plane containing the points whose coordinates are:
   - **(a)** $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$
   - **(b)** $(1, 1, 1)$, $(0, 0, 0)$, and $(0, 0, -1)$
   - **(c)** $(3, 4, 0)$, $(2, 5, 0)$, and $(1, -5, 0)$
3. Write parametric coordinate equations for the plane $\overleftrightarrow{A[\vec{b}, \vec{c}]}$ where [in each of the following parts] the first triple gives the coordinates of A and the second and third triples give the components of $\vec{b}$ and $\vec{c}$, respectively.
   - **(a)** $(1, 0, 0)$, $(-1, 1, 0)$, $(-1, 0, 1)$
   - **(b)** $(1, 1, 1)$, $(-1, -1, -1)$, $(-1, -1, -2)$
   - **(c)** $(3, 4, 0)$, $(-1, 1, 0)$, $(-2, -9, 0)$

   [Compare your answers in Exercises 2 and 3.]
4. Suppose you had a problem to solve which involved only points in a given plane $\overleftrightarrow{ABC}$. If you wished to use coordinates and hoped to make your algebraic work as simple as possible, what would be a good choice for O and for $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$?
5. Use coordinates to show that, in $\triangle ABC$, the medians from A and from C intersect at a point which is $\frac{2}{3}$ of the way along either of them from vertex to midpoint of opposite side.

Part B

When studying coordinate equations of lines we found that it was possible to "eliminate" the parameter from the parametric equations of a line so as to obtain the two-point equations of the line. Similarly, it is possible to eliminate both parameters from the parametric coordinate equations of a plane and, thus, describe the coordinates of points belonging to the plane without using parameters. We start with a rather simple example.

At least the first two exercises of Part A should be used to illustrate the discussion on page 430. The algebra involved in Part B can become very involved for some students. These exercises are best treated under teacher direction.

Answers for Part A

1. (a) The components of $B - A$ and $C - A$ are $(1, 3, 4)$ and $(2, -1, -2)$, respectively. Since neither vector is $\vec{0}$, linear dependence would require that each component of one be the same multiple of the corresponding component of the other. Since this is not the case, $(B - A, C - A)$ is linearly independent and, so, $\{A, B, C\}$ is noncollinear.
   - (b) $(1, 3, 4)$; $(2, -1, -2)$; $(s + 2t, 3s - t, 4s - 2t)$
   - (c) $(1 + s + 2t, 2 + 3s - t, 3 + 4s - 2t)$
   - (d) $1 + s + 2t$; $2 + 3s - t$; $3 + 4s - 2t$
   - (e) [The translations $B - A$ and $C - A$ are the most natural choice.] $\vec{u}_1 + \vec{u}_2 3 + \vec{u}_3 4$ and $\vec{u}_1 2 - \vec{u}_2 - \vec{u}_3 2$
2. (a) / (b) / (c)

$$\begin{array}{lll} \text{(a)}\ x_1 = 1 - s - t & \text{(b)}\ x_1 = 1 - s - t & \text{(c)}\ x_1 = 3 - s - 2t \\ \quad\ \ x_2 = s & \quad\ \ x_2 = 1 - s - t & \quad\ \ x_2 = 4 + s - 9t \\ \quad\ \ x_3 = t & \quad\ \ x_3 = 1 - s - 2t & \quad\ \ x_3 = 0 \end{array}$$

3. [Same answers as for Exercise 2.]
4. $O = A$, $\vec{u}_1 = B - A$, $\vec{u}_2 = C - A$, $\vec{u}_3 \notin [\vec{u}_1, \vec{u}_2]$ [The choice of $\vec{u}_3$ is immaterial, as long as $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is linearly independent. It would never be referred to in solving such a problem, so there would be no need to make a specific choice.]
5. To make use of symmetry, it is helpful to modify the procedure described in Exercise 4 and choose $O = B$, $\vec{u}_1 = A - B$, and $\vec{u}_2 = C - B$. The coordinates of A, B, and C are, then $(1, 0, 0)$, $(0, 0, 0)$, and $(0, 1, 0)$. Those of the midpoints of $\overline{BC}$ and $\overline{BA}$ are $(0, \frac{1}{2}, 0)$ and $(\frac{1}{2}, 0, 0)$. The coordinates of the point 2/3 of the way from A to the midpoint of $\overline{BC}$ are $(1 + \frac{2}{3} \cdot -1,\ 0 + \frac{2}{3} \cdot \frac{1}{2},\ 0 + \frac{2}{3} \cdot 0)$. Those of the point 2/3 of the way from C to the midpoint of $\overline{BA}$ are $(0 + \frac{2}{3} \cdot \frac{1}{2},\ 1 + \frac{2}{3} \cdot -1,\ 0 + \frac{2}{3} \cdot 0)$. So, the points are the same.

1. Consider points A, B, and C whose coordinates are $(1, -1, 1)$, $(3, -1, 2)$, and $(2, 2, 1)$.
   (a) Write parametric coordinate equations for $\overline{ABC}$.
   (b) Solve two of these equations for 's' and 't'. [*Hint*: Choose the two equations for which this is easiest to do.]
   (c) Use the equations you obtained in part (b) to eliminate 's' and 't' from the third of the parametric equations for $\overline{ABC}$.
   (d) Why must the coordinates (x_1, x_2, x_3) of any point of $\overline{ABC}$ satisfy the equation you obtained in part (c)?
   (e) Why must any point whose coordinates satisfy the equation of part (c) be a point in $\overline{ABC}$? [*Hint*: Given coordinates (x_1, x_2, x_3) which satisfy the equation of part (c), how can you compute values of 's' and 't' from which you can get the given coordinates by using the parametric equations from part (a)?]

*

The only thing special about Exercise 1 is that it is particularly easy to solve two of the parametric equations of $\overline{ABC}$ for 's' and 't'. [Why?] The parametric equations of any plane are of the form:

$$\begin{cases} x_1 = a_1 + b_1 s + c_1 t \\ x_2 = a_2 + b_2 s + c_2 t \\ x_3 = a_3 + b_3 s + c_3 t \end{cases}$$

[What is (a_1, a_2, a_3)? What are (b_1, b_2, b_3) and (c_1, c_2, c_3)?] These equations can be rewritten in the form:

$$\begin{cases} b_1 s + c_1 t = x_1 - a_1 \\ b_2 s + c_2 t = x_2 - a_2 \\ b_3 s + c_3 t = s_3 - a_3 \end{cases}$$

and, whenever it is possible to solve some two of these equations for 's' and 't' then substitution in the third equation yields a single equation which is satisfied by the coordinates of just those points which belong to the given plane. [Whether or not it is possible to solve some two of the equations for 's' and 't' depends on what the triples (b_1, b_2, b_3) and (c_1, c_2, c_3) happen to be. Make a conjecture as to the relation between the vectors having these components in case *no* two of the equations can be solved for 's' and 't'.]

*

2. For each of the planes $\overline{A[\vec{b}, \vec{c}]}$ of Exercise 3, Part A, find a single equation representing the plane.

Part C

1. What points have coordinates which satisfy the equation:

$$0x_1 + 0x_2 + 0x_3 = d$$

if $d \neq 0$? If $d = 0$?

Answers for Part B

1. (a) $x_1 = 1 + 2s + t$
   $x_2 = -1 + 0s + 3t$
   $x_3 = 1 + s + 0t$
   (b) From the second and third equations, $s = x_3 - 1$ and $t = (x_2 + 1)/3$.
   (c) Substituting from the results of part (b) into the first equation and simplifying:

$$3x_1 - x_2 - 6x_3 = -2$$

   (d) For any point of $\overline{ABC}$, its coordinates must be given by (a) for a proper choice of values of 's' and 't'. These values can be computed in terms of the second and third coordinates of the point by using the equations in part (b). The equation of part (c) is just another way of saying that these values of 's' and 't' and the first coordinate of the point satisfy the first of the equations in (a).
   (e) If x_1, x_2, and x_3 satisfy the equation in part (c) and s and t are determined from x_2 and x_3 by the equations of part (b) then x_2, x_3, s, and t will, automatically, satisfy the second and third equations of part (a) and, because x_1, x_2, x_3 satisfy the equation of part (c), x_1, s, and t will satisfy the first equation of part (a).

[A correct conjecture is that no two of the equations can be solved for 's' and 't' if and only if the vectors with components (a_1, a_2, a_3) and (b_1, b_2, b_3) are linearly dependent. It is just in this case that $\overline{ABC}$ is not a plane.]

2. (a) $x_1 + x_2 + x_3 = 1$ (b) $x_1 - x_2 = 0$ (c) $x_3 = 0$

Answers for Part C

1. No points.; All points of $\mathcal{E}$.

2. Consider the equation: $2x_1 - 3x_2 - x_3 = 5$.
   (a) Show that the parametric equations

$$\begin{cases} x_1 = \frac{5}{2} + \frac{3}{2}s + \frac{1}{2}t \\ x_2 = \quad s \\ x_3 = \quad t \end{cases}$$

   represent a plane whose points are just those which satisfy the given equation. [*Hint*: Does it matter what letters we use as parameters? Could we use 'x_2' in place of 's'?]
   (b) Give two other sets of parametric equations, each of which describes the same plane.

*3. Show that the equation:

$$a_1x_1 + a_2x_2 + a_3x_3 = d$$

is an equation of a plane if and only if not all of a_1, a_2, and a_3 are zero. [*Hint*: For the only-if part consider the case where all of a_1, a_2, and a_3 are zero. From this, you should be able to show that the equation is not one of a plane. For the if part consider the case where not all of a_1, a_2, and a_3 are zero and make use of the notions discussed in Exercise 2 and Part B.]

10.08 Determinants

In Chapter 4, on (pages 173–175), we discussed methods for solving systems of linear equations. Systems such as:

(1)
$$\begin{aligned} 8x + 2y &= 17 \\ 5x + 2y &= 11 \end{aligned}$$

are said to be *independent* because the equations of the system have just one common solution. (What is the solution of (1)?) Systems such as:

(2)
$$\begin{aligned} 3x - 4y &= 6 \\ 6x - 8y &= 12 \end{aligned}$$

are said to be *dependent* because the equations of the system have the same solutions, that is, are equivalent. And, systems such as:

(3)
$$\begin{aligned} 6x - 8y &= 12 \\ 6x - 8y &= 13 \end{aligned}$$

are said to be *inconsistent* because the equations of the system have no common solution.

Answers for Part C

2. (a) Using 'x_2' in place of 's' and 'x_3' in place of 't' as parameters, the parametric equations become:

$$\begin{cases} x_1 = \frac{5}{2} + \frac{3}{2}x_2 + \frac{1}{2}x_3 \\ x_2 = x_2 \\ x_3 = x_3 \end{cases}$$

This last system is equivalent to the single equation '$2x_1 - 3x_2 - x_3 = 5$'. The solutions of the parametric equations are the solutions of the given equation, and the parametric equations represent the plane represented by the given equation.

(b) An example of each of two typical kinds of other sets of parametric equations which students might offer:

$$\begin{cases} x_1 = \quad s \\ x_2 = -\frac{5}{3} + \frac{2}{3}s - \frac{1}{3}t \\ x_3 = \quad t \end{cases} \text{ and: } \begin{cases} x_1 = \frac{10}{4} + \frac{6}{4}s + \frac{2}{4}t \\ x_2 = \quad s \\ x_3 = \quad t \end{cases}$$

However, if the student chose to locate three points of the plane described by the given equation such as, for example, the points whose coordinates are (1, −1, 0), (0, −1, −2), and (2, 0, −1), that student might offer the set of parametric equations:

$$\begin{cases} x_1 = 1 - s + t \\ x_2 = -1 + t \\ x_3 = -2s - t \end{cases}$$

In such cases it may be well to ask the student to demonstrate that his parametric equations do represent the same plane as that of the given equation. As an exercise, the students could be asked to use the parametric equations to write a single equation which describes the same set of points. They may do this, of course, by "eliminating the parameters" from the given parametric equations.

3. Suppose, first, that the given equation is an equation of a plane. Now, if all of a_1, a_2, and a_3 are zero, then the given equation is satisfied by the coordinates of each point of $\mathcal{E}$ provided $d = 0$ and no point of $\mathcal{E}$ provided $d \neq 0$. In either case, the given equation is not one of a plane, a contradiction. So, not all of a_1, a_2, and a_3 are zero.

Suppose, next, that not all of a_1, a_2, and a_3 are zero. In case $a_1 \neq 0$, the system of parametric equations '$x_1 = d/a_1 - s/a_1 - t/a_1$', '$x_2 = s$', '$x_3 = t$' is equivalent to the single equation '$a_1x_1 + a_2x_2 + a_3x_3 = d$'. Since the system of parametric equations represent a plane, so does the single equation. A similar result occurs in case $a_2 \neq 0$ and in case $a_3 \neq 0$. Thus, in any case, '$a_1x_1 + a_2x_2 + a_3x_3 = d$' is an equation of a plane.

If you did not review solutions of simultaneous linear equations while studying Chapter 4, we suggest that, before beginning this section, you review these topics with your students. Pages 173 - 175 should help you with this task.

The solution to system (1) is (1, 3).

As mentioned earlier on page 174, there is an easy way to tell whether a system is independent. The system:

$$(4)\qquad \begin{aligned} a_1x + b_1y &= c_1\\ a_2x + b_2y &= c_2\end{aligned}$$

is independent if and only if (the determinant)

$$(5)\qquad \begin{vmatrix} a_1 & b_1\\ a_2 & b_2\end{vmatrix} \neq 0.$$

[If this determinant is 0 then either the system is dependent (the equations are equivalent) or the system is inconsistent (has no solution).]

The operation indicated by the vertical bars is called the determinant operation. For the pairs (a_1, b_1) and (a_2, b_2) the value of the determinant operation is the number $a_1b_2 - a_2b_1$. That is,

$$\begin{vmatrix} a_1 & b_1\\ a_2 & b_2\end{vmatrix} = a_1b_2 - a_2b_1.$$

The sequences (a_1, b_1) and (a_2, b_2) are called *the rows* of the determinant; (a_1, a_2) and (b_1, b_2) are called *the columns* of the determinant.

When the value of the determinant for the system (4) is not 0, the common solution of the equations in (4) is given by the formulas:

$$x = \frac{\begin{vmatrix} c_1 & b_1\\ c_2 & b_2\end{vmatrix}}{\begin{vmatrix} a_1 & b_1\\ a_2 & b_2\end{vmatrix}},\quad y = \frac{\begin{vmatrix} a_1 & c_1\\ a_2 & c_2\end{vmatrix}}{\begin{vmatrix} a_1 & b_1\\ a_2 & b_2\end{vmatrix}}$$

Determinants have several simple properties. Some of these you should discover in doing the exercises of Part A.

Exercises

Part A

Simplify [or: evaluate] each of the given determinants.

1. (a) $\begin{vmatrix} 2 & 3\\ 1 & 4\end{vmatrix}$ **(b)** $\begin{vmatrix} 2 & 1\\ 3 & 4\end{vmatrix}$

2. (a) $\begin{vmatrix} 3 & -5\\ 7 & 2\end{vmatrix}$ **(b)** $\begin{vmatrix} 7 & 2\\ 3 & -5\end{vmatrix}$ **(c)** $\begin{vmatrix} 2 & 7\\ -5 & 3\end{vmatrix}$

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Answers for Part A

1. (a) 5 [2·4 - 3·1] (b) 5 [2·4 - 1·3]

2. (a) 41 [3·2 - −5·7] (b) −41 [7·−5 - 2·3] (c) 41 [2·3 - 7·−5]

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3. (a) −2 [3·4 - 7·2] (b) 4 [−6·4 - −14·2] (c) 4 [3·−8 - 7·−4]

4. (a) 34 [6·8 - 2·7] (b) 34 [(4·8 - 4·7) + (2·8 - −2·7)]

5. (a) 2 [1·8 - 2·3] (b) 2 [1·(8 - 2·3) - 2(3 - 1·3)]

*

Note that "determinanting" [as far as we introduce it here] is a function from pairs of pairs of real numbers to real numbers. [Compare with squaring as a function from real numbers to real numbers.] So, the determinant of a pair of pairs is a real number — just as the square of a real number is a real number. The word 'determinant' is sometimes used to refer to the symbol formed of a pair of vertical lines enclosing a square array of numerals. We shall occasionally use the word in this sense. We do so, for example, when we speak of the value of a determinant.

Third order determinants are — better, the third order determinanting function is — introduced on page 462.

3. (a) $\begin{vmatrix} 3 & 7 \\ 2 & 4 \end{vmatrix}$ **(b)** $\begin{vmatrix} -6 & -14 \\ 2 & 4 \end{vmatrix}$ **(c)** $\begin{vmatrix} 3 & 7 \\ -4 & -8 \end{vmatrix}$

4. (a) $\begin{vmatrix} 6 & 2 \\ 7 & 8 \end{vmatrix}$ **(b)** $\begin{vmatrix} 4 & 4 \\ 7 & 8 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 7 & 8 \end{vmatrix}$

5. (a) $\begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix}$ **(b)** $\begin{vmatrix} 1 & 2 \\ 3 - 1 \cdot 3 & 8 - 2 \cdot 3 \end{vmatrix}$

Part B

Each of the five exercises in Part A illustrates one or more rules for computing with determinants. For example, Exercise 1 suggests that two determinants have the same value if the first and second columns of one are, respectively, the first and second rows of the other. Briefly, "interchanging the rows with the columns of a determinant does not change its value." This, as well as the rules illustrated in the other exercises, is easily derived from the definition.

Prove each of the following theorems.

1. $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

2. (a) $\begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ **(b)** $\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

[*Note*: It is easy to prove both part (a) and part (b) merely by using the definition. It is still worthwhile, however, to notice that part (b) can be derived from part (a) by using Exercise 1:

$$\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Justify each of the three steps in this argument.]

3. (a) $\begin{vmatrix} a_1c & a_2c \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}c$ **(b)** $\begin{vmatrix} a_1c & a_2 \\ b_1c & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}c$

[Prove part (b) by using Exercise 1 and part (a). Prove part (c), below, by using Exercise 2(b) and part (a).]

(c) $\begin{vmatrix} a_1 & a_2 \\ b_1c & b_2c \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}c$ **(d)** $\begin{vmatrix} a_1 & a_2c \\ b_1 & b_2c \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}c$

4. (a) $\begin{vmatrix} a_1 + c_1 & a_2 + c_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} c_1 & c_2 \\ b_1 & b_2 \end{vmatrix}$

(b) State three similar results which you can derive from part (a) by using either Exercise 1 or Exercise 2.

Answers for Part B

1. $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2 = a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

2. (a) $\begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} = a_2b_1 - a_1b_2 = -(a_1b_2 - a_2b_1) = -\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

(b) The proof is given in the hint. The justifications for the steps are, in turn, Exercise 1, Exercise 2(a), and Exercise 1.

3. (a) $\begin{vmatrix} a_1c & a_2c \\ b_1 & b_2 \end{vmatrix} = (a_1c)b_2 - (a_2c)b_1 = (a_1b_2 - a_2b_1)c = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}c$

(b) $\begin{vmatrix} a_1c & a_2 \\ b_1c & b_2 \end{vmatrix} = \begin{vmatrix} a_1c & b_1c \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}c = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}c$

(c) $\begin{vmatrix} a_1 & a_2 \\ b_1c & b_2c \end{vmatrix} = -\begin{vmatrix} b_1c & b_2c \\ a_1 & a_2 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix}c = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}c$

(d) $\begin{vmatrix} a_1 & a_2c \\ b_1 & b_2c \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2c & b_2c \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}c = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}c$

4. (a)

$$\begin{vmatrix} a_1 + c_1 & a_2 + c_2 \\ b_1 & b_2 \end{vmatrix} = (a_1 + c_1)b_2 - (a_2 + c_2)b_1$$

$$= (a_1b_2 - a_2b_1) + (c_1b_2 - c_2b_1)$$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} c_1 & c_2 \\ b_1 & b_2 \end{vmatrix}$$

(b) $\begin{vmatrix} a_1 & a_2 \\ b_1 + c_1 & b_2 + c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$

$\begin{vmatrix} a_1 & a_2 + c_1 \\ b_1 & b_2 + c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 \\ b_1 & c_2 \end{vmatrix}$

$\begin{vmatrix} a_1 + c_1 & a_2 \\ b_1 + c_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} c_1 & a_2 \\ c_2 & b_2 \end{vmatrix}$

5. (a) $\begin{vmatrix} a_1 & a_2 \\ b_1 + a_1c & b_2 + a_2c \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

[*Hint*: Use one of your results from Exercise 4, then use Exercise 3(c). Notice that Exercise 2(b) tells you that

$$\begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} = 0. \text{ Why?]}$$

(b) State three similar results which you can derive from part (a) by using Exercise 1 or Exercise 2.

*

Many of the applications we shall make of determinants depend on the following fundamental theorem concerning a system of two equations in 'x' and 'y':

> Theorem A
>
> The system of equations:
>
> $$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$
>
> has a unique solution if and only if
>
> $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$
>
> and, in this case, the given system of equations is equivalent to:
>
> $$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},\ y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

[See Part B on pages 174–175] This is easy to remember if you compare the common denominator of the two fractions with the given system of equations and then note how the numerators of the fractions compare with this denominator. It may also be helpful to compare this theorem with a simpler one:

> The equation '$ax = c$' has a unique solution if and only if $a \neq 0$; and, in this case, the given equation is equivalent to '$x = c/a$'.

Answers for Part B [cont.]

5. (a) $\begin{vmatrix} a_1 & a_2 \\ b_1 + a_1c & b_2 + a_2c \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ a_1c & a_2c \end{vmatrix}$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} c = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

(b) $\begin{vmatrix} a_1 + b_1c & a_2 + b_2c \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$; $\begin{vmatrix} a_1 & a_2 + a_1c \\ b_1 & b_2 + b_1c \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

$$\begin{vmatrix} a_1 + a_2c & a_2 \\ b_1 + b_2c & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

There is a frequently useful corollary of Theorem A which deals with solutions of a system of equations like:

$$(*) \qquad \begin{cases} a_1x + b_1y = 0 \\ a_2x + b_2y = 0 \end{cases}$$

Such a system always has at least one solution. [Explain.] But, there are many situations in which we wish to know whether there are numbers x and y which are *not* both 0 and which satisfy the given equations (*). In other words, we wish to know whether (*) has a solution other than (0,0) or, as it is sometimes put, whether (*) has a *nontrivial* solution. Now, since (0, 0) is a solution of (*) it follows that (*) has a nontrivial solution if and only if (*) does not have a unique solution. So, as a corollary to the first part of Theorem A we have:

> **Corollary**
>
> The system of equations:
>
> $$\begin{cases} a_1x + b_1y = 0 \\ a_2x + b_2y = 0 \end{cases}$$
>
> has a nontrivial solution if and only if
>
> $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

Part C

1. For each of the following systems of equations, use Theorem A to determine whether the system has a unique solution and, in case it does, to find this solution.

(a) $\begin{cases} 4x + 2y = 7 \\ 5x + 3y = 8 \end{cases}$ **(b)** $\begin{cases} 4x + 2y = 9 \\ 6x + 3y = 7 \end{cases}$

(c) $\begin{cases} 3x - 7y = 12 \\ 2x + 5y = 8 \end{cases}$ **(d)** $\begin{cases} -5x + 6y = 11 \\ 4x - 5y = 0 \end{cases}$

2. Which of the following systems have nontrivial solutions?

(a) $\begin{cases} 15x + 21y = 0 \\ 10x + 14y = 0 \end{cases}$ **(b)** $\begin{cases} 15x - 26y = 0 \\ 4x - 7y = 0 \end{cases}$

3. Explain why the following is equivalent to the corollary to Theorem A:

$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0$ if and only if there are numbers x and y, not both 0, such that $(a_1x + b_1y = 0$ and $a_2x + b_2y = 0)$.

[*Hint*: Recall Exercise 1 of Part B.]

System (*) has the solution (0, 0). So, it has at least one solution.

Answers for Part C

1. (a) Since $\begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix} = 2 \neq 0$, the system has a unique solution. It is $(5/2, -3/2)$. [By Theorem A, the solution is found by solving:

$$x = \frac{\begin{vmatrix} 7 & 2 \\ 8 & 3 \end{vmatrix}}{2} \text{ and } y = \frac{\begin{vmatrix} 4 & 7 \\ 5 & 8 \end{vmatrix}}{2} \quad]$$

(b) No unique solution.

(c) Unique solution. It is (4, 0).

(d) Unique solution. It is (−55, −44).

2. (a) Has a nontrivial solution, for $\begin{vmatrix} 15 & 21 \\ 10 & 14 \end{vmatrix} = 0$.

(b) The only solution is (0, 0). That is, there are no nontrivial solutions.

3. To say that there are numbers x and y, not both zero, such that $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$ is to say that the system:

$$\begin{cases} a_1x + b_1y = 0 \\ a_2x + b_2y = 0 \end{cases}$$

has a solution other than (0, 0) — that is, has a nontrivial solution. Also, $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$.

10.09 Determinants and Equations of Planes

Suppose that $(\vec{c}, \vec{d})$ is linearly independent and that, for certain numbers a_1, a_2, b_1, and b_2,

$$(*) \qquad \vec{a} = \vec{c}a_1 + \vec{d}a_2 \quad \text{and} \quad \vec{b} = \vec{c}b_1 + \vec{d}b_2.$$

What we wish to know is whether or not $(\vec{a}, \vec{b})$ is linearly independent. In other words, what we wish to know is whether or not the equation:

$$\vec{a}a + \vec{b}b = \vec{0}$$

has any solution other than (0, 0). [Explain.] In view of (*), this amounts to asking whether or not the equation:

$$\vec{c}(a_1a + b_1b) + \vec{d}(a_2a + b_2b) = \vec{0}$$

has any solution other than (0, 0). [Explain.] Since $(\vec{c}, \vec{d})$ is linearly independent, this amounts to asking whether or not the system:

$$\begin{cases} a_1a + b_1b = 0 \\ a_2a + b_2b = 0 \end{cases}$$

has a solution other than (0, 0). Since (0, 0) is [obviously] a solution of this system, what we wish to know is merely whether or not this system has a unique solution. What is the answer?

Exercises

Part A

Assuming that $(\vec{c}, \vec{d})$ is linearly independent, use determinants to determine which of the given pairs are linearly independent.

Sample. $(\vec{c}1 + \vec{d}2, \vec{c}3 + \vec{d}4)$

Solution. $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$

Answer. ? ['linearly independent' or 'linearly dependent']

1. $(\vec{c}5 + \vec{d}, \vec{c}5 + \vec{d} \cdot -1)$
2. $(\vec{c}3, \vec{c}3 - \vec{d}4)$
3. $(\vec{c}2 + \vec{d}6, \vec{c} + \vec{d}3)$
4. $(\vec{d}, \vec{c})$ [i.e.: $(\vec{c}0 + \vec{d}1, \vec{c}1 + \vec{d}0)$]
5. $(\vec{d}9, \vec{c}4 - \vec{d})$
6. $(\vec{c} + \vec{d}, \vec{c} - \vec{d})$
7. $(\vec{c}5 + \vec{d}7, \vec{c} \cdot -5 - \vec{d}7)$
8. $(-\vec{c}2 - \vec{d}2, -\vec{c}2 + \vec{d}2)$
9. $(\vec{c}9 + \vec{d}11, \vec{c}9 + \vec{d}11)$
10. $(\vec{0}, \vec{c}61 + \vec{d} \cdot -125)$

It may be helpful for you to review the commentary for pages 373 - 374. The exercises in Part A of this section are the same as those in the Background Topic on those pages.

Here are some suggestions for the use of Parts A - F, pages 438 - 447. Following an illustrated discussion of the text preceding Part A, Parts A and B make a reasonable homework assignment. We recommend that you use Part C in class to illustrate the application of Theorem 10-14. Following this, Part D can be used for homework but we recommend that you permit your students to team up for this. The algebra needed in these exercises is a little involved, and there is no point to doing subsequent exercises based on the results of an algebraic error. Part E is another class exercise to illustrate applications of Theorem 10-15. Finally, Part F can be used as an individual homework assignment.

Answers for Part A

1. linearly independent

$$\begin{vmatrix} 5 & 5 \\ 1 & -1 \end{vmatrix} = -5 - 5 = -10 \neq 0$$

2. linearly independent

$$\begin{vmatrix} 3 & 3 \\ 0 & -4 \end{vmatrix} = -12 \neq 0$$

3. linearly dependent

$$\begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix} = 6 - 6 = 0$$

4. linearly independent

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1 \neq 0$$

5. linearly independent

$$\begin{vmatrix} 0 & 4 \\ 9 & -1 \end{vmatrix} = 0 - 36 = -36 \neq 0$$

6. linearly independent

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2 \neq 0$$

7. linearly dependent

$$\begin{vmatrix} 5 & -5 \\ 7 & -7 \end{vmatrix} = -35 - -35 = 0$$

8. linearly independent

$$\begin{vmatrix} -2 & -2 \\ -2 & 2 \end{vmatrix} = -4 - 4 = -8 \neq 0$$

9. linearly dependent

$$\begin{vmatrix} 9 & 9 \\ 11 & 11 \end{vmatrix} = 99 - 99 = 0$$

10. linearly dependent

$$\begin{vmatrix} 0 & 61 \\ 0 & -125 \end{vmatrix} = 0 - 0 = 0$$

Part B

In Exercise 3 of Part C on page 437, you showed that one corollary to Theorem A is:

$$(*)\qquad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0 \text{ if and only if there are numbers } x \text{ and } y, \text{ not both } 0,$$

such that $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$

1. Use (*) to prove the following

> **Theorem 10-13** For $(\vec{u}_1, \vec{u}_2)$ linearly independent, $(\vec{u}_1a_1 + \vec{u}_2a_2, \vec{u}_1b_1 + \vec{u}_2b_2)$ is linearly dependent $\Longleftrightarrow$
>
> $$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0.$$

[*Hint*: Complete: $(\vec{u}_1a_1 + \vec{u}_2a_2)x + (\vec{u}_1b_1 + \vec{u}_2b_2)y = \vec{u}_1(\_\_\_\_\_\_) + \vec{u}_2(\_\_\_\_\_\_)$]

2. Use Exercise 1 and an exercise from Part B on page 435 to show that $(\vec{a}, \vec{b} + \vec{a}c)$ is linearly dependent if and only if $(\vec{a}, \vec{b})$ is linearly dependent. [*Hint* For any $\vec{a}$ and $\vec{b}$, there are vectors—say $\vec{u}_1$ and $\vec{u}_2$—such that $(\vec{u}_1, \vec{u}_2)$ is linearly independent and $\{\vec{a}, \vec{b}\} \subseteq [\vec{u}_1, \vec{u}_2]$.]

3. (a) One solution (x, y) of '$3x - 2y = 0$' is $(2, 3)$. What are some other solutions?

   (b) Use Exercise 1 to prove:
   For $(a_1, a_2) \neq (0, 0)$,

   $$a_1x - a_2y = 0 \Longleftrightarrow \exists_t\, (x = a_2t \text{ and } y = a_1t)$$

   [*Hint*: Choose any $\vec{u}_1$ and $\vec{u}_2$ such that $(\vec{u}_1, \vec{u}_2)$ is linearly independent and let $\vec{a} = \vec{u}_1a_1 + \vec{u}_2a_2$ and $\vec{b} = \vec{u}_1y + \vec{u}_2x$. Recall that, for $\vec{a} \neq \vec{0}$, $(\vec{a}, \vec{b})$ is linearly dependent if and only if $\vec{b} \in [\vec{a}]$.]

   (c) Describe *all* solutions of the equation '$5x + 3y = 0$'.

4. Suppose that $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is linearly independent and that

   $$\vec{a} = \vec{u}_1a_1 + \vec{u}_2a_2 + \vec{u}_3a_3 \text{ and } \vec{b} = \vec{u}_1b_1 + \vec{u}_2b_2 + \vec{u}_3b_3.$$

   (a) Show that $(\vec{a}, \vec{b})$ is linearly dependent if and only if the system:

   $$(*)\qquad \begin{cases} a_1x + b_1y = 0 \\ a_2x + b_2y = 0 \\ a_3x + b_3y = 0 \end{cases}$$

   has a nontrivial solution.

Answers for Part B

1. Suppose that $(\vec{u}_1, \vec{u}_2)$ is linearly independent. Now, the sequence $(\vec{u}_1a_1 + \vec{u}_2a_2, \vec{u}_1b_1 + \vec{u}_2b_2)$ is linearly dependent if and only if there are numbers x and y, not both 0, such that $(\vec{u}_1a_1 + \vec{u}_2a_2)x + (\vec{u}_1b_1 + \vec{u}_2b_2)y = \vec{0}$ or, equivalently, such that $\vec{u}_1(a_1x + b_1y) + \vec{u}_2(a_2x + b_2y) = \vec{0}$. Now, since $(\vec{u}_1, \vec{u}_2)$ is linearly independent, the latter is the case if and only if $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$, and these last equations have a nontrivial solution if and only if

   $$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0.$$

2. Let $(\vec{u}_1, \vec{u}_2)$ be linearly independent and such that $\{\vec{a}, \vec{b}\} \subseteq [\vec{u}_1, \vec{u}_2]$. Then, for some numbers—say, a_1, a_2, b_1, and b_2—$\vec{a} = \vec{u}_1a_1 + \vec{u}_2a_2$ and $\vec{b} = \vec{u}_1b_1 + \vec{u}_2b_2$. So, for any c, $\vec{b} + \vec{a}c = \vec{u}_1(b_1 + a_1c) + \vec{u}_2(b_2 + a_2c)$. It follows by Theorem 13-1 that $(\vec{a}, \vec{b})$ is linearly dependent if and only if

   $$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0$$

   and $(\vec{a}, \vec{b} + \vec{a}c)$ is linearly dependent if and only if

   $$\begin{vmatrix} a_1 & a_2 \\ b_1 + a_1c & b_2 + a_2c \end{vmatrix} = 0.$$

   But, by Exercise 5(a) on page 436 the two determinants have the same value. Hence, one determinant is 0 if and only if the other is 0 and, so, one sequence is linearly dependent if and only if the other is linearly dependent.

3. (a) All solutions of the given equation are of the form '(2a, 3a)' for some a, That is, (p, q) is a solution of '$3x - 2y = 0$' if and only if $\exists_t\ (p, q) = (2t, 3t)$.

   (b) Choose $\vec{u}_1$ and $\vec{u}_2$ so that $(\vec{u}_1, \vec{u}_2)$ is linearly independent and let $\vec{a} = \vec{u}_1a_1 + \vec{u}_2a_2$ and $\vec{b} = \vec{u}_1y + \vec{u}_2x$. By Theorem 10-13, $a_1x - a_2y = 0$ if and only if $(\vec{a}, \vec{b})$ is linearly dependent. For $(a_1, a_2) \neq (0, 0)$, $\vec{a} \neq \vec{0}$ and, so, $(\vec{a}, \vec{b})$ is linearly dependent if and only if there exists a real number t such that $\vec{b} = \vec{a}t$ — that is, [since $(\vec{u}_1, \vec{u}_2)$ is linearly independent] such that $y = a_1t$ and $x = a_2t$. So, for $(a_1, a_2) \neq (0, 0)$, $a_1x - a_2y = 0$ if and only if $\exists_t\ (x = a_2t \text{ and } y = a_1t)$. [Note that we have used a result concerning vector spaces to prove one which concerns only real numbers.]

   (c) All (p, q) such that $\exists_t\ (p = -3t \text{ and } q = 5t)$.

(b) Show that if $(\vec{a}, \vec{b})$ is linearly dependent then each of the determinants

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, \text{ and } \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}$$

is 0.

(c) Show that if each of the determinants in part (b) is 0 then $(b_1, -a_1)$, $(b_2, -a_2)$, and $(b_3, -a_3)$ are solutions of (*). [*Hint*: $(b_1, -a_1)$ is obviously a solution of the first of the three equations in (*). Show that it is also a solution of the second equation if and only if a certain one of the determinants in part (b) is 0, and that it is a solution of the third equation if and only if another of these determinants is 0. Give similar arguments for $(b_2, -a_2)$ and $(b_3, -a_3)$.]

(d) Show that if the three determinants in part (b) are 0 then (*) has at least one nontrivial solution. [*Hint*: Assuming that the determinants are 0, you know three ways of finding a solution of (*). All that remains is to show that (*) still has a nontrivial solution in case all three of these ways give the trivial solution (0, 0).]

*

From Exercise 4 and Exercise 1 of Part B [page 435] we obtain a very useful theorem:

> **Theorem 10-14** For $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ linearly independent, $(\vec{u}_1 a_1 + \vec{u}_2 a_2 + \vec{u}_3 a_3, \vec{u}_1 b_1 + \vec{u}_2 b_2 + \vec{u}_3 b_3)$ is linearly dependent
>
> $$\Longleftrightarrow \quad \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right) = (0, 0, 0).$$

Explain. Compare this theorem with Theorem 10-13. [The advantage of writing the determinants in this order, instead of in the order given in Exercise 4(b), will appear later.]

Determinants like those in Theorem 10-14 appear in two ways when we introduce a coordinate system and use equations to represent planes. As you know, given a coordinate system on an origin $O \in \mathscr{E}$ and a basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ for $\mathscr{T}$, a plane π can be described by giving parametric equations:

$$(1) \qquad \begin{cases} x_1 = a_1 + p_1 s + q_1 t \\ x_2 = a_2 + p_2 s + q_2 t \\ x_3 = a_3 + p_3 s + q_3 t \end{cases}$$

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4. (a) $(\vec{a}, \vec{b})$ is linearly dependent if and only if there are numbers x and y, not both 0, such that $\vec{a}x + \vec{b}y = \vec{0}$. This is so if and only if $\vec{u}_1(a_1x + b_1y) + \vec{u}_2(a_2x + b_2y) + \vec{u}_3(a_3x + b_3y) = \vec{0}$. Since $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is linearly independent, we have that $(\vec{a}, \vec{b})$ is linearly dependent if and only if there are numbers x and y, not both 0, such that $a_1x + b_1y = 0$, $a_2x + b_2y = 0$, and $a_3x + b_3y = 0$. Since the latter is the case if and only if (*) has a nontrivial solution, the proof is finished.

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(b) Suppose that $(\vec{a}, \vec{b})$ is linearly dependent. Then, by (a), the system (*) has a nontrivial solution. So, each two equations in (*) has a nontrivial solution. Hence, each of the given determinants is 0.

(c) Suppose that each of the given determinants is 0. Since $(b_1, -a_1)$ is a solution of the first equation, we will be [essentially] finished if we can show that it is also a solution of each of the other equations in (*). In making use of the second equation, we see that $a_2b_1 + b_2 \cdot -a_1 = a_2b_1 - b_2a_1$ $= \begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$. Similarly, $a_3b_1 - b_3a_1 = 0$ so that $(b_1, -a_1)$ is a solution of (*). Similarly, $(b_2, -a_2)$ and $(b_3, -a_3)$ are solutions of (*).

(d) If at least one of $\vec{a}$ and $\vec{b}$ is non-$\vec{0}$ then one of the three solutions from part (c) is nontrivial. If both $\vec{a}$ and $\vec{b}$ are $\vec{0}$, then each ordered pair of reals (x, y) is a solution of (*). So, in any case, there is a nontrivial solution of (*).

*

The order in which the determinants are given in Theorem 10-14 is a desirable one since, as we shall see they are, in this order, the $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ components of a vector which is interestingly related to the sequence $(\vec{u}_1a_1 + \vec{u}_2a_2 + \vec{u}_3a_3, \vec{u}_1b_1 + \vec{u}_2b_2 + \vec{u}_3b_3)$. In particular, if $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is orthonormal then the first vector, if not $\vec{0}$, generates the orthogonal complement of the bidirection which has the given sequence as a basis. More generally, see part (b) of Theorem 10-15 on page 445.

In these equations (a_1, a_2, a_3) are the coordinates of a point $A \in \pi$, while (p_1, p_2, p_3) and (q_1, q_2, q_3) are components of vectors $\vec{p}$ and $\vec{q}$ – that is,

$$(2) \quad \vec{p} = \vec{u}_1 p_1 + \vec{u}_2 p_2 + \vec{u}_3 p_3 \text{ and } \vec{q} = \vec{u}_1 q_1 + \vec{u}_2 q_2 + \vec{u}_3 q_3$$

– such that $[\vec{p}, \vec{q}]$ is the bidirection of π. A point belongs to π if and only if its coordinates are the values of the right sides of equations (1) for some values of the parameters 's' and 't'. Of course, a system like (1) need not represent a plane. Such a system will represent a plane if and only if the vectors $\vec{p}$ and $\vec{q}$ given by (2) are linearly independent. So, one consequence of Theorem 10 – 14 is that the system (1) of parametric equations represents a plane if and only if

$$(3) \qquad \left(\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix}, \begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix}, \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} \right) \neq (0, 0, 0).$$

Note that, with respect to the same coordinate system, any plane can be described by each of many systems of parametric equations like (1). [Explain.] We shall see in the next section how to tell whether two such systems represent the same plane.

Part C

1. In the following, you are given systems of equations like (1). You are to decide which of the systems are parametric equations for a plane and which are not.

(a) $\begin{cases} x_1 = 4 + 2s - 3t \\ x_2 = 5 - 2s - 3t \\ x_3 = 7 + 4s + 6t \end{cases}$ **(b)** $\begin{cases} x_1 = 7 + 3s - 5t \\ x_2 = 6 - s \\ x_3 = 4 + t \end{cases}$

(c) $\begin{cases} x_1 = -3 + 10s - 4t \\ x_2 = -2 + 5s - 2t \\ x_3 = 1 - 5s + 2t \end{cases}$ **(d)** $\begin{cases} x_1 = 3 \\ x_2 = 5 + s \\ x_3 = 8 - t \end{cases}$

2. For each of the parametric equations in Exercise 1 which are parametric equations for a plane, give the components of two vectors – say, $\vec{p}$ and $\vec{q}$ – such that $[\vec{p}, \vec{q}]$ is the bidirection of the plane.

•3. For each part of Exercise 1 which gives parametric equations for a plane, write parametric equations for the plane which passes through the origin and is parallel to that plane.

*

From our work in Part B on pages 431 – 432, we can see that an equation like:

$$(4) \qquad x_1 m_1 + x_2 m_2 + x_3 m_3 = f$$

represents a plane if and only if

$$(5) \qquad (m_1, m_2, m_3) \neq (0, 0, 0).$$

Explanations called for in the text: Exercise 4(b), together with interchanging rows and columns of the given determinants, establishes the only-if part of Theorem 10-14. Exercises 4(c) and 4(d) in conjunction with Exercise 4(a), and interchanging rows and columns of the given determinants, establishes the if part of Theorem 10-14.

Choosing any point of a given plane and any basis for the bidirection of that plane yields three ordered triples — say, (a_1, a_2, a_3), (p_1, p_2, p_3), and (q_1, q_2, q_3) — which may be used to write equations like (1) for that plane.

Answers for Part C

1. (a) Plane, since $\left(\begin{vmatrix} -2 & 4 \\ -3 & 6 \end{vmatrix}, \begin{vmatrix} 4 & 2 \\ 6 & -3 \end{vmatrix}, \begin{vmatrix} 2 & -2 \\ -3 & -3 \end{vmatrix} \right) = (0, -24, -12) \neq (0, 0, 0).$

(b) Plane, since $\left(\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 3 \\ 1 & -5 \end{vmatrix}, \begin{vmatrix} 3 & -1 \\ -5 & 0 \end{vmatrix} \right) = (-1, -3, -5) \neq (0, 0, 0).$

(c) Not a plane, since $\left(\begin{vmatrix} 5 & -5 \\ -2 & 2 \end{vmatrix}, \begin{vmatrix} -5 & 10 \\ 2 & -4 \end{vmatrix}, \begin{vmatrix} 10 & 5 \\ -4 & -2 \end{vmatrix} \right) = (0, 0, 0).$

(d) Plane, since $\left(\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \begin{vmatrix} 0 & 0 \\ -1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \right) = (-1, 0, 0) \neq (0, 0, 0).$

2. Making use of sentence (2), we have:
   - (i) for part (a) $(2, -2, 4)$ and $(-3, -3, 6)$
   - (ii) for part (b) $(3, -1, 0)$ and $(-5, 0, 1)$
   - (iii) for part (d) $(0, 1, 0)$ and $(0, 0, -1)$

   [Of course in each case any two independent vectors which are linear combinations of the given two will do. But, it requires a bit more work than was done here to get such answers.]

3. [There are, of course, many answers for each part. We give the most direct one.]

(a) $\begin{cases} x_1 = 2s - 3t \\ x_2 = -2s - 3t \\ x_3 = 4s + 6t \end{cases}$ (b) $\begin{cases} x_1 = 3s - 5t \\ x_2 = -s \\ x_3 = t \end{cases}$ (d) $\begin{cases} x_1 = 0 \\ x_2 = s \\ x_3 = -t \end{cases}$

If, say, $m_1 \neq 0$, (4) is equivalent to the system:

$$\begin{cases} x_1 = f/m_1 + s(-m_2/m_1) + t(-m_3/m_1) \\ x_2 = s \\ x_3 = t \end{cases}$$

and the vectors with components $(-m_2/m_1, 1, 0)$ and $(-m_3/m_1, 0, 1)$ are linearly independent since

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \neq 0.$$

Similar arguments apply to the cases in which $m_2 \neq 0$ and $m_3 \neq 0$.

If condition (5) is not satisfied then (4) is satisfied by the coordinates of any point if $f = 0$ or of no point if $f \neq 0$.

Sample Quiz

1. Here is a system of equations:

$$\begin{cases} a_1x + a_2y = a_3 \\ b_1x + b_2y = b_3 \end{cases}$$

Solve for 'x' and 'y', expressing your answer in determinant form.

2. Consider the system of equations:

$$\begin{cases} 3x - 5y = a \\ 8x + by = 16 \end{cases}$$

(a) Solve the given system for 'x' and 'y'.

(b) Give values for 'a' and 'b' such that the solution set of the system is empty.

(c) Give values for 'a' and 'b' such that the solution set of the system is a linear function.

(d) Give values for 'a' and 'b' such that the solution set of the system contains exactly one ordered pair of real numbers.

Key to Sample Quiz

1.

$$x = \frac{\begin{vmatrix} a_3 & a_2 \\ b_3 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}$$

2. (a)

$$x = \frac{\begin{vmatrix} a & -5 \\ 16 & b \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 8 & b \end{vmatrix}} = \frac{ab + 80}{3b + 40}, \quad y = \frac{48 - 8a}{3b + 40} \quad [b \neq -40/3]$$

(b) Choose $b = -40/3$ and $a \neq 6$.

(c) Choose $b = -40/3$ and $a = 6$.

(d) Choose $b \neq -40/3$ and a any real number.

[Recall how to transform an equation like (4), which satisfies (5), into an equivalent system of parametric equations which satisfies (3). What points satisfy (4) if condition (5) is not satisfied?] If (4) represents a plane π containing the point A with coordinates (a_1, a_2, a_3) then $f = a_1m_1 + a_2m_2 + a_3m_3$ and, so, (4) is equivalent to:

$$(6) \qquad (x_1 - a_1)m_1 + (x_2 - a_2)m_2 + (x_3 - a_3)m_3 = 0$$

Conversely, any equation like (6) for which (5) is satisfied describes some plane which contains the point A. We need to find an easy way of determining when two equations like (6) [or like (4)] describe the same plane. But, first, there is a more basic question to answer: Does every plane have an equation like (4)? Equivalently: Does every plane which contains A have an equation like (6)? The answer to these questions is 'Yes.'. Briefly, any plane can be described by parametric equations like (1) and, by eliminating the parameters from these equations, one can obtain from them an equation like (4) which describes the same plane. Although you can already "reduce" a particular system of parametric equations like (1) to an equation like (4) [by solving two of the parametric equations for 's' and 't' and then substituting in the third], there is much to be gained by carrying out this procedure "in general"—that is, without using special numerical values for the nine variables 'a_1', . . . , and 'q_3' in the equations (1).

Part D

Consider the parametric equations:

$$(1) \qquad \begin{cases} x_1 = a_1 + p_1s + q_1t \\ x_2 = a_2 + p_2s + q_2t \\ x_3 = a_3 + p_3s + q_3t \end{cases}$$

and assume that the condition:

$$(3) \qquad \left(\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix}, \begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix}, \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}\right) \neq (0, 0, 0)$$

is satisfied.

1. Assumption (3) says that at least one of three determinants is not 0. This suggests considering three cases—that in which the first determinant is not 0, that in which the second is not 0, and that in which the third is not 0. Explain why, in each case, two of the parametric equations in (1) can be solved for 's' and 't'.

Answers for Part D

1. This follows directly from Theorem A. [Here is one such explanation: Suppose, first, that $\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix} \neq 0$. Under this assumption, the equations:

$$\begin{cases} x_2 = a_2 + p_2s + q_2t \\ x_3 = a_3 + p_3s + q_3t \end{cases}$$

or, the equivalent equations:

$$\begin{cases} x_2 - a_2 = p_2s + q_2t \\ x_3 - a_3 = p_3s + q_3t \end{cases}$$

can be solved, by use of Theorem A, for 's' and 't'.]

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2. Consider the system: $\begin{cases} p_1s + q_1t = x_1 - a_1 \\ p_2s + q_2t = x_2 - a_2 \end{cases}$

which is equivalent to the first two equations in (1) on page 440. Since $\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \neq 0$, it follows, by Theorem A, that this systen has a unique solution, and is equivalent to:

$$s = \frac{\begin{vmatrix} x_1 - a_1 & q_1 \\ x_2 - a_2 & q_2 \end{vmatrix}}{\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}}, \quad t = \frac{\begin{vmatrix} p_1 & x_1 - a_1 \\ p_2 & x_2 - a_2 \end{vmatrix}}{\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}}$$

2. Let's consider the third case—that in which

$$(7)\qquad \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} \neq 0.$$

Use determinants to solve two of the parametric equations in (1) for '*s*' and '*t*'. [*Hint*: Rewrite those equations as '$p_1 s + q_1 t = x_1 - a_1$', etc.]

3. Substitute the expressions you obtained for '*s*' and '*t*' in Exercise 2 into the third parametric equation like (1). [*Hint*: Since we are trying to obtain an equation like:

$$(6)\qquad (x_1 - a_1)m_1 + (x_2 - a_2)m_2 + (x_3 - a_3)m_3 = 0,$$

it should save trouble later if, before substituting, you rewrite the third parametric equation in (1) as '$-p_3 s - q_3 t + (x_3 - a_3) = 0$'.]

4. Simplify the equation you obtained in Exercise 3 by "clearing of fractions." Then use the definition of 'determinant' to rewrite the result in the form of (6).
5. The result of Exercise 4 should be an equation like (6) in which 'm_3', at least, is replaced by a determinant. If you have not already done so, transform your result into one in which 'm_1' and 'm_2' are also replaced by determinants.
6. We began by noticing in Exercise 1 that condition (3) suggested three cases; from Exercise 2 on, we have been dealing with the third case. Looking at the result you obtained in Exercise 5 for this case, what equation do you guess you would have obtained in the second case? In the first case? If you have any doubts as to the correctness of your guess, repeat your work in Exercises 2–5 for one of the other cases.

*

In the preceding exercises you have seen that, under the assumption (7) of Exercise 2, the system consisting of the first two parametric equations [in (1)] is equivalent to:

$$(8)\qquad s = \begin{vmatrix} x_1 - a_1 & q_1 \\ x_2 - a_2 & q_2 \end{vmatrix} \Bigg/ \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}, \quad t = \begin{vmatrix} p_1 & x_1 - a_1 \\ p_2 & x_2 - a_2 \end{vmatrix} \Bigg/ \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}$$

So, the system (1) of parametric equations is equivalent to the system consisting of these two equations and the equation you obtained in Exercise 3. [Explain.] The latter equation is equivalent to:

$$(9)\qquad (x_1 - a_1)\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix} + (x_2 - a_2)\begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix} + (x_3 - a_3)\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = 0$$

Answers for Part D [cont.]

3. Following the directions in the hint, we obtain the equation:

$$-p_3 \cdot \frac{\begin{vmatrix} x_1 - a_1 & q_1 \\ x_2 - a_2 & q_2 \end{vmatrix}}{\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}} - q_3 \cdot \frac{\begin{vmatrix} p_1 & x_1 - a_1 \\ p_2 & x_2 - a_2 \end{vmatrix}}{\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}} + (x_3 - a_3) = 0$$

4. "Clearing fractions" and simplifying, we obtain, in turn:

$$-p_3 \cdot \begin{vmatrix} x_1 - a_1 & q_1 \\ x_2 - a_2 & q_2 \end{vmatrix} - q_3 \begin{vmatrix} p_1 & x_1 - a_1 \\ p_2 & x_2 - a_2 \end{vmatrix} + (x_3 - a_3)\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} = 0$$

$$-p_3\big((x_1 - a_1)q_2 - (x_2 - a_2)q_1\big) - q_3\big(p_1(x_2 - a_2) - p_2(x_1 - a_1)\big) + (x_3 - a_3)\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} = 0$$

$$(x_1 - a_1)(p_2 q_3 - p_3 q_2) + (x_2 - a_2)(p_3 q_1 - p_1 q_3) + (x_3 - a_3)\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} = 0$$

$$(x_1 - a_1)\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix} + (x_2 - a_2)\begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix} + (x_3 - a_3)\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = 0$$

The last result is an equation in the form of (6).

5. [The final result should be the last result in the solution of Exercise 4.]
6. You would obtain an equivalent equation in each case. To see this it should be sufficient to note the argument for the second case can be obtained from that just given for the first case by a cyclic substitution of subscripts — '3' for '2', '1' for '3', and '2' for '1'. Now, note that the same substitutions transform the result obtained in Exercise 5 into an equivalent equation. Finally, the third case can be obtained from the second by the same substitution of subscripts.

Answers to questions:

Given coordinates which satisfy (9) the corresponding values of 's' and 't' can be found by substitution in (8).

$\overleftrightarrow{A[\vec{p}, \vec{q}]}$ is a line if $(\vec{p}, \vec{q})$ is linearly dependent and $\vec{p}$ and $\vec{q}$ are not both $\vec{0}$. It is $\{A\}$ in case $\vec{p} = \vec{0} = \vec{q}$. If $(\vec{p}, \vec{q})$ is linearly dependent then the set described by (9) is $\mathcal{E}$. [So, (9) describes $\overleftrightarrow{A[\vec{p}, \vec{q}]}$ only if $(\vec{p}, \vec{q})$ is linearly independent.]

[Your answer for Exercise [illegible] may not be the same as (9). But, from what you know about determinants, it should be obviously equivalent to (9).] It follows that, under the assumption (7), the system (1) of parametric equations is equivalent to the system consisting of the equations in (8) and (9). In particular, if this assumption is satisfied, coordinates (x_1, x_2, x_3) satisfy (9) if and only if they can be obtained from the parametric equations by choosing appropriate values for 's' and 't'. [If you had coordinates which satisfy (9), how might you find corresponding values for 's' and 't'?]

Finally, in answering Exercise 6, you should have discovered that the procedure which led to equation (9) under the assumption (7) would lead to the same equation in each of the other two cases covered by (3). Hence,

> the plane which is described by the parametric equations (1) [in case (3) is satisfied] is also described by the single equation (9).

So, since any plane can be described by parametric equations it follows that any plane can be described by a single equation—and, in (9), we have a simple rule for writing an equation which describes the plane $\overline{A[\vec{p}, \vec{q}]}$, given the coordinates of A and the components of $\vec{p}$ and $\vec{q}$. Since $\overline{ABC} = \overline{A[B - A, C - A]}$ we can also use (9) to obtain an equation for $\overline{ABC}$, given the coordinates of A, B, and C. [Of course, the preceding remarks apply only if $(\vec{p}, \vec{q})$ is linearly independent and $\{A, B, C\}$ is noncollinear. What kind of set is $\overline{A[\vec{p}, \vec{q}]}$ if $(\vec{p}, \vec{q})$ is linearly dependent? What set is described by (9) in this case?]

As an example, consider the parametric equations:

$$(*) \qquad \begin{cases} x_1 = -3 + 10s + 4t \\ x_2 = -2 + 5s - 2t \\ x_3 = 1 - 5s + 2t \end{cases}$$

As we have seen, it is easily checked that these are parametric equations for a plane. Since $(-3, -2, 1)$ is a solution for (*), we can make use of (9) as follows to write a single equation for this plane:

$$(x_1 - -3)\begin{vmatrix} 5 & -5 \\ -2 & 2 \end{vmatrix} + (x_2 - -2)\begin{vmatrix} -5 & 10 \\ 2 & 4 \end{vmatrix} + (x_3 - 1)\begin{vmatrix} 10 & 5 \\ 4 & -2 \end{vmatrix} = 0$$

Simplifying this, we obtain, in turn:

$$(x_1 + 3) \cdot 0 + (x_2 + 2) \cdot -40 + (x_3 - 1) \cdot -40 = 0$$
$$(x_2 + 2) + (x_3 - 1) = 0$$
$$x_2 + x_3 = -1$$

The last is an equation which is equivalent to one of the form of sentences (4), namely:

$$x_1m_1 + x_2m_2 + x_3m_3 = f$$

and describes the same plane as does the system of equations (*).

Our results on describing planes by equations may be summarized in:

Theorem 10–15 Suppose that, with respect to a given coordinate system, the coordinates of A are (a_1, a_2, a_3) and the components of $\vec{p}$ and $\vec{q}$ are (p_1, p_2, p_3) and (q_1, q_2, q_3), respectively. With respect to the given coordinate system

(a) the parametric equations:

$$\begin{cases} x_1 = a_1 + p_1s + q_1t \\ x_2 = a_2 + p_2s + q_2t \\ x_3 = a_3 + p_3s + q_3t \end{cases}$$

describe the set $\overline{A[\vec{p}, \vec{q}]}$; and this set is a plane if and only if

$$(\dagger) \quad \left(\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix}, \begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix}, \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}\right) \neq (0, 0, 0),$$

and

(b) the single equation:

$$(x_1 - a_1)\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix} + (x_2 - a_2)\begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix} + (x_3 - a_3)\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = 0$$

represents $\overline{A[\vec{p}, \vec{q}]}$ if and only if this set is a plane—that is, if and only if the condition (†) is satisfied.

The advantage of writing the three determinants in (†) in the order given should be clear when you look at the order in which these determinants appear in the single equation for the plane $\overline{A[\vec{p}, \vec{q}]}$ given in part (b) of the theorem.

Part E

1. In each of the following parts you are given, first, the coordinates of a point A and, second, the components of two vectors $\vec{p}$ and $\vec{q}$.
   (i) Use determinants to check the vectors for linear independence.
   (ii) In case the vectors are linearly independent, write parametric equations for the plane $\overleftrightarrow{A[\vec{p}, \vec{q}]}$ and, also, write a single equation for this plane.
   (a) (2, −1, 3); (1, 4, 6), (−2, 3, 5)
   (b) (8, 7, −5); (4, −2, 6), (−6, 3, −9)
   (c) (4, −3, 0); (−2, −5, 6), (2, 5, −4)
   (d) (9, −6, −3); (7, −14, 0), (−3, 6, 5)
2. In each of the following parts you are given coordinates of three points A, B, and C.
   (i) Use determinants to check the points for noncollinearity.
   (ii) In case the points are noncollinear, write parametric equations for the plane $\overleftrightarrow{ABC}$ and, also, write a single equation for this plane.
   (a) (2, −5, 9), (4, 1, 15), (1, −7, 6)
   (b) (4, 6, −4), (6, 3, 1), (10, 10, −6)
   (c) (−3, −3, 3), (3, 1, 1), (−1, −6, 8)
   (d) (−7, 9, −7), (−1, 0, 8), (−9, 12, −12)

*

We have, now, three ways of describing a plane with respect to a given coordinate system. We may use parametric equations like:

$$(1)\qquad \begin{cases} x_1 = a_1 + p_1 s + q_1 t \\ x_2 = a_2 + p_2 s + q_2 t \\ x_3 = a_3 + p_3 s + q_3 t \end{cases}$$

or we may use a single equation like:

$$(6)\qquad (x_1 - a_1)m_1 + (x_2 - a_2)m_2 + (x_3 - a_3)m_3 = 0,$$

or we may use a single equation like:

$$(4)\qquad x_1 m_1 + x_2 m_2 + x_3 m_3 = f$$

In (1) and (6), (a_1, a_2, a_3) are the coordinates of a point A which belongs to the described set. In (1), (p_1, p_2, p_3) and (q_1, q_2, q_3) are components of vectors $\vec{p}$ and $\vec{q}$—

$$\vec{p} = \vec{u}_1 p_1 + \vec{u}_2 p_2 + \vec{u}_3 p_3 \text{ and } \vec{q} = \vec{u}_1 q_1 + \vec{u}_2 q_2 + \vec{u}_3 q_3$$

Answers for Part E

1. (a) $(\vec{p}, \vec{q})$ is linearly independent, since $\left(\begin{vmatrix} 4 & 6 \\ 3 & 5 \end{vmatrix}, \begin{vmatrix} 6 & 1 \\ 5 & -2 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ -2 & 3 \end{vmatrix}\right)$ $= (2, -17, 11) \neq (0, 0, 0)$. Parametric equations for $\overleftrightarrow{A[\vec{p}, \vec{q}]}$ are:

$$\begin{cases} x_1 = 2 + r - 2s \\ x_2 = -1 + 4r + 3s \\ x_3 = 3 + 6r + 5s \end{cases}$$

A single equation for $\overleftrightarrow{A[\vec{p}, \vec{q}]}$ is:

$$2x_1 - 17x_2 + 11x_3 = 54$$

[This was obtained by simplifying '$(x_1 - 2)\cdot 2 + (x_2 + 1)\cdot -17 + (x_3 - 3)\cdot 11 = 0$'.]

(b) $(\vec{p}, \vec{q})$ is linearly dependent, since $\left(\begin{vmatrix} -2 & 6 \\ 3 & -9 \end{vmatrix}, \begin{vmatrix} 6 & 4 \\ -9 & -6 \end{vmatrix}, \begin{vmatrix} 4 & -2 \\ -6 & 3 \end{vmatrix}\right)$ $= (0, 0, 0)$. So, $\overleftrightarrow{A[\vec{p}, \vec{q}]}$ is not a plane.

(c) $(\vec{p}, \vec{q})$ is linearly independent, since $\left(\begin{vmatrix} -5 & 6 \\ 5 & -4 \end{vmatrix}, \begin{vmatrix} 6 & -2 \\ -4 & 2 \end{vmatrix}, \begin{vmatrix} -2 & -5 \\ 2 & 5 \end{vmatrix}\right)$ $= (-10, 4, 0) \neq (0, 0, 0)$. Parametric equations for $\overleftrightarrow{A[\vec{p}, \vec{q}]}$ are:

$$\begin{cases} x_1 = 4 - 2r + 2s \\ x_2 = -3 - 5r + 5s \\ x_3 = 6r - 4s \end{cases}$$

A single equation for $\overleftrightarrow{A[\vec{p}, \vec{q}]}$ is:

$$5x_1 - 2x_2 = 26$$

[This was obtained by simplifying '$(x_1 - 4)\cdot -10 + (x_2 + 3)\cdot 4 + x_3\cdot 0 = 0$'.]

(d) $(\vec{p}, \vec{q})$ is linearly independent, since $\left(\begin{vmatrix} -14 & 0 \\ 6 & 5 \end{vmatrix}, \begin{vmatrix} 0 & 7 \\ 5 & -3 \end{vmatrix}, \begin{vmatrix} 7 & -14 \\ -3 & 6 \end{vmatrix}\right)$ $= (-70, -35, 0) \neq (0, 0, 0)$. Parametric equations for $\overleftrightarrow{A[\vec{p}, \vec{q}]}$ are:

$$\begin{cases} x_1 = 9 + 7r - 3s \\ x_2 = -6 - 14r + 6s \\ x_3 = -3 + 5s \end{cases}$$

A single equation for $\overleftrightarrow{A[\vec{p}, \vec{q}]}$ is:

$$2x_1 + x_2 = 12$$

[This was obtained by simplifying '$(x_1 - 9)\cdot -70 + (x_2 + 6)\cdot -35 + (x_3 + 3)\cdot 0 = 0$'.]

Answers for Part E [cont.]

2. (a) B - A and C - A have components (2, 6, 6) and (-1, -2, -3), respectively. Since $\left(\begin{vmatrix} 6 & 6 \\ -2 & -3 \end{vmatrix}, \begin{vmatrix} 6 & 2 \\ -3 & -1 \end{vmatrix}, \begin{vmatrix} 2 & 6 \\ -1 & -2 \end{vmatrix} \right)$ $= (-6, 0, 2) \neq (0, 0, 0)$, (B - A, C - A) is linearly independent. So, {A, B, C} is noncollinear.

Parametric equations for $\overline{ABC}$ are:

$$\begin{cases} x_1 = 2 + 2r - s \\ x_2 = -5 + 6r - 2s \\ x_3 = 9 + 6r - 3s \end{cases}$$

A single equation for $\overline{ABC}$ is:

$$3x_1 - x_3 = -3$$

which is obtained by simplifying '$(x_1 - 2) \cdot -6 + (x_2 + 5) \cdot 0 + (x_3 - 9) \cdot 2 = 0$'.

(b) B - A and C - A have components (2, -3, 5) and (6, 4, -2), respectively. Since $\left(\begin{vmatrix} -3 & 5 \\ 4 & -2 \end{vmatrix}, \begin{vmatrix} 5 & 2 \\ -2 & 6 \end{vmatrix}, \begin{vmatrix} 2 & -3 \\ 6 & -4 \end{vmatrix} \right)$ $= (-14, 34, 26) \neq (0, 0, 0)$, (B - A, C - A) is linearly independent. So, {A, B, C} is noncollinear.

Parametric equations for $\overline{ABC}$ are:

$$\begin{cases} x_1 = 4 + 2r + 6s \\ x_2 = 6 - 3r + 4s \\ x_3 = -4 + 5r - 2s \end{cases}$$

A single equation for $\overline{ABC}$ is:

$$7x_1 - 17x_2 - 13x_3 = -22$$

which is obtained by simplifying '$(x_1 - 4) \cdot -14 + (x_2 - 6) \cdot 34 + (x_3 + 4) \cdot 26 = 0$'.

(c) $\begin{cases} x_1 = -3 + 6r + 2s \\ x_2 = -3 + 4r - 3s \\ x_3 = 3 - 2r + 5s \end{cases}$; $7x_1 - 17x_2 - 13x_3 = -9$

(d) B - A and C - A have components (6, -9, 15) and (-2, 3, -5), respectively. Since $\left(\begin{vmatrix} -9 & 15 \\ 3 & -5 \end{vmatrix}, \begin{vmatrix} 15 & 6 \\ -5 & -2 \end{vmatrix}, \begin{vmatrix} 6 & -9 \\ -2 & 3 \end{vmatrix} \right)$ $= (0, 0, 0)$, (B - A, C - A) is linearly dependent. So, {A, B, C} is collinear.

such that the bidirection of the set described by (1) is $[\vec{p}, \vec{q}]$. And, in studying (6) and (4) it will be convenient to consider the vector $\vec{m}$ whose components are (m_1, m_2, m_3):

$$\vec{m} = \vec{u}_1 m_1 + \vec{u}_2 m_2 + \vec{u}_3 m_3$$

As we have seen, the condition that either (6) or (4) describes a plane is, just, that $\vec{m} \neq \vec{0}$. The condition under which the parametric equations (1) describe a plane is given in Theorem 10-15(a). Part (b) of this theorem tells how to find an equation like (6) for a plane which has previously been described by parametric equations [like (1)]. Conversely, given an equation like (6) which describes a plane—that is, one for which $\vec{m} \neq \vec{0}$—it is easy to find parametric equations [like (1)] which describe the same plane. If, for example, $m_1 \neq 0$, we can obtain such equations by replacing '$(x_2 - a_2)$' and '$(x_3 - a_3)$' in the following equations by 's' and 't':

$$\left|\begin{array}{l} x_1 = a_1 - \dfrac{m_2}{m_1}(x_2 - a_2) - \dfrac{m_3}{m_1}(x_3 - a_3) \\ x_2 = a_2 + 1 \cdot (x_2 - a_2) \\ x_3 = a_3 + 1 \cdot (x_3 - a_3) \end{array}\right.$$

[Explain why this works, and show how to obtain parametric equations like (1) for a plane described by an equation like (4) for which $m_1 \neq 0$.] Finally, given an equation like (6) it is easy to find an equivalent equation like (4) merely by taking

$$f = a_1 m_1 + a_2 m_2 + a_3 m_3.$$

And, given an equation like (4) which represents a plane it is easy to find an equivalent equation like (6) by taking for (a_1, a_2, a_3) the coordinates of any point of this plane. Note that, in these procedures for changing from (6) to (4) and from (4) to (6), the vector $\vec{m}$ is not changed.

Part F

In each exercise you are given equations like (1), (6), or (4), page 446. In each case, check whether the given equations describe a plane. If they do, write equations of each of the other two kinds which describe the same plane.

1. $\begin{cases} x_1 = 1 - 2s + 3t \\ x_2 = 3s - \frac{9}{2}t \\ x_3 = 2 + 4s - 6t \end{cases}$

2. $\begin{cases} x_1 = 5 + 6s - 2t \\ x_2 = -7 - 3s + t \\ x_3 = 6 + 8s + \frac{8}{3}t \end{cases}$

3. $(x_1 - 1)2 - (x_2 + 3)4 + (x_3 - 4)3 = 0$

4. $3x_1 + 2x_2 - 4x_3 = 6$

5. $x_1 - x_2 - x_3 = 0$

6. $x_1 - 3x_2 - 2x_3 = 0$

Answers for Part F

1. $\vec{p}$ and $\vec{q}$ have components $(-2, 3, 4)$ and $(3, -9/2, -6)$, respectively.

   Since $\left(\begin{vmatrix} 3 & 4 \\ -9/2 & -6 \end{vmatrix}, \begin{vmatrix} 4 & -2 \\ -6 & 3 \end{vmatrix}, \begin{vmatrix} -2 & 3 \\ 3 & -9/2 \end{vmatrix}\right) = (0, 0, 0)$, $(\vec{p}, \vec{q})$ is linearly dependent. So, the given parametric equations do not describe a plane.

2. $\vec{p}$ and $\vec{q}$ have components $(6, -3, 8)$ and $(-2, 1, 8/3)$, respectively.

   Since $\left(\begin{vmatrix} -3 & 8 \\ 1 & 8/3 \end{vmatrix}, \begin{vmatrix} 8 & 6 \\ 8/3 & -2 \end{vmatrix}, \begin{vmatrix} 6 & -3 \\ -2 & 1 \end{vmatrix}\right) = (-16, -32, 0) \neq (0, 0, 0)$, $(\vec{p}, \vec{q})$ is linearly independent. So, the given parametric equations describe a plane.

   A single equation like (6) which describes the same plane is:

   $$(x_1 - 5) \cdot -16 + (x_2 + 7) \cdot -32 + (x_3 - 6) \cdot 0 = 0$$

   A single equation like (4) which describes the same plane is:

   $$x_1 + 2x_2 = -9$$

3. Since $(m_1, m_2, m_3) = (2, -4, 3) \neq (0, 0, 0)$, the given equation describes a plane.

   Now, since the equation:

   $$x_1 = 1 + (x_2 + 3)2 + (x_3 - 4) \cdot -3/2$$

   is equivalent to the given equation, parametric equations which describe the same plane as does the given equation are:

   $$\begin{cases} x_1 = 1 + 2r - \frac{3}{2}s \\ x_2 = -3 + r \\ x_3 = 4 + s \end{cases}$$

   [These are obtained by letting $r = x_2 + 3$ and $s = x_3 - 4$.]

   A single equation like (4) which describes the same plane is:

   $$2x_1 - 4x_2 + 3x_3 = 26$$

4. Since $(m_1, m_2, m_3) = (3, 2, -4) \neq (0, 0, 0)$, the given equation describes a plane.

   An equation like (6) which describes the same plane is:

   $$(x_1 - 2)3 + (x_2 - 2)2 + (x_3 - 1) \cdot -4 = 0$$

   [There are, of course, many correct answers.] Since the equation:

   $$x_1 = 2 - \frac{2}{3}x_2 + \frac{4}{3}x_3$$

   is equivalent to the given equation, parametric equations which describe the same plane as does the given equation are:

   $$\begin{cases} x_1 = 2 - \frac{2}{3}s + \frac{4}{3}t \\ x_2 = s \\ x_3 = \end{cases}$$

   [These are obtained by letting $s = x_2$ and $t = x_3$.]

Answers for Part F [cont.]

5. Since $(m_1, m_2, m_3) = (1, -1, -1) \neq (0, 0, 0)$, the given equation describes a plane.

An equation like (6) which describes the same plane is:

$$(x_1 - 1) \cdot 1 + (x_2 - 1) \cdot -1 + (x_3 - 0) \cdot -1 = 0$$

[There are, of course, many correct answers.] Since the given equation is equivalent to '$x_1 = x_2 + x_3$', parametric equations which describe the same plane as does the given equation are:

$$\begin{cases} x_1 = r + s \\ x_2 = r \\ x_3 = \phantom{r +{}} s \end{cases}$$

[These are obtained by letting $r = x_2$ and $s = x_3$.]

6. Since $(m_1, m_2, m_3) = (1, -3, -2) \neq (0, 0, 0)$, the given equation describes a plane.

An equation like (6) which describes the same plane is:

$$(x_1 - 0) \cdot 1 + (x_2 - 2) \cdot -3 + (x_3 + 3) \cdot -2 = 0$$

[There are, of course, many correct answers.] Since the given equation is equivalent to '$x_1 = 3x_2 + 2x_3$', parametric equations which describe the same plane are:

$$\begin{cases} x_1 = 3r + 2s \\ x_2 = r \\ x_3 = s \end{cases}$$

[Making use of the equation like (6), it is easy to obtain the following parametric equations for this same plane:

$$\begin{cases} x_1 = \phantom{-3 +{}} 3r + 2s \\ x_2 = 2 + r \\ x_3 = -3 + s \end{cases}$$

There are, of course, many correct answers, no one of which can be construed to be the "best" one.]

10.10 Determinants and Equations of Lines

There is one more problem concerning the description of planes by equations which remains to be solved. Suppose that we are given two descriptions of planes each of which is like (1), (4), or (6) on page 446. How can we tell whether they describe the same plane or different planes? Since a plane is parallel to itself and since no *two* parallel planes have a point in common, we can solve this problem if we know how to tell whether two descriptions of planes describe parallel or nonparallel planes. [Explain.] Since it is easy to change from one kind of description to the other it will be sufficient to solve this problem for equations like (6). As we shall see, the same procedure will apply to equations like (4) – or to two equations one of which is like (4) and the other like (6). So, let's consider two equations:

$$(6) \qquad (x_1 - a_1)m_1 + (x_2 - a_2)m_2 + (x_3 - a_3)m_3 = 0$$
$$(6') \qquad (x_1 - b_1)n_1 + (x_2 - b_2)n_2 + (x_3 - b_3)n_3 = 0$$

It is reasonable to suspect that, in case these equations represent planes π and σ, the direction of π will be determined in some way by the vector $\vec{m}$ and that of σ will be determined in the same way by the vector $\vec{n}$, where

$$\vec{m} = \vec{u}_1 m_1 + \vec{u}_2 m_2 + \vec{u}_3 m_3 \text{ and } \vec{n} = \vec{u}_1 n_1 + \vec{u}_2 n_2 + \vec{u}_3 n_3.$$

In fact, we might guess that $[\pi]$ is determined by $[\vec{m}]$ and $[\sigma]$ by $[\vec{n}]$. If this is correct then it will be the case that $\pi \parallel \sigma$ if and only if $[\vec{m}] = [\vec{n}]$. [Explain.] Since (6) and (6′) describe planes if and only if $\vec{m}$ and $\vec{n}$ are non-$\vec{0}$, we may state our guess by saying that

(*) (6) and (6′) describe parallel planes if and only if $\vec{m}$ and $\vec{n}$ are non-$\vec{0}$ and $(\vec{m}, \vec{n})$ is linearly dependent.

[Explain.]

We can obtain some more confidence in the correctness of (*) by noting that its if-part is almost obvious. To make it quite obvious, note that (6) and (6′) are equivalent, respectively, to

$$(4) \qquad x_1 m_1 + x_2 m_2 + x_3 m_3 = f$$
$$\text{and:} \quad (4') \qquad x_1 n_1 + x_2 n_2 + x_3 n_3 = g,$$

The discussion which begins here is a lengthy one! In it we show how to determine if two equations like (6) and (6′) describe the same or different planes. For your students it is the result, stated and illustrated on page 453 that is important. It is a very easy result to apply, a very difficult one to obtain. Much of the work on the ensuing pages should be developed by the teacher and class together. It is much too involved for most students to do alone. However, when guided by the teacher, students are able to understand the argument, and to contribute small parts of it from time to time. A student should not be asked to reproduce this argument.

The explanation asked for near the beginning of section 10.10 might be that in case the descriptions describe nonparallel planes they describe different planes, and in case they describe parallel planes they describe the same plane if and only if there is a point whose coordinates satisfy both.

Explanations called for in text:

If $[\pi]$ is determined by $[\vec{m}]$ and $[\sigma]$ is determined [in the same way] by $[\vec{n}]$ then $[\pi] = [\sigma]$ if and only if $[\vec{m}] = [\vec{n}]$. So, by definition, $\pi \parallel \sigma$ if and only if $[\vec{m}] = [\vec{n}]$.

For $\vec{m} \neq \vec{0} \neq \vec{n}$, $[\vec{m}] = [\vec{n}]$ if and only if $(\vec{m}, \vec{n})$ is linearly dependent.

Equation (4′), if $\vec{n} = \vec{m}k$, is equivalent to '$x_1(m_1k) + x_2(m_2k) + x_3(m_3k) = g$' and for $k \neq 0$, this is equivalent to (4″).

If $f \neq g/k$ then the planes described by (4) and (4″) have no common point; if $f = g/k$ then (4) and (4″) describe the same plane. In either case, the plane described by (4) is parallel to the plane described by (4″).

Having established $(*_1)$ all we need do is establish its converse. So it is sufficient to establish the contrapositive of the converse of $(*_1)$. Now, this latter is equivalent to:

> If $(\vec{m}, \vec{n})$ is linearly independent or $\vec{m} = \vec{0}$ or $\vec{n} = \vec{0}$ then (6) and (6′) do not describe parallel planes.

We already know that if $\vec{m} = \vec{0}$ or $\vec{n} = \vec{0}$ then (6) and (6′) do not [both] describe planes. So, what remains to be proved is:

> If $(\vec{m}, \vec{n})$ is linearly independent then (6) and (6′) do not describe parallel planes.

Clearly, this follows from the stronger statement $(*_2)$. Hence, it is sufficient to prove $(*_2)$.

where $f = a_1m_1 + a_2m_2 + a_3m_3$ and $g = b_1n_1 + b_2n_2 + b_3n_3$. Now, assuming that $\vec{m}$ and $\vec{n}$ are non-$\vec{0}$ and $(\vec{m}, \vec{n})$ is linearly dependent, it follows that there is some number – say, k – such that $\vec{n} = \vec{m}k$ and $k \neq 0$. [Explain.] Hence, equation (4′) is equivalent to:

$$(4'') \qquad x_1m_1 + x_2m_2 + x_3m_3 = g/k \quad \text{[Explain.]}$$

It should now be obvious that, when $\vec{m}$ and $\vec{n}$ are non-$\vec{0}$ dependent vectors, the plane described by (4) is parallel to the plane described by (4″). [Consider two cases, that in which $f \neq g/k$ and that in which $f = g/k$.] Since (6) is equivalent to (4) and (6′) to (4″), we have established the if-part of $(^*)$:

$(^*_1)$ If $\vec{m}$ and $\vec{n}$ are non-$\vec{0}$ and $(\vec{m}, \vec{n})$ is linearly dependent then (6) and (6′) describe parallel planes.

To complete the proof of $(^*)$ it will be sufficient to show that

$(^*_2)$ if $(\vec{m}, \vec{n})$ is linearly independent then (6) and (6′) describe nonparallel planes.

[Explain.] To guess how we might be able to prove $(^*_2)$, note that we could replace 'nonparallel planes' by 'planes whose intersection is a line'. So, assuming that $(\vec{m}, \vec{n})$ is linearly independent, we might try to show that any three points whose coordinates satisfy both (6) and (6′) are collinear. This would lead to the consideration of six equations – three obtained by substituting the coordinates of the three points in (6), and three obtained in the same way from (6′). It wouldn't be impossible to prove $(^*_2)$ in this way, but there should be something simpler. Looking for a simpler way, we might remember that the collinearity of three points amounts, by definition, to the linear dependence of two translations. [Explain.] Also, if the three points belong to a plane then the two translations belong to the bidirection of that plane. So, we could prove $(^*_2)$ by showing that if $(\vec{m}, \vec{n})$ is linearly independent then any two translations which belong both to the bidirection of the plane described by (6) and to the bidirection of the plane described by (6′) are linearly dependent. This gives us an easy problem to start on: What condition must the components of a vector $\vec{c}$ satisfy if $\vec{c}$ belongs to the bidirection of the plane described by (6)? It turns out, as we shall see, that this condition is a rather simple one and, after learning a bit more about using determinants to solve pairs of equations, it will be easy to prove $(^*_2)$. Moreover, it will be easy to compute the components of a non-$\vec{0}$ vector in the direction of the line of intersection of the two planes.

Exercises

Part A

In these exercises we shall assume that $(\vec{m}, \vec{n})$ is linearly independent and we shall consider the planes π and σ which are described by the equations:

$$(6) \qquad (x_1 - a_1)m_1 + (x_2 - a_2)m_2 + (x_3 - a_3)m_3 = 0$$

and:

$$(6') \qquad (x_1 - b_1)n_1 + (x_2 - b_2)n_2 + (x_3 - b_3)n_3 = 0,$$

respectively.

1. Why does it follow from our assumption that equations (6) and (6′) do describe planes?

2. What are the coordinates of a point which you can be sure lies on the plane π described by equation (6)?

3. **(a)** Suppose that $\vec{c} \in [\pi]$ and that $P \in \pi$. What can you say about $P + \vec{c}$?

(b) Suppose that P and $P + \vec{c}$ both belong to π. What can you say about $\vec{c}$?

4. You know that the point A whose coordinates are (a_1, a_2, a_3) belongs to π. You also know that $\vec{c} \in [\pi]$ if and only if $A + \vec{c} \in \pi$. Assuming that

$$\vec{c} = \vec{u}_1 c_1 + \vec{u}_2 c_2 + \vec{u}_3 c_3,$$

what are the coordinates of $A + \vec{c}$? Derive from (6) an equation which says that $A + \vec{c} \in \pi$.

5. What is an equation which is satisfied by (c_1, c_2, c_3) if and only if (c_1, c_2, c_3) are the components of a vector $\vec{c}$ which belongs to the bidirection of the plane described by (6)?

6. Repeat Exercise 5 for the plane described by (6′).

*

You have seen in the preceding exercises that (c_1, c_2, c_3) are the components of a vector—say, $\vec{c}$—in $[\pi] \cap [\sigma]$ if and only if (c_1, c_2, c_3) is a solution of the system:

$$(10) \qquad \begin{cases} c_1m_1 + c_2m_2 + c_3m_3 = 0 \\ c_1n_1 + c_2n_2 + c_3n_3 = 0 \end{cases}$$

In order to prove $(*_2)$ on page 449, we wish to show that, if $(\vec{m}, \vec{n})$ is linearly independent, all such vectors $\vec{c}$ are linearly dependent. In terms of the components of $\vec{m}$ and $\vec{n}$, the condition that $(\vec{m}, \vec{n})$ is linearly independent is:

$$(11) \qquad \left(\begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix}, \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix}, \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} \right) \neq (0, 0, 0)$$

It may require the majority of a class period to complete the discussion on pages 448 - 449. In view of this, we recommend just Part A as a homework assignment. The material found on pages 451 - 455, including Part B, is best treated in class, so that emphasis can be placed on the result derived (Theorem 10-16 and its corollary) rather than on the derivation itself. After a brief illustration, Parts C and D make one reasonable homework assignment and Parts E and F make another.

Answers for Part A

1. Because the linear independence of $(\vec{m}, \vec{n})$ implies that neither $\vec{m}$ nor $\vec{n}$ is $\vec{0}$.

2. (a_1, a_2, a_3)

3. (a) $P + \vec{c} \in \pi$

   (b) $\vec{c} \in [\pi]$

4. The coordinates of $A + \vec{c}$ are $(a_1 + c_1,\ a_2 + c_2,\ a_3 + c_3)$. [This follows from the fact that $A + \vec{c} = O + \{(A - O) + \vec{c}\}$, for then $A + \vec{c} = O + \vec{u}_1(a_1 + c_1) + \vec{u}_2(a_2 + c_2) + \vec{u}_3(a_3 + c_3)$. The latter result gives the coordinates of $A + \vec{c}$.] Making use of this in (6), we obtain, in turn:

$$\big((a_1 + c_1) - a_1\big)m_1 + \big((a_2 + c_2) - a_2\big)m_2 + \big((a_3 + c_3) - a_3\big)m_3 = 0$$

$$c_1m_1 + c_2m_2 + c_3m_3 = 0$$

So, $A + \vec{c} \in \pi$ if and only if this last equation is satisfied by the components, (c_1, c_2, c_3), of $\vec{c}$.

5. $c_1m_1 + c_2m_2 + c_3m_3 = 0.$

6. $c_1n_1 + c_2n_2 + c_3n_3 = 0.$

[By what theorem?] So, our problem is to discover what we can about the solutions of a system like (10), assuming that the condition (11) is satisfied.

Since we shall deal with other systems like (10), we shall restate our problem in the "*a*, *b*, *c*, *x*, *y*, *z*-notation" we used earlier in this chapter. [See, for example, Theorem A on page 436.] In these terms, we are interested in the solutions (x, y, z) of the system:

$$(12) \qquad \begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \end{cases}$$

subject to the condition:

$$(13) \qquad \left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right) \neq (0, 0, 0)$$

Solving (12), subject to (13), turns out to be rather similar to what you did in Part D of Section 10.09 in order to transform the parametric equations (1), subject to (3), into the equation (9).

Part B

In these exercises you will learn how to find all solutions of the system (12), above, subject to the condition (13).

1. Assumption (13) says that at least one of three determinants is not 0. This suggests considering three cases—that in which the first determinant is not 0, etc. Explain why, in each case, the equations in (12) can be solved for two of the variables '*x*', '*y*', and '*z*' in terms of the third.
2. Let's consider the third case—that in which

$$(14) \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

   Use determinants to solve (12) for '*x*' and '*y*'. [*Hint*: Rewrite the equations as '$a_1x + b_1y = -c_1z$', etc.]
3. On the basis of your answer for Exercise 2, show that (12) is equivalent to the system:

$$x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} z, \quad y = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} z$$

   [*Hint*: Recall the properties of determinants which you established in Part B of Section 10.09.]

The theorem referred to in connection with (11) is Theorem 10-14.

Answers for Part B

1. Given the condition (13), the system (12) reduces to one of the form:

$$\begin{cases} p_1x_1 + p_2x_2 = p_3 \\ q_1x_1 + q_2x_2 = q_3 \end{cases}$$

   where $\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}$ is a nonzero term in (13). [For example, given that $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \neq 0$, the system (12) reduces to the following:

$$\begin{cases} b_1y + c_1z = -a_1x \\ b_2y + c_2z = -a_2x \end{cases}$$

   In this case $p_1 = b_1$, $x_1 = y$, $p_2 = c_1$, $x_2 = z$, $p_3 = -a_1x$, etc.]
2. Given the condition (14), we are asked to solve the system:

$$\begin{cases} a_1x + b_1y = -c_1z \\ a_2x + b_2y = -c_2z \end{cases}$$

   This system is equivalent to:

$$x = \begin{vmatrix} -c_1z & b_1 \\ -c_2z & b_2 \end{vmatrix} \Big/ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad y = \begin{vmatrix} a_1 & -c_1z \\ a_2 & -c_2z \end{vmatrix} \Big/ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

3. The system (12) is equivalent to the system given in this exercise because

$$\begin{vmatrix} -c_1z & b_1 \\ -c_2z & b_2 \end{vmatrix} = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \cdot -z = -\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \cdot -z = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} z$$

   and

$$\begin{vmatrix} a_1 & -c_1z \\ a_2 & -c_2z \end{vmatrix} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \cdot -z = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} z.$$

4. For any solution (x, y, z) of (12) there is a number t such that

$$t = z \Big/ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Show that, in terms of such a number t, the solution in question is given by:

$$(*) \qquad x = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} t, \quad y = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} t, \quad z = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} t$$

5. Show, by substitution in the system of Exercise 3, that, for any value of 't', the numbers determined by (*) satisfy the equations (12).

6. You have shown, subject to the assumption (14) of Exercise 2, that numbers (x, y, z) satisfy (12) if and only if

$$\exists_t \left(x = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} t, \ y = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} t, \text{ and } z = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} t \right).$$

Do you think that this also holds in the other two cases covered by (13)? Explain.

*

The result proved in Part B can be stated as follows:

> Theorem B
>
> For
>
> $$\left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right) \neq (0, 0, 0),$$
>
> $$(a_1x + b_1y + c_1z = 0 \text{ and } a_2x + b_2y + c_2z = 0)$$
>
> $$\Longleftrightarrow$$
>
> $$\exists_t \left(x = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} t, y = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} t, \text{ and } z = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} t \right).$$

[This result is similar to that in Exercise 3(b) of Part B on page 439.]

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Answers for Part B [cont.]

4. In terms of the number t, $z = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} t$ and, using this result in the equations in Exercise 3, we see that

$$x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \cdot \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} t = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} t$$

and

$$y = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \cdot \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} t = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} t.$$

Hence, the solution in question is given by (*).

5. Substituting from (*) into the system of Exercise 3 results in two equations of the form 'a = a'. This shows that, for any t, the numbers determined by (*) satisfy the system of Exercise 3. Since this system is equivalent to (12) — subject to the condition (14) — we know that, for any t, the numbers determined by (*) satisfy (12).

6. Yes. Solving (12) under the assumption that either the second or the first of the determinants in (13) is nonzero will yield the same equations (*). The argument given in TC 443(2) for Exercise 6 of Part D applies here also.

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Details which are left to the reader:

The equations (**) describe different planes because, for example, (5, −4, 6) are the coordinates of a point which is in the plane described by the first equation but [since $(5 + 6) \cdot -6 + (-4 + 4) \cdot -4 + (6 + 2) \cdot 4 \neq 0$] does not belong to the plane described by the second equation.

Explanation called for in text:

If (6) and (6′) describe nonparallel planes then $\vec{m} \neq \vec{0} \neq \vec{n}$ and (6) and (6′) do not describe parallel planes. So, by the contrapositive of $(*_1)$, if (6) and (6′) describe nonparallel planes then $(\vec{m}, \vec{n})$ is linearly independent. [The contrapositive of $(*_1)$ is equivalent to:

> If (6) and (6′) do not describe parallel planes then $(\vec{m}, \vec{n})$ is linearly independent or $\vec{m} = \vec{0}$ or $\vec{n} = \vec{0}$.

The assumption that (6) and (6′) describe nonparallel planes implies the antecedent of this conditional as well as '$\vec{m} \neq \vec{0}$ and $\vec{n} \neq \vec{0}$'.]

We can now apply this result to the situation dealt with in Part A [page 450]. If the vectors $\vec{m}$ and $\vec{n}$ whose components are (m_1, m_2, m_3) and (n_1, n_2, n_3) are non-$\vec{0}$ then each of the equations:

$$(x_1 - a_1)m_1 + (x_2 - a_2)m_2 + (x_3 - a_3)m_3 = 0 \tag{6}$$

$$(x_1 - b_1)n_1 + (x_2 - b_2)n_2 + (x_3 - b_3)n_3 = 0 \tag{6'}$$

describes a plane. A vector $\vec{c}$ with components (c_1, c_2, c_3) belongs to the intersection of the bidirections of these planes if and only if

$$m_1c_1 + m_2c_2 + m_3c_3 = 0 \text{ and } n_1c_1 + n_2c_2 + n_3c_3 = 0.$$

[We have modified equations (10) slightly to conform more closely with the statement of Theorem B.] Also, by Theorem 10-14, $(\vec{m}, \vec{n})$ is linearly independent if and only if

$$\left(\begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix}, \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix}, \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix}\right) \neq (0, 0, 0). \tag{11}$$

[Of course, if $(\vec{m}, \vec{n})$ is linearly independent then $\vec{m}$ and $\vec{n}$ are non-$\vec{0}$.] So, if $(\vec{m}, \vec{n})$ is linearly independent then (6) and (6′) describe planes and, by Theorem B, $\vec{c}$ belongs to the intersection of the bidirections of these planes if and only if

$$\exists_t \left(c_1 = \begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix} t,\ c_2 = \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix} t, \text{ and } c_3 = \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} t \right).$$

In other words, the intersection of the bidirections of the two planes is precisely the direction of the non-$\vec{0}$ vector whose components are the determinants given in the left side of (11). In particular, the intersection of planes described by (6) and (6′) is a line—that is, the planes are nonparallel.

Although the discussion on pages 448-453, showing how to determine whether two equations describe the same plane, is quite long and involved, the result is easy to apply. For example, consider the equations:

$$\begin{cases} (x_1 - 5)3 + (x_2 + 4)2 + (x_3 - 6) \cdot -2 = 0 \\ (x_1 + 6) \cdot -2 + (x_2 - 5)3 + (x_3 + 2)9 = 0 \end{cases} \tag{*}$$

We compute the determinants, in this case, as described in (11), and note that

$$\left(\begin{vmatrix} 2 & -2 \\ 3 & 9 \end{vmatrix}, \begin{vmatrix} -2 & 3 \\ 9 & -2 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ -2 & 3 \end{vmatrix}\right) = (24, -23, 13) \neq (0, 0, 0).$$

So, the equations (*) describe nonparallel planes and these planes intersect in a line whose direction is that of the vector with components (24, −23, 13).

As a second example consider the equations:

$$\begin{cases} (x_1 - 5)3 + (x_2 + 4)2 + (x_3 - 6) \cdot -2 = 0 \\ (x_1 + 6) \cdot -6 + (x_2 + 4) \cdot -4 + (x_3 + 2)4 = 0 \end{cases} \tag{**}$$

Computing the determinants described in (11), we note that

$$\left(\begin{vmatrix} 2 & -2 \\ -4 & 4 \end{vmatrix}, \begin{vmatrix} -2 & 3 \\ 4 & -6 \end{vmatrix}, \begin{vmatrix} 6 & 2 \\ -6 & -4 \end{vmatrix}\right) = (0, 0, 0).$$

So, the equations (**) describe parallel planes. [It is left to the reader to determine whether the equations (**) describe distinct planes.]

Since we have seen earlier that if (6) and (6′) describe nonparallel planes then $(\vec{m}, \vec{n})$ is linearly independent [Explain; see $(*_1)$ on page 449], we have the following:

> **Theorem 10-16** Suppose that, with respect to a given coordinate system, the components of $\vec{m}$ and $\vec{n}$ are (m_1, m_2, m_3) and (n_1, n_2, n_3) respectively. With respect to the given coordinate system, the equations:
>
> $$\begin{cases} (x_1 - a_1)m_1 + (x_2 - a_2)m_2 + (x_3 - a_3)m_3 = 0 \\ (x_1 - b_1)n_1 + (x_2 - b_2)n_2 + (x_3 - b_3)n_3 = 0 \end{cases} \tag{i}$$
>
> describe nonparallel planes if and only if $(\vec{m}, \vec{n})$ is linearly independent; and, in this case,
>
> $$\left(\begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix}, \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix}, \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix}\right) \tag{ii}$$
>
> are the components of a non-$\vec{0}$ vector in the direction of the line of intersection of the planes.

We also have the corollaries:

Corollary 1 The equations (i) describe parallel planes if and only if $\vec{m}$ and $\vec{n}$ are non-$\vec{0}$ and $(\vec{m}, \vec{n})$ is linearly dependent.

Corollary 2 If $(\vec{m}, \vec{n})$ is linearly independent then the line of intersection of the planes described by the equations (i) is, itself, described by the system:

$$\begin{cases} x_1 = c_1 + \begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix} r \\ x_2 = c_2 + \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix} r \\ x_3 = c_3 + \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} r \end{cases}$$

where (c_1, c_2, c_3) are the coordinates of any chosen point C which is common to the two planes.

[Explain.] Both Theorem 10-16 and its corollaries apply equally well if equations (i) are replaced by:

$$\begin{cases} x_1m_1 + x_2m_2 + x_3m_3 = f \\ x_1n_1 + x_2n_2 + x_3n_3 = g \end{cases} \qquad \text{[Explain.]}$$

Part C

Here are six descriptions of planes.

$$\pi_1\text{: } 4x_1 - 6x_2 + 2x_3 = 5$$
$$\pi_2\text{: } 4x_1 + 6x_2 - 2x_3 = 7$$
$$\pi_3\text{: } 6x_1 - 9x_2 + 3x_3 = 10$$

$$\pi_4\text{: } \begin{cases} x_1 = 1 + 4r - 5s \\ x_2 = -3r + 4s \\ x_3 = 2 - r + 2s \end{cases}$$

$$\pi_5\text{: } \begin{cases} x_1 = \frac{7}{3} + r + 3s \\ x_2 = 4 + r + 2s \\ x_3 = \frac{32}{3} + r \end{cases}$$

$$\pi_6\text{: } \begin{cases} x_1 = 1 + 2r - 3s \\ x_2 = -r + 2s \\ x_3 = 2 + r \end{cases}$$

1. Which of these planes are parallel? [*Hint*: Find single equations which describe π_4, π_5, and π_6.]
2. Do any two of these descriptions describe the same plane?
3. Find an equation for the plane parallel to π_2 which contains the point whose coordinates are (1, 2, 3).

Answers for Part C

Taking the hint in Exercise 1, single equations for π_4, π_5, and π_6 are:

$$2x_1 + 3x_2 - x_3 = 0 \qquad [\text{for } \pi_4]$$
$$2x_1 - 3x_2 + x_3 = 10/3 \qquad [\text{for } \pi_5]$$
$$2x_1 + 3x_2 - x_3 = 0 \qquad [\text{for } \pi_6]$$

1. π_1, π_3, and π_5 are parallel; π_2, π_4, and π_6 are parallel
2. Yes. $\pi_4 = \pi_6$ and $\pi_3 = \pi_5$
3. One such equation is '$2x_1 + 3x_2 - x_3 = 5$'. [This is obtained by simplifying the equation '$(x_1 - 1)4 + (x_2 - 2)6 + (x_3 - 3)\cdot -2 = 0$'.]

4. One set of parametric equations for the required plane is:

$$\begin{cases} x_1 = 1 + 2r - 3s \\ x_2 = 2 - r + 2s \\ x_3 = 3 + r \end{cases}$$

[The plane described by these equations clearly has the same bidirection as π_6, namely $[\vec{p}, \vec{q}]$, where $\vec{p}$ and $\vec{q}$ have components (2, −1, 1) and (−3, 2, 0), respectively, and also very clearly contains the point with coordinates (1, 2, 3).]

5. $\pi_1 \cap \pi_2$ intersects the third coordinate plane in the point whose coordinates (x, y, 0) satisfy the system:

$$\begin{cases} 4x - 6y = 5 \\ 4x + 6y = 7 \end{cases}$$

So, $\pi_1 \cap \pi_2$ contains the point with coordinates (3/2, 1/6, 0). Also, $\pi_1 \cap \pi_2$ has the direction of the vector $\vec{m}$ whose components are

$$\left(\begin{vmatrix} -6 & 2 \\ 6 & -2 \end{vmatrix}, \begin{vmatrix} 2 & 4 \\ -2 & 4 \end{vmatrix}, \begin{vmatrix} 4 & -6 \\ 4 & 6 \end{vmatrix} \right) = (0, 16, 48).$$

Thus, by Corollary 2 of Theorem 10-16, parametric equations for $\pi_1 \cap \pi_2$ are:

$$\begin{cases} x_1 = 3/2 \\ x_2 = 1/6 + 16r \\ x_3 = 48r \end{cases}$$

6. Any line parallel to both π_1 and π_2 is parallel to $\pi_1 \cap \pi_2$. So, by Exercise 5, any such line is in the direction of the vector whose components are (0, 16, 48). Thus, parametric equations for the required line are:

$$\begin{cases} x_1 = 37 \\ x_2 = -96 + 16r \\ x_3 = 42 + 48r \end{cases}$$

4. Find parametric equations for the plane parallel to π_6 which contains the point with coordinates (1, 2, 3).
5. Find parametric equations for the line of intersection of π_1 and π_2. [*Hint*: This is the line $\overleftrightarrow{A[\vec{r}]}$ for any point $A \in \pi_1 \cap \pi_2$ and any non-$\vec{0}$ vector $\vec{r} \in [\pi_1] \cap [\pi_2]$. An easy choice for A is the point at which $\pi_1 \cap \pi_2$ intersects one of the coordinate planes.]
6. Find parametric equations for the line which is parallel to π_1 and to π_2 and contains the point with coordinates (37, −96, 42).

Part D

1. Show that, for

$$\left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right) \neq (0, 0, 0)$$

the solutions of the system:

$$(*) \qquad \begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$$

are given parametrically by:

$$x = x_0 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} t, \quad y = y_0 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} t, \quad z = z_0 + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} t,$$

where (x_0, y_0, z_0) is any chosen solution of (*). [*Hint*: Interpret the equations in (*) as equations of planes with respect to some coordinate system. Then proceed as in Exercise 5 of Part C.]

2. Find several solutions of each system:

(a) $\begin{cases} 3x - 5y + 7z = 2 \\ 2x + y - z = 3 \end{cases}$ (b) $\begin{cases} x + 2y - 3z = 0 \\ 3x - y + 4z = 0 \end{cases}$

Part E

Here are descriptions of a plane π and of lines l, m, and n

$$\pi\colon\ 5x_1 + 3x_2 - 4x_3 = 1$$

$$l\colon \begin{cases} x_1 = 1 + r \\ x_2 = 3r \\ x_3 = 3 + 2r \end{cases}, \qquad m\colon \begin{cases} x_1 = 1 + 3r \\ x_2 = 2 - r \\ x_3 = -2 + 3r \end{cases}, \qquad n\colon \begin{cases} x_1 = 2 + 2r \\ x_2 = 1 + 6r \\ x_3 = 3 + 7r \end{cases}$$

1. Find the coordinates of a point in which l intersects π. [*Hint*: Begin by finding a value of 'r' such that the corresponding coordinates given by the equations for l satisfy the equation for π. Do this by obtaining an equation whose solution is such a value of 'r'.]
2. Show that $m \parallel \pi$. [*Hint*: By a procedure like that use in Exercise 1, show that $m \cap \pi = \emptyset$.]
3. Show that $n \subseteq \pi$.

Answers for Part D

1. Since the determinant triple is not (0, 0, 0), it gives the components of a vector in the direction of the line of intersection of the planes described by (*). And, if (x_0, y_0, z_0) is any solution of (*), it follows that (x_0, y_0, z_0) are the coordinates of a point in the intersection of the planes described by (*). Hence, by Corollary 2 of Theorem 10-16, the given equations are parametric equations for the line of intersection of these planes, and the coordinates of all such points are the solutions for (*).

2. (a) The line of intersection of the planes described by the given equations intersects the third coordinate plane in the point (x, y, 0) which satisfies the system:

$$\begin{cases} 3x - 5y = 2 \\ 2x + y = 3 \end{cases}$$

This point, therefore, has coordinates (17/13, 5/13, 0). The line of intersection of the given planes is in the direction of $\vec{m}$, whose components are $\left(\begin{vmatrix} -5 & 7 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} 7 & 3 \\ -1 & 2 \end{vmatrix}, \begin{vmatrix} 3 & -5 \\ 2 & 1 \end{vmatrix}\right) = (-2, 17, 13)$.

So, all the solutions of the given system are of the form:

(17/13 - 2r, 5/13 + 17r, 13r), for some r.

(b) An analysis similar to that in part (a) shows that all the solutions of the given system are of the form:

(5r, -13r, -7r), for some r

Answers for Part E

1. If ℓ intersects π then there is an r such that 5(1 + r) + 3(3r) - 4(3 + 2r) = 1. The only such value of 'r' is 4/3. So, the coordinates of the point of intersection of ℓ and π are (7/3, 4, 17/3).
2. If m intersects π then there is an r such that 5(1 + 3r) + 3(2 - r) - 4(-2 + 3r) = 1, that is, such that 19 = 1. Since 19 ≠ 1, m does not intersect π. Hence m $\parallel$ π.
3. The points common to n and π are such that 5(2 + 2r) + 3(1 + 6r) - 4(3 + 7r) = 1, for some r. This is the case if and only if 1 = 1. So, for each r, the point whose coordinates are (2 + 2r, 1 + 6r, 3 + 7r) is a point of π. Thus, each point of n is a point of π. That is, n $\subseteq$ π.

Part F

1. Prove:

> **Theorem 10-17** Suppose that, with respect to a given coordinate system, the components of $\vec{l}$ and $\vec{p}$ are (l_1, l_2, l_3) and (p_1, p_2, p_3), respectively. With respect to the given coordinate system, the equations:
>
> $$\text{(i)} \qquad l_1x_1 + l_2x_2 + l_3x_3 = e$$
>
> and:
>
> $$\text{(ii)} \qquad \begin{cases} x_1 = a_1 + p_1r \\ x_2 = a_2 + p_2r \\ x_3 = a_3 + p_3r \end{cases}$$
>
> describe a plane and a line which are parallel if and only if $\vec{l}$ and $\vec{p}$ are non-$\vec{0}$ and
>
> $$l_1p_1 + l_2p_2 + l_3p_3 = 0;$$
>
> and, in this case, the line is a subset of the plane if and only if
>
> $$l_1a_1 + l_2a_2 + l_3a_3 = e.$$

2. Prove:

> **Corollary** The equations (i) and (ii) represent a plane and a transversal to this plane if and only if $l_1p_1 + l_2p_2 + l_3p_3 \neq 0$.

3. In each of the following, you are given equations for a plane π and a line l. You are to determine whether $\pi \parallel l$.

(a) π: $3x_1 + 5x_2 - 2x_3 = 7$

$$l: \begin{cases} x_1 = 2 + 2r \\ x_2 = 1 - 4r \\ x_3 = 2 - 7r \end{cases}$$

(b) π: $6x_1 - 7x_2 - 3x_3 = 12$

$$l: \begin{cases} x_1 = 8 + 2r \\ x_2 = 6 + r \\ x_3 = 11 + 6r \end{cases}$$

(c) π: $4x_1 + 7x_2 - x_3 = -5$

$$l: \begin{cases} x_1 = 4 + 3r \\ x_2 = -2 - 2r \\ x_3 = -7 + 3r \end{cases}$$

(d) π: $-9x_1 - 4x_2 + 2x_3 = 16$

$$l: \begin{cases} x_1 = 3 - 2r \\ x_2 = -5 + 2r \\ x_3 = 8 - 5r \end{cases}$$

4. **(a)** For those parts of Exercise 3 in which $\pi \parallel l$, determine whether $l \subseteq \pi$.

(b) For those parts of Exercise 3 in which $\pi \nparallel l$, determine the coordinates of the point in $l \cap \pi$.

Answers for Part F

1. Suppose, first, that (i) and (ii) describe a plane and a line which are parallel. Then, both $(\ell_1, \ell_2, \ell_3) \neq (0, 0, 0)$ and $(p_1, p_2, p_3) \neq (0, 0, 0)$. So, both $\vec{\ell}$ and $\vec{p}$ are non-$\vec{0}$. Then, either there is no r such that $\ell_1(a_1 + p_1r) + \ell_2(a_2 + p_2r) + \ell_3(a_3 + p_3r) = e$, that is, such that

$$(*) \quad (\ell_1p_1 + \ell_2p_2 + \ell_3p_3)r = e - (\ell_1a_1 + \ell_2a_2 + \ell_3a_3),$$

or every value of 'r' satisfies this equation. It follows that $\ell_1p_1 + \ell_2p_2 + \ell_3p_3 = 0$ and either $e \neq \ell_1a_1 + \ell_2a_2 + \ell_3a_3$ or $e = \ell_1a_1 + \ell_2a_2 + \ell_3a_3$, respectively. This proves the only if-part of the first part of the theorem and, also, proves the second part of the theorem.

Next, suppose that $\vec{\ell} \neq \vec{0} \neq \vec{p}$ and that $\ell_1p_1 + \ell_2p_2 + \ell_3p_3 = 0$. It follows that (i) and (ii) describe a line and a plane and, since (*) does not have a unique solution, that this line and plane are parallel.

2. By the theorem, if (i) and (ii) do not describe a parallel plane and line then $\vec{p} = \vec{0}$ or $\vec{\ell} = \vec{0}$ or $\ell_1p_1 + \ell_2p_2 + \ell_3p_3 \neq 0$. So, if (i) and (ii) describe a plane and a transversal then $\ell_1p_1 + \ell_2p_2 + \ell_3p_3 \neq 0$. [For, in this case, $\vec{p} \neq \vec{0} \neq \vec{\ell}$.] On the other hand, if $\ell_1p_1 + \ell_2p_2 + \ell_3p_3 \neq 0$ then $\vec{\ell} \neq \vec{0} \neq \vec{p}$ and, so, (i) and (ii) describe a plane and a line which, by the theorem, are not parallel. Hence, in this case, the line is a transversal of the plane.

The structure of the proof of the corollary is a bit tricky. If we symbolize the first part of the theorem by '$p \Longleftrightarrow (q \text{ and } r)$' and use 's' for '(i) and (ii) describe a plane and a transversal' then, in addition to the theorem, we have '$(\sim p \text{ and } q) \Longleftrightarrow s$' and '$\sim r \Longrightarrow q$'. Using the first of these and the if-part of the theorem we can derive [see below] '$s \Longrightarrow \sim r$'. Using both and the only if-part of the theorem we can derive [see below] '$\sim r \Longrightarrow s$'.

$$\dfrac{(\sim p \text{ and } q) \Longleftrightarrow s \qquad \dfrac{\dfrac{(q \text{ and } r) \Longrightarrow p}{\sim p \Longrightarrow (\sim q \text{ or } \sim r)}}{(\sim p \text{ and } q) \Longrightarrow \sim r}}{s \Longrightarrow \sim r}$$

$$\dfrac{(\sim p \text{ and } q) \Longleftrightarrow s \qquad \dfrac{\sim r \Longrightarrow q \qquad \dfrac{\dfrac{p \Longrightarrow (q \text{ and } r)}{(\sim q \text{ or } \sim r) \Longrightarrow \sim p}}{\sim r \Longrightarrow \sim p}}{\sim r \Longrightarrow (\sim p \text{ and } q)}}{\sim r \Longrightarrow s}$$

3. (a) $3 \cdot 2 + 5 \cdot -4 + -2 \cdot -7 = 0$. So, $\pi \parallel \ell$.
   (b) $6 \cdot 2 + -7 \cdot 1 + -3 \cdot 6 = -13 \neq 0$. So, $\pi \nparallel \ell$.
   (c) $4 \cdot 3 + 7 \cdot -2 + -1 \cdot 3 = -5 \neq 0$. So, $\pi \nparallel \ell$.
   (d) $-9 \cdot -2 + -4 \cdot 2 + 2 \cdot -5 = 0$. So, $\pi \parallel \ell$.

10.11 Third Order Determinants

By Theorem 10–16 we know that if $(\vec{m}, \vec{n})$ is linearly independent then the equations:

$$x_1m_1 + x_2m_2 + x_3m_3 = f$$
$$x_1n_1 + x_2n_2 + x_3n_3 = g$$

describe two planes whose intersection is a line. By the corollary to Theorem 10–16 we know that if $\vec{m}$ and $\vec{n}$ are non-$\vec{0}$ and $(\vec{m}, \vec{n})$ is linearly dependent then each equation describes a plane and the plane described by one equation is parallel to the plane described by the other. We also know that if $\vec{m}$, say, is $\vec{0}$ then the first equation describes $\varnothing$ [in case $f \neq 0$] or $\mathscr{E}$ [in case $f = 0$]. These geometric results can be used to obtain information concerning the solutions of a system:

$$(1) \qquad \begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$$

As we saw in Part D, such a system has a "one-parameter family of solutions" in case

$$(2) \qquad \left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right) \neq (0, 0, 0).$$

The preceding geometrical argument shows what may happen if (2) is not satisfied. There are three possibilities:

(i) The system (1) may have no solutions.
(ii) The system (1) may have a "two-parameter family of solutions".
(iii) Every triple (x, y, z) may be a solution of (1).

[Explain.]

A similar geometric analysis will lead us to an understanding of the solutions of a system consisting of *three* equations like those in (1). To begin with, recall the corollary to Theorem 10–17 in which you showed that a plane described by:

$$(3) \qquad x_1l_1 + x_2l_2 + x_3l_3 = e$$

and a line described by parametric equations:

$$(4) \qquad \begin{cases} x_1 = a_1 + p_1r \\ x_2 = a_2 + p_2r \\ x_3 = a_3 + p_3r \end{cases}$$

intersect in a single point if and only if

$$l_1p_1 + l_2p_2 + l_3p_3 \neq 0.$$

Answers for Part F [cont.]

4. (a) In 3(a), $\pi \parallel \ell$. Since $3 \cdot 2 + 5 \cdot 1 - 2 \cdot 2 = 7$, $\ell \subseteq \pi$.

In 3(d), $\pi \parallel \ell$. Since $-9 \cdot 3 - 4 \cdot -5 + 2 \cdot 8 = 9 \neq 16$, $\ell \not\subseteq \pi$.

(b) In 3(b), $\pi \not\parallel \ell$. So, for some unique r, $6(8 + 2r) - 7(6 + r) - 3(11 + 6r) = 12$. Simplifying, $r = -3$. The coordinates of the point in $\ell \cap \pi$ are $(2, 3, -7)$.

In 3(c), $\pi \not\parallel \ell$. So, for some unique r, $4(4 + 3r) + 7(-2 - 2r) - (-7 + 3r) = -5$. Simplifying, $r = 14/5$. So, the coordinates of the point in $\ell \cap \pi$ are $(62/5, -38/5, 7/5)$.

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Explanation asked for at end of first paragraph:

Suppose (2) is not satisfied. [This corresponds to the preceding situation in which $(\vec{m}, \vec{n})$ is linearly dependent.] If $(a_1, b_1, c_1) \neq (0, 0, 0)$ then there is a number — say, k — such that $a_2 = a_1k$, $b_2 = b_1k$, and $c_2 = c_1k$. In this case if $d_2 \neq d_1k$ the system has no solution, while if $d_2 = d_1k$ the solutions of the system are those of the first equation and, so, constitute a two-parameter family. [If, for example, $a_1 \neq 0$, the solutions of the first equation can be found by assigning values to the two variables — or parameters — 'y' and 'z' and computing the corresponding value of 'x'.] Similar remarks apply in case $(a_2, b_2, c_2) \neq (0,0,0)$. Suppose, then, that $(a_1, b_1, c_1) = (0, 0, 0) = (a_2, b_2, c_2)$. In this case if d_1 or d_2 is not 0 the system has no solutions, while if $d_1 = 0 = d_2$ the system is satisfied by any triple.

[Under what conditions are the plane and line parallel? Under what further condition is the line a subset of the plane?] Let's apply this result to the case in which the line is given to us, not by (4), but as the intersection of two planes described by the equations:

$$x_1m_1 + x_2m_2 + x_3m_3 = f \qquad (5)$$
$$x_1n_1 + x_2n_2 + x_3n_3 = g$$

In this case we know from Theorem 10–16 that the line has parametric equations, like (4), where

$$p_1 = \begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix}, \; p_2 = \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix}, \text{ and } p_3 = \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix}.$$

So, in this case, the plane and line intersect in a unique point if and only if

$$(6) \qquad l_1 \begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix} + l_2 \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix} + l_3 \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} \neq 0.$$

What we have shown is that in case

$$(7) \qquad (l_1, l_2, l_3) \neq (0, 0, 0) \text{ and } \left(\begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix}, \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix}, \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} \right) \neq (0, 0, 0)$$

the system consisting of the three equations in (3) and (5) has a unique solution if and only if (6) is satisfied. Now, if (6) is satisfied then so is (7). Hence, if (6) is satisfied then our system of three equations has a unique solution.

As an example, consider the system of equations:

$$(*) \qquad \begin{cases} 3x_1 - 2x_2 + 5x_3 = 7 \\ 4x_1 + 3x_2 + 6x_3 = 3 \\ 8x_1 - 7x_2 - 2x_3 = 5 \end{cases}$$

To determine whether the system (*) has a unique solution, we make use of (6) to do the following computations, and note that

$$3\begin{vmatrix} 3 & 6 \\ -7 & -2 \end{vmatrix} + -2\begin{vmatrix} 6 & 4 \\ -2 & 8 \end{vmatrix} + 5\begin{vmatrix} 4 & 3 \\ 8 & -7 \end{vmatrix}$$
$$= 3 \cdot 36 + -2 \cdot 56 + 5 \cdot -52 = -264 \neq 0.$$

Answer to question.

The plane and line are parallel if and only if $\ell_1p_1 + \ell_2p_2 + \ell_3p_3 = 0$. [Implicitly, since it is assumed that the equations describe a plane and a line, $\vec{\ell} \neq \vec{0} \neq \vec{p}$. The line is a subset of the plane if and only if, in addition, $\ell_1a_1 + \ell_2a_2 + \ell_3a_3 = e$.]

Thus, we know that the system of equations (*) has a unique solution. [As a matter of fact, we are well on our way to finding the solution. In a short time we shall be able to make use of these computations to help us obtain it.]

Suppose, on the other hand, that the system consisting of (3) and (5) has a unique solution. We know that (3) describes either a plane or $\varnothing$ or $\mathscr{E}$ and that (5) describes either a line or a plane or $\varnothing$ or $\mathscr{E}$. Since, together, they describe a set consisting of a single point it follows that (3) must describe a plane and (5) must describe a line. So, if the system consisting of (3) and (5) has a unique solution then (7) is satisfied. But we have seen that in case (7) is satisfied it follows that if the system has a unique solution then (6) holds.

The preceding argument shows that we can forget about (7); the system consisting of the equations in (3) and (5) has a unique solution if and only if (6) is satisfied. In our "*a, b, c, d, x, y, z*-notation" this means that the system of equations:

$$\text{(8)}\qquad \begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

has a unique solution if and only if

$$\text{(9)}\qquad a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \neq 0.$$

This result should remind you of Theorem A, the fundamental theorem concerning solutions of a system of equations in two variables:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

which has been the basis for all our work in Sections 10.08 through 10.11. In fact, by a definition which we shall state shortly, sentence (9) can be replaced by:

$$\text{(10)}\qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

and, as in the case of Theorem A, our new result can be completed to give another fundamental theorem concerning the solution of equations in three variables:

We shall not ask students to prove the second part of Theorem C. A proof by direct substitution is messy and the "natural" way to prove the theorem requires a knowledge of fourth order determinants or their equivalent. As an indication of this "natural" proof we give here the analogous proof for the second part of Theorem A by using properties of third order determinants.

We wish to show that, if $a_1b_2 - b_1a_2 \neq 0$, the system:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

has

$$\left(\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \Big/ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},\ \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \Big/ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right)$$

as a solution. Substituting in the first of the two equations and clearing of fractions we have:

$$a_1\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} + b_1\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = c_1\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

From properties of second order determinants this is equivalent to:

$$a_1\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_1\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_1\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$

By the definition of third order determinants, this is equivalent to:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

But this is the case since a third order determinant with a repeated row is 0. A similar argument shows that the reputed solution satisfies the second of the two equations.

Once fourth order determinants are defined and it is shown that such a determinant with a repeated row is 0, a similar proof can be given for the second part of Theorem C. [One also needs to know the result concerning interchanging columns of third order determinants which is given in Exercise 2 of Part B on page 463.]

Answer to question in the text:

$$\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} d_2 & c_2 \\ d_3 & c_3 \end{vmatrix} + d_1\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & d_2 \\ a_3 & d_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & d_2 \\ b_3 & d_3 \end{vmatrix} + b_1\begin{vmatrix} d_2 & a_2 \\ d_3 & a_3 \end{vmatrix} + d_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Theorem C

The system of equations:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

has a unique solution if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

and, in this case, the given system of equations is equivalent to:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\ y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\ z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Note that, at least by analogy with (9) and (10), the numerator of the first of these three fractions:

$$\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

is an abbreviation for:

$$(**) \qquad d_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} c_2 & d_2 \\ c_3 & d_3 \end{vmatrix} + c_1 \begin{vmatrix} d_2 & b_2 \\ d_3 & b_3 \end{vmatrix}$$

[What are the other numerators abbreviations for ?]

The definition of third order determinanting makes use of what is sometimes called "the expansion of a third order determinant with respect to the elements of its first row." [As remarked in Exercise 1 of Part A on the following page, students should remember the definition in these terms, rather than in terms of 'a's, 'b's, and 'c's.] Exercise 2 of Part B can be used to justify the similar expressions of a third order determinant with respect to the elements of its second or third row; Exercise 1 then justifies expanding such a determinant according to the elements of any of its columns. You may wish to indicate these applications of Exercises 1 and 2 and to give the usual mnemonic for recalling what signs to use in such an expansion:

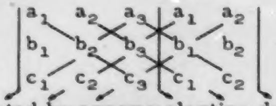
$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

For example:

$$\begin{aligned} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= + a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= - b_1\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \\ &= - c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \end{aligned}$$

etc.

It is probably better not to introduce the mnemonic:



in which products indicated by arrows slanting down to the right are added and those indicated by arrows slanting down to the left are subtracted. A somewhat similar device does work for second order determinants, but there is no analogous device for determinants of higher order. So, if the device in question is introduced for evaluating third order determinants, students are bound to formulate an incorrect generalization which you will not have an opportunity to correct.

We recommend that you use Part A as a class illustration of the use of third order determinants. Parts B and C can then be used for homework. Including Part D in this assignment will make it rather long but since Part D alone is rather short you may wish to combine these three parts. One way to make the combination of Parts B - D more reasonable for homework would be to have students do only one of Exercises 1, 2, and 4 of Part C.

As a basis for the following exercises we need a definition of *the determinant of the triple* $((a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3))$ *of* [3-termed] *sequences*. In analogy with the definition on page 434 of the determinant of $((a_1, a_2), (b_1, b_2))$, this new definition is usually stated as:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

[Note that the '−' on the right side of this definition can be replaced by '+' if one also interchanges the columns in the middle determinant.] The determinants we have dealt with up to now are called *second order determinants*; the new ones are called *third order determinants*. As in the case of second order determinants, the *rows* of the third order determinant given above are the sequences (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3); its *columns* are the sequences (a_1, b_1, c_1), (a_2, b_2, c_2), and (a_3, b_3, c_3). As you will see, third order determinants have all the properties you established in Part B of Section 10.08 for second order determinants.

Exercises

Part A

1. Use the definition of third order determinants to show that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

[*Hint*: Instead of making replacements like 'b_1' for 'a_2' in the definition, interpret the definition in terms of rows and columns. For example, in applying the definition to any determinant, you begin by multiplying the first term in the first row by the determinant of the sequences obtained by deleting the first terms from the second and third rows. Now, look at the definition and describe the second step.]

2. (a) Show that

$$\begin{vmatrix} 2 & 3 & -4 \\ -1 & -2 & 6 \\ 5 & 7 & 2 \end{vmatrix} = -8.$$

Answers for Part A

1. By definition, and what we know of second order determinants,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

= left side of (9).

[The second step in evaluating a third order determinant is to subtract the product of the second term in the first row by the determinant of the sequences obtained by deleting the second terms from the second and third rows; the third step is to add the product of --- etc.]

2. (a)

$$\begin{vmatrix} 2 & 3 & -4 \\ -1 & -2 & 6 \\ 5 & 7 & 2 \end{vmatrix} = 2 \begin{vmatrix} -2 & 6 \\ 7 & 2 \end{vmatrix} - 3 \begin{vmatrix} -1 & 6 \\ 5 & 2 \end{vmatrix} + -4 \begin{vmatrix} -1 & -2 \\ 5 & 7 \end{vmatrix}$$

$$= 2 \cdot -46 - 3 \cdot -32 + -4 \cdot 3$$

$$= -8$$

(b) Use Theorem C to find the solution of the system:

$$\begin{cases} 2x + 3y - 4z = 1 \\ -x - 2y + 6z = 2 \\ 5x + 7y + 2z = 3 \end{cases}$$

(c) Check the result of part (b) by substituting in the given equations.

3. (a) Evaluate the determinant:

$$\begin{vmatrix} 2 & -1 & 5 \\ 3 & -2 & 7 \\ -4 & 6 & 2 \end{vmatrix}$$

(b) Compare the determinant in part (a) with that in Exercise 2(a). If you have evaluated both correctly, the results should not surprise you. Explain.

Part B

Prove each of the following.

1. $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

[*Hint*: Use the definition or (**) to express the left side in terms of three second order determinants. Transform your result into the similar expression for the right side given in the definition or (**).]

2. (a) $$\begin{vmatrix} a_2 & a_1 & a_3 \\ b_2 & b_1 & b_3 \\ c_2 & c_1 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(b) $$\begin{vmatrix} a_1 & a_3 & a_2 \\ b_1 & b_3 & b_2 \\ c_1 & c_3 & c_2 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(c) $$\begin{vmatrix} a_3 & a_2 & a_1 \\ b_3 & b_2 & b_1 \\ c_3 & c_2 & c_1 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

[*Hint*: For parts (a) and (b), proceed as in Exercise 1, recalling Exercise 2(a) of Part B in Section 10.08. Part (c) can be established in the same way. It can also be derived from parts (a) and (b).]

(b) By Theorem C and the result in (a), we have that

$$x = \frac{\begin{vmatrix} 1 & 3 & -4 \\ 2 & -2 & 6 \\ 3 & 7 & 2 \end{vmatrix}}{-8}, \quad y = \frac{\begin{vmatrix} 2 & 1 & -4 \\ -1 & 2 & 6 \\ 5 & 3 & 2 \end{vmatrix}}{-8}, \quad z = \frac{\begin{vmatrix} 2 & 3 & 1 \\ -1 & -2 & 2 \\ 5 & 7 & 3 \end{vmatrix}}{-8}$$

so that the solution of the system is $(21/2, -7, -1/4)$.

(c) $$2 \cdot \frac{21}{2} + 3 \cdot -7 - 4 \cdot -\frac{1}{4} = 21 - 21 + 1 = 1$$

$$-\frac{21}{2} - 2 \cdot -7 + 6 \cdot -\frac{1}{4} = -\frac{21}{2} + 14 - \frac{3}{2} = 2$$

$$5 \cdot \frac{21}{2} + 7 \cdot -7 + 2 \cdot -\frac{1}{4} = \frac{105}{2} - 49 - \frac{1}{2} = 3$$

3. (a) $$\begin{vmatrix} 2 & -1 & 5 \\ 3 & -2 & 7 \\ -4 & 6 & 2 \end{vmatrix} = 2\begin{vmatrix} -2 & 7 \\ 6 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 7 \\ -4 & 2 \end{vmatrix} + 5\begin{vmatrix} 3 & -2 \\ -4 & 6 \end{vmatrix}$$

$$= 2 \cdot -46 + 34 + 5 \cdot 10 = -8.$$

(b) The results are the same. The determinant in 3(a) is obtained from that in 2(a) by interchanging the rows with the columns in 2(a). This [as we should have expected] results in a determinant of equal value.

Answers for Part B

1. $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1(c_2a_3 - a_2c_3) + c_1(a_2b_3 - b_2a_3)$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2(c_1b_3 - b_1c_3) + a_3(b_1c_2 - c_1b_2)$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2\begin{vmatrix} c_1 & b_1 \\ c_3 & b_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Answers for Part B [cont.]

2. (a)
$$\begin{vmatrix} a_2 & a_1 & a_3 \\ b_2 & b_1 & b_3 \\ c_2 & c_1 & c_3 \end{vmatrix} = a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_1\begin{vmatrix} b_3 & b_2 \\ c_3 & c_2 \end{vmatrix} + a_3\begin{vmatrix} b_2 & b_1 \\ c_2 & c_1 \end{vmatrix}$$
$$= -a_2\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} - a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$
$$= -\left(a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right)$$
$$= -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(b)
$$\begin{vmatrix} a_1 & a_3 & a_2 \\ b_1 & b_3 & b_2 \\ c_1 & c_3 & c_2 \end{vmatrix} = a_1\begin{vmatrix} b_3 & b_2 \\ c_3 & c_2 \end{vmatrix} + a_3\begin{vmatrix} b_2 & b_1 \\ c_2 & c_1 \end{vmatrix} + a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$$
$$= -a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} - a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$
$$= -\left(a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right)$$
$$= -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(c)
$$\begin{vmatrix} a_3 & a_2 & a_1 \\ b_3 & b_2 & b_1 \\ c_3 & c_2 & c_1 \end{vmatrix} = -\begin{vmatrix} a_3 & a_1 & a_2 \\ b_3 & b_1 & b_2 \\ c_3 & c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_3 & a_2 \\ b_1 & b_3 & b_2 \\ c_1 & c_3 & c_2 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

As indicated in the commentary for page 460 Exercises 1 and 2, together with the definition, justify the expansion of a third order determinant with respect to the elements of any row or column.

3. In Exercise 3 of Part A, you evaluated the determinant:

$$\begin{vmatrix} 2 & -1 & 5 \\ 3 & -2 & 7 \\ -4 & 6 & 2 \end{vmatrix}$$

Without doing any more extensive computing, evaluate each of the following:

(a) $\begin{vmatrix} -1 & 2 & 5 \\ -2 & 3 & 7 \\ 6 & -4 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 5 & -1 & 2 \\ 7 & -2 & 3 \\ 2 & 6 & -4 \end{vmatrix}$ (c) $\begin{vmatrix} 3 & 2 & -4 \\ -2 & -1 & 6 \\ 7 & 5 & 2 \end{vmatrix}$

4. In Exercise 2 you have seen that interchanging two columns of a given third order determinant results in a determinant whose value is the opposite of that of the given determinant. What other property must third order determinants have in consequence of this and the result in Exercise 1?

5. (a) $$\begin{vmatrix} a_1 + d_1 & a_2 + d_2 & a_3 + d_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(b) What follows from this together with Exercises 1 and 2?

6. (a) Evaluate these determinants:

$$\begin{vmatrix} 2 & 1 & 3 \\ 5 & -1 & 2 \\ 3 & 2 & 2 \end{vmatrix} \qquad \begin{vmatrix} 3 & -1 & -2 \\ 5 & -1 & 2 \\ 3 & 2 & 2 \end{vmatrix}$$

(b) Make use of the results in part (a) to evaluate the following:

(i) $\begin{vmatrix} 5 & 0 & 1 \\ 5 & -1 & 2 \\ 3 & 2 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & 5 & 3 \\ -1 & -1 & 2 \\ -2 & 2 & 2 \end{vmatrix}$

(iii) $\begin{vmatrix} -2 & 3 & -1 \\ 2 & 5 & -1 \\ 2 & 3 & 2 \end{vmatrix}$ (iv) $\begin{vmatrix} 2 & 1 & 3 \\ 45 & -9 & 18 \\ 3 & 2 & 2 \end{vmatrix}$

7. (a) What can you say about a third order determinant which has the same sequence for two of its rows?

(b) About a determinant which has the sequence (0, 0, 0) as one of its rows?

Answers for Part B [cont.]

3. (a) 8 [by 2(a)]
   (b) 8 [by 2(c)]
   (c) 8 [by 1 and 2]

4. Interchanging two rows of a given third order determinant results in a determinant whose value is the opposite of that of the given determinant.

5. (a)
$$\begin{vmatrix} a_1 + d_1 & a_2 + d_2 & a_3 + d_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (a_1 + d_1)\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + (a_2 + d_2)\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + (a_3 + d_3)\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} + d_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + d_2\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + d_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(b)
$$\begin{vmatrix} a_1 + a_4 & a_2 & a_3 \\ b_1 + b_4 & b_2 & b_3 \\ c_1 + c_4 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_4 & a_2 & a_3 \\ b_4 & b_2 & b_3 \\ c_4 & c_2 & c_3 \end{vmatrix}$$

[And four similar results in which the terms of the second or third row or column are indicated sums.]

6. (a)
$$\begin{vmatrix} 2 & 1 & 3 \\ 5 & -1 & 2 \\ 3 & 2 & 2 \end{vmatrix} = 2 \cdot -6 + 1 \cdot -4 + 3 \cdot 13 = 23$$

$$\begin{vmatrix} 3 & -1 & -2 \\ 5 & -1 & 2 \\ 3 & 2 & 2 \end{vmatrix} = 3 \cdot -6 + -1 \cdot -4 + -2 \cdot 13 = -40$$

(b) (i) −17 [23 + −40, by Ex. 5(a)]
   (ii) −40 [by Ex. 1]
   (iii) −40 [by Ex. 2]
   (iv) 207 [23 · 9, by repeated use of Ex. 5(b)]

7. (a) Such a determinant has value 0 [for by interchanging these rows, we see that it is equal to its opposite].

(b) Such a determinant has value 0 [for it is equal to or the opposite of a determinant whose first row is (0, 0, 0), and, from the definition, such a determinant is easily seen to have value 0].

Part C

Here is an important corollary of the first part of Theorem C on page 461:

Corollary The system of equations:

$$\begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{cases}$$

has a nontrivial solution—that is, has a solution other than (0, 0, 0)—if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

1. Show that this corollary is implied by Theorem C. [*Hint*: See the discussion preceding Part C of Section 10.08.]
2. Prove:

Theorem 10-18 For $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ linearly independent,

$$(\vec{u}_1a_1 + \vec{u}_2a_2 + \vec{u}_3a_3, \vec{u}_1b_1 + \vec{u}_2b_2 + \vec{u}_3b_3, \vec{u}_1c_1 + \vec{u}_2c_2 + \vec{u}_3c_3)$$

is linearly dependent

$$\Longleftrightarrow$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

[*Hint*: See Exercise 1 of Part B in Section 10.09.]

3. In each part you are given the components, with respect to some basis, of three vectors. Decide whether the given vectors are linearly dependent or linearly independent.
   - **(a)** (3, 4, −2), (5, −6, 3), (1, 14, −7)
   - **(b)** (3, −2, 6), (1, 0, 2), (4, 6, −1)
   - **(c)** (1, −2, 3), (0, 3, 5), (0, 0, −2)
   - **(d)** (−2, 3, −1), (3, −1, −2), (−1, −2, 3)

Answers for Part C

1. Clearly, the given system has the solution (0, 0, 0). By Theorem C, this solution is unique — that is, is the only solution — if and only if the given determinant is not 0. Thus, this solution is not unique — that is, there are other solutions — if and only if the given determinant is 0. But, any solution different from (0, 0, 0) is nontrivial.

2. $(\vec{u}_1a_1 + \vec{u}_2a_2 + \vec{u}_3a_3, \vec{u}_1b_1 + \vec{u}_2b_2 + \vec{u}_3b_3, \vec{u}_1c_1 + \vec{u}_2c_2 + \vec{u}_3c_3)$ is linearly dependent if and only if there are numbers x, y, and z, not all zero, such that

$$(\vec{u}_1a_1 + \vec{u}_2a_2 + \vec{u}_3a_3)x + (\vec{u}_1b_1 + \vec{u}_2b_2 + \vec{u}_3b_3)y + (\vec{u}_1c_1 + \vec{u}_2c_2 + \vec{u}_3c_3)z = \vec{0}.$$

The latter is the case if and only if

$$\vec{u}_1(a_1x + b_1y + c_1z) + \vec{u}_2(a_2x + b_2y + c_2z) \\ \vec{u}_3(a_3x + b_3y + c_3z) = \vec{0}.$$

Since $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is linearly independent, this is the case if and only if

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0.$$

And, by the corollary to Theorem C and Exercise 1 of Part A, this system has a nontrivial solution if and only if $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$

Hence, the theorem.

3. (a) $\begin{vmatrix} 3 & 4 & -2 \\ 5 & -6 & 3 \\ 1 & 14 & -7 \end{vmatrix} = 3\cdot 0 + 4\cdot 38 - 2\cdot 76 = 0$

So, the given vectors are linearly dependent.

(b) $\begin{vmatrix} 3 & -2 & 6 \\ 1 & 0 & 2 \\ 4 & 6 & -1 \end{vmatrix} = 3\cdot -12 + -2\cdot 9 + 6\cdot 6 = -18 \neq 0$

So, the given vectors are linearly independent.

(c) $\begin{vmatrix} 1 & -2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{vmatrix} = 1\cdot -6 + -2\cdot 0 + 3\cdot 0 = -6 \neq 0$

So, the given vectors are linearly independent.

(d) $\begin{vmatrix} -2 & 3 & -1 \\ 3 & -1 & -2 \\ -1 & -2 & 3 \end{vmatrix} = -2\cdot -7 + 3\cdot -7 + -1\cdot -7 = 0$

So, the given vectors are linearly dependent.

4. Use Theorem C on page 461 and its corollary to show that three planes intersect in a single point if and only if they are not all parallel to the same line. [*Hint*: Suppose that the planes are described, with respect to some coordinate system, by the equations:

$$\begin{cases} l_1x_1 + l_2x_2 + l_3x_3 = e \\ m_1x_1 + m_2x_2 + m_3x_3 = f \\ n_1x_1 + n_2x_2 + n_3x_3 = g \end{cases}$$

What conditions must be satisfied by components (p_1, p_2, p_3) of a vector which belongs to each of the directions of these planes?]

5. Which of the following systems of equations have nontrivial solutions?

(a) $\begin{cases} x_1 + 2x_2 - 3x_3 = 0 \\ 4x_1 - 6x_2 + 2x_3 = 0 \\ 2x_1 - 3x_2 + x_3 = 0 \end{cases}$
(b) $\begin{cases} -2x_1 + 3x_2 - 4x_3 = 0 \\ 12x_1 - 9x_2 - 2x_3 = 0 \\ 4x_1 - 6x_2 + 8x_3 = 0 \end{cases}$

(c) $\begin{cases} 3x_1 - 5x_2 + 7x_3 = 0 \\ 2x_1 + 3x_2 - 4x_3 = 0 \\ -2x_1 - 5x_2 - x_3 = 0 \end{cases}$
(d) $\begin{cases} x_1 + 6x_3 = 0 \\ x_1 + 2x_2 = 0 \\ x_2 - 3x_3 = 0 \end{cases}$

*

Suppose that π and σ are planes which are described by the parametric equations:

$$\begin{cases} x_1 = c_1 + p_1r + q_1s \\ x_2 = c_2 + p_2r + q_2s \\ x_3 = c_3 + p_3r + q_3s \end{cases} \text{and: } \begin{cases} x_1 = d_1 + a_1r + b_1s \\ x_2 = d_2 + a_2r + b_2s \\ x_3 = d_3 + a_3r + b_3s \end{cases}$$

One way to determine whether or not $\pi \parallel \sigma$ is to obtain single equations for π and σ and proceed from there. [Explain how, having done this, we can decide whether or not $\pi \parallel \sigma$.] Another way is to note that $[\pi] = [\vec{p}, \vec{q}]$, where $\vec{p}$ and $\vec{q}$ have components (p_1, p_2, p_3) and (q_1, q_2, q_3) and that, similarly, $[\sigma] = [\vec{a}, \vec{b}]$, where $\vec{a}$ and $\vec{b}$ have components (a_1, a_2, a_3) and (b_1, b_2, b_3) respectively. Now, since $(\vec{a}, \vec{b})$ is linearly independent, we can show that $[\pi] = [\sigma]$ — and, so, that $\pi \parallel \sigma$ — by showing that $\{\vec{a}, \vec{b}\} \subseteq [\vec{p}, \vec{q}]$. Since $(\vec{p}, \vec{q})$ is linearly independent, this will be the case if and only if each of $(\vec{a}, \vec{p}, \vec{q})$ and $(\vec{b}, \vec{p}, \vec{q})$ is linearly dependent. As we are now in a position to make use of third order determinants to check the linear dependence of these sequences [see Theorem 10-18], we can make use of the information in the parametric equations to decide whether or not $\pi \parallel \sigma$.

Answers for Part C [cont.]

4. Using the equations in the hint the three planes are parallel to a line whose direction is that of the non-$\vec{0}$ vector with components (p_1, p_2, p_3) if and only if

$$\begin{aligned} \ell_1p_1 + \ell_2p_2 + \ell_3p_3 &= 0, \\ m_1p_1 + m_2p_2 + m_3p_3 &= 0, \text{ and} \\ n_1p_1 + n_2p_2 + n_3p_3 &= 0. \end{aligned}$$

The condition that this system of equations not have a non-$\vec{0}$ solution (p_1, p_2, p_3) is the same as the condition that the equations in the hint have a unique solution. So, the planes are not parallel to any line if and only if they intersect in a single point.

5. In each case all we need to do is make use of the corollary to Theorem C.

(a) $\begin{vmatrix} 1 & 2 & -3 \\ 4 & -6 & 2 \\ 2 & -3 & 1 \end{vmatrix} = 0$

System has a nontrivial solution.

(b) $\begin{vmatrix} -2 & 3 & -4 \\ 12 & -9 & -2 \\ 4 & -6 & 8 \end{vmatrix} = 0$

System has a nontrivial solution.

(c) $\begin{vmatrix} 3 & -5 & 7 \\ 2 & 3 & -4 \\ -2 & -5 & -1 \end{vmatrix} = -147 \neq 0$

System has no nontrivial solutions.

(d) $\begin{vmatrix} 1 & 0 & 6 \\ 1 & 2 & 0 \\ 0 & 1 & -3 \end{vmatrix} = 0$

System has a nontrivial solution.

Part D

In each of the following you are given parametric equations for planes π and σ. Determine, in each part, whether or not $\pi \parallel \sigma$ by evaluating two third order determinants.

1. $\pi: \begin{cases} x_1 = 2 + 3r - s \\ x_2 = -1 + r + 3s \\ x_3 = 17 - r + 2s \end{cases}$ $\sigma: \begin{cases} x_1 = -1 - r + 3s \\ x_2 = 2 + r + 13s \\ x_3 = 3 + r + 5s \end{cases}$

2. $\pi: \begin{cases} x_1 = 5 - r - 2s \\ x_2 = -2 + 5r + s \\ x_3 = -3 - 3r - 5s \end{cases}$ $\sigma: \begin{cases} x_1 = 2 + 2r + s \\ x_2 = -4 - 3r \\ x_3 = 3 + 2s \end{cases}$

3. $\pi: \begin{cases} x_1 = 4 + 2r + s \\ x_2 = 5 + 2s \\ x_3 = 7 \end{cases}$ $\sigma: \begin{cases} x_1 = 2 + r - s \\ x_2 = 2 + 3s \\ x_3 = 1 + 5r + s \end{cases}$

10.12 Chapter Summary

Vocabulary Summary

n-dimensional vector space
basis for $\mathscr{T}$
dimension of a vector space
components of a vector
determinant
    second order
    third order
    rows
    columns

3-dimensional space of points
span $\mathscr{T}$
coordinate system
coordinates of a point
parameter
parametric equations
    of a line
    of a plane
nontrivial solution

Postulates

1. **(a)** $B - A \in \mathscr{T}$ **(b)** $A + \vec{a} \in \mathscr{E}$
2. **(a)** $A + (B - A) = B$ **(b)** $\vec{a} = (A + \vec{a}) - A$
3. $(B - A) + (C - B) = C - A$

4_0. **(a)** $\vec{a} + \vec{b} \in \mathscr{T}$ **(b)** $\vec{0} \in \mathscr{T}$ **(c)** $-\vec{a} \in \mathscr{T}$ **(d)** $\overrightarrow{aa} \in \mathscr{T}$
4_1. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
4_2. $\vec{a} + \vec{0} = \vec{a}$
4_3. $\vec{a} + -\vec{a} = \vec{0}$
4_4. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
‖ 4‴. $\mathscr{T}$, under function composition, is a commutative group.
4_5. $\vec{a}1 = \vec{a}$
4_6. $\vec{a}(b + c) = \overrightarrow{ab} + \overrightarrow{ac}$
4_7. $(\vec{a} + \vec{b})c = \overrightarrow{ac} + \overrightarrow{bc}$
4_8. $(\overrightarrow{ab})c = \vec{a}(bc)$
‖ 4″. $\mathscr{T}$, under function composition, is a vector space over $\mathscr{R}$.
4_9. There are three linearly independent members of $\mathscr{T}$.
4_{10}. There are not four linearly independent members of $\mathscr{T}$.
‖ 4′. $\mathscr{T}$, under function composition, is a 3-dimensional vector space over $\mathscr{R}$.

Answers for Part D

1. Let $\vec{p}$ and $\vec{q}$ be the vectors whose components are $(3, 1, -1)$ and $(-1, 3, 2)$, respectively, so that $[\pi] = [\vec{p}, \vec{q}]$. Similarly, $[\sigma] = [\vec{a}, \vec{b}]$, where $\vec{a}$ and $\vec{b}$ have components $(-1, 1, 1)$ and $(3, 13, 5)$, respectively. Since

$$\begin{vmatrix} -1 & 1 & 1 \\ 3 & 1 & -1 \\ -1 & 3 & 2 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} 3 & 13 & 5 \\ 3 & 1 & -1 \\ -1 & 3 & 2 \end{vmatrix} = 0$$

it follows that $[\vec{a}, \vec{b}] = [\vec{p}, \vec{q}]$. So, $\pi \parallel \sigma$.

2. [We use the notation established in Exercise 1.] Since

$$\begin{vmatrix} 2 & -3 & 0 \\ -1 & 5 & -3 \\ -2 & 1 & -5 \end{vmatrix} = -45 \neq 0 \quad \text{and} \quad \begin{vmatrix} 1 & 0 & 2 \\ -1 & 5 & -3 \\ -2 & 1 & -5 \end{vmatrix} = -4 \neq 0$$

it follows that $[\vec{a}, \vec{b}] \neq [\vec{p}, \vec{q}]$. So, $\pi \nparallel \sigma$.

3. Since $\begin{vmatrix} 1 & 0 & 5 \\ 2 & 0 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 20 \neq 0$ and $\begin{vmatrix} -1 & 3 & 1 \\ 2 & 0 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 4 \neq 0$

it follows that $[\vec{a}, \vec{b}] \neq [\vec{p}, \vec{q}]$. So, $\pi \nparallel \sigma$.

Definitions

10–1. $(\vec{a}_1, \ldots, \vec{a}_n)$ is a basis for $\mathscr{T}$ if and only if (i) $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly independent, and (ii) $[\vec{a}_1, \ldots, \vec{a}_n] = \mathscr{T}$.

10–2. If $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is a basis for $\mathscr{T}$ and $\vec{a} = \vec{u}_1 a_1 + \vec{u}_2 a_2 + \vec{u}_3 a_3$ then a_1, a_2, and a_3 are, respectively, the first, second, and third components of $\vec{a}$ with respect to the given basis. Also, (a_1, a_2, a_3) is the *component-triple* of $\vec{a}$ with respect to this basis.

10–3. If $O \in \mathscr{E}$, $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is a basis for $\mathscr{T}$, and $A = O + (\vec{u}_1 a_1 + \vec{u}_2 a_2 + \vec{u}_3 a_3)$ then a_1, a_2, and a_3 are, respectively, the first, second, and third coordinates of A with respect to the given point and given basis. Also, (a_1, a_2, a_3) is the *coordinate-triple* of A with respect to this point and basis.

Other Theorems

10–1. There are four noncoplanar points.

10–2. The intersection of two nonparallel planes is a line.

10–3. A line which is parallel to each of two nonparallel planes is parallel to their intersection.

10–4. A line and a plane which are not parallel intersect at a single point.

Corollary 1. A line which is a transversal of one plane is a transversal of any parallel plane.

Corollary 2. Parallel lines are transversals of the same plane.

10–5. A plane which intersects one of two parallel planes intersects the other, and the intersections are parallel lines.

10–6. (a) $\sigma \parallel \pi \longleftrightarrow (\sigma = \pi \text{ or } \sigma \cap \pi = \varnothing)$
(b) $l \parallel \pi \longleftrightarrow (l \subseteq \pi \text{ or } l \cap \pi = \varnothing)$

10–7. Each 3-termed linearly independent sequence of translations is a basis for $\mathscr{T}$.

10–8. Each basis for $\mathscr{T}$ is a 3-termed linearly independent sequence of translations.

10–9. $[\vec{a}_1, \vec{a}_2, \vec{a}_3] = \mathscr{T} \longleftrightarrow (\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is linearly independent.

10–10. (a) Each non-$\vec{0}$ translation is a term of some basis for $\mathscr{T}$.
(b) Each two linearly independent translations are terms of some basis for $\mathscr{T}$.

10–11. For any basis for $\mathscr{T}$, (a) each component of $\vec{0}$ is 0, (b) each component of $-\vec{a}$ is the opposite of the corresponding component of $\vec{a}$, (c) each component of $\vec{a} + \vec{b}$ is the sum of the corresponding components of $\vec{a}$ and $\vec{b}$, (d) each component of $\vec{a}a$ is the product of the corresponding component of $\vec{a}$ by a.

10–12. If $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are position vectors of noncoplanar points and $a + b + c + d = 0$ then $\vec{a}a + \vec{b}b + \vec{c}c + \vec{d}d = \vec{0}$ if and only if $a = 0$, $b = 0$, $c = 0$, and $d = 0$.

10–13. For $(\vec{u}_1, \vec{u}_2)$ linearly independent, $(\vec{u}_1 a_1 + \vec{u}_2 a_2, \vec{u}_1 b_1 + \vec{u}_2 b_2)$ is linearly dependent if and only if $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0$.

10–14. For $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ linearly independent, $(\vec{u}_1 a_1 + \vec{u}_2 a_2 + \vec{u}_3 a_3, \vec{u}_1 b_1 + \vec{u}_2 b_2 + \vec{u}_3 b_3)$ is linearly dependent if and only if

$$\left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right) = (0, 0, 0).$$

10–15. Suppose that, with respect to a given coordinate system, the coordinates of A are (a_1, a_2, a_3) and the components of $\vec{p}$ and $\vec{q}$ are (p_1, p_2, p_3) and (q_1, q_2, q_3), respectively. With respect to the given coordinate system
(a) the parametric equations:

$$\begin{cases} x_1 = a_1 + p_1 s + q_1 t \\ x_2 = a_2 + p_2 s + q_2 t \\ x_3 = a_3 + p_3 s + q_3 t \end{cases}$$

describe the set $\overrightarrow{A[\vec{p}, \vec{q}]}$; and this set is a plane if and only if

$$(\dagger) \qquad \left(\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix}, \begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix}, \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}\right) \neq (0, 0, 0),$$

and
(b) the single equation:

$$(x_1 - a_1)\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix} + (x_2 - a_2)\begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix} + (x_3 - a_3)\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = 0$$

represents $\overrightarrow{A[\vec{p}, \vec{q}]}$ if and only if the condition (†) is satisfied.

10–16. Suppose that, with respect to a given coordinate system, the components of $\vec{m}$ and $\vec{n}$ are (m_1, m_2, m_3) and (n_1, n_2, n_3), respectively. With respect to the given coordinate system, the equations:

$$\text{(i)} \quad \begin{cases} (x_1 - a_1)m_1 + (x_2 - a_2)m_2 + (x_3 - a_3)m_3 = 0 \\ (x_1 - b_1)n_1 + (x_2 - b_2)n_2 + (x_3 - b_3)n_3 = 0 \end{cases}$$

describe nonparallel planes if and only if $(\vec{m}, \vec{n})$ is linearly independent; and, in this case,

$$\text{(ii)} \quad \left(\begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix}, \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix}, \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix}\right)$$

are the components of a non-$\vec{0}$ vector in the direction of the line of intersection of the planes.

Corollary 1. The equations (i) describe parallel planes if and only if $\vec{m}$ and $\vec{n}$ are non-$\vec{0}$ and $(\vec{m}, \vec{n})$ is linearly dependent.

Corollary 2. If $(\vec{m}, \vec{n})$ is linearly independent then the line of intersection of the planes described by the equations (i) is, itself, described by the system:

$$\begin{cases} x_1 = c_1 + \begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix} r \\ x_2 = c_2 + \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix} r \\ x_3 = c_3 + \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} r \end{cases}$$

where (c_1, c_2, c_3) are the coordinates of any chosen point C which is common to the two planes.

10–17. Suppose that, with respect to a given coordinate system, the components of $\vec{l}$ and $\vec{p}$ are (l_1, l_2, l_3) and (p_1, p_2, p_3), respectively. With respect to the given coordinate system, the equations:

(i) $$l_1x_1 + l_2x_2 + l_3x_3 = e$$

and:

(ii) $$\begin{cases} x_1 = a_1 + p_1r \\ x_2 = a_2 + p_2r \\ x_3 = a_3 + p_3r \end{cases}$$

describe a plane and a line which are parallel if and only if $\vec{l}$ and $\vec{p}$ are non-$\vec{0}$ and

$$l_1p_1 + l_2p_2 + l_3p_3 = 0;$$

and, in this case, the line is a subset of the plane if and only if

$$l_1a_1 + l_2a_2 + l_3a_3 = e.$$

Corollary. The equations (i) and (ii) represent a plane and a transversal to this plane if and only if $l_1p_1 + l_2p_2 + l_3p_3 \neq 0$.

10–18. For $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ linearly independent, $(\vec{u}_1a_1 + \vec{u}_2a_2 + \vec{u}_3a_3, \vec{u}_1b_1 + \vec{u}_2b_2 + \vec{u}_3b_3, \vec{u}_1c_1 + \vec{u}_2c_2 + \vec{u}_3c_3)$ is linearly dependent if and only if $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$.

Theorems About Equations

A. The system of equations:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

has a unique solution if and only if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$; and, in this case, the given system of equations is equivalent to:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Corollary. The system of equations:

$$\begin{cases} a_1x + b_1y = 0 \\ a_2x + b_2y = 0 \end{cases}$$

has a nontrivial solution if and only if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$.

B. For $\left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right) \neq (0, 0, 0)$, $(a_1x + b_1y + c_1z = 0$ and $a_2x + b_2y + c_2z = 0)$ if and only if $\exists_t \left(x = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} t, y = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} t \text{ and } z = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} t\right)$.

C. The system of equations:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

has a unique solution if and only if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ and, in this case, the given system of equations is equivalent to:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Corollary. The system of equations:

$$\begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{cases}$$

has a nontrivial solution if and only if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Chapter Test

1. Given that $(\vec{a}, \vec{b}, \vec{c})$ is a basis for $\mathcal{T}$, tell whether or not $(\vec{a} - \vec{b}, \vec{b} + \vec{c}, \vec{a} + \vec{b})$ is a basis for $\mathcal{T}$. Justify your answer.
2. **(a)** Show that $[\vec{a}, \vec{b}]$ is a vector space (over the real numbers).
   (b) Tell what is needed in order to say that $[\vec{a}, \vec{b}]$ is a 2-dimensional vector space.
3. Suppose that the coordinates of points A, B, C, and D with respect to the coordinate system determined by basis $(\vec{a}, \vec{b}, \vec{c})$ for $\mathcal{T}$ and origin O are as follows:

   A: $(2, -1, 3)$ B: $(4, -5, 7)$
   C: $(-1, 4, -2)$ D: $(-3, 7, -5)$

   (a) Give the coordinates of the point which divides the segment from A to B in $2 : 1$.
   (b) Write parametric equations for the line $\overleftrightarrow{AC}$.
   (c) Write parametric equations for the line $\overleftrightarrow{BD}$.
   (d) Exactly one of these statements is true:
   (i) $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$
   (ii) $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are skew lines.
   (iii) $\overleftrightarrow{AC} \cap \overleftrightarrow{BD} = \{P\}$, for some P
   If you feel either (i) or (ii) is true, tell why it is. If you feel that (iii) is true, give the coordinates of P.

Key for Chapter Test

1. Let a, b, and c be numbers such that $(\vec{a} - \vec{b})a + (\vec{b} + \vec{c})b + (\vec{a} + \vec{b})c = \vec{0}$. Then, $\vec{a}(a + c) + \vec{b}(-a + b + c) + \vec{c}b = \vec{0}$. Since $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent, $a + c = 0$ and $-a + b + c = 0$ and $b = 0$. So, $c = a$ and $a + c = 0$. So, $a = 0$, $b = 0$, and $c = 0$. Hence, $(\vec{a} - \vec{b}, \vec{b} + \vec{c}, \vec{a} + \vec{b})$ is a basis for $\mathcal{T}$.

2. (a) To show that $[\vec{a}, \vec{b}]$ is a vector space over the real numbers, it is enough to show that the closure postulates are satisfied. This is so, for the remaining postulates must be satisfied by virtue of the fact that $[\vec{a}, \vec{b}] \subseteq \mathcal{T}$.

   Let $\vec{c}$ and $\vec{d}$ belong to $[\vec{a}, \vec{b}]$. Then, $\vec{c} = \vec{a}c_1 + \vec{b}c_2$ and $\vec{d} = \vec{a}d_1 + \vec{b}d_2$, for some real numbers c_1, c_2, d_1, and d_2. So, $\vec{c} + \vec{d} = \vec{a}(c_1 + d_1) + \vec{b}(c_2 + d_2)$. Hence, $\vec{c} + \vec{d} \in [\vec{a}, \vec{b}]$.

   $\vec{0} \in [\vec{a}, \vec{b}]$, for $\vec{0} = \vec{a}0 + \vec{b}0$.

   Let $\vec{c} \in [\vec{a}, \vec{b}]$. Then $\vec{c} = \vec{a}c_1 + \vec{b}c_2$, for some c_1 and c_2. So, $-\vec{c} = \vec{a} \cdot -c_1 + \vec{b} \cdot -c_2$ so that $-\vec{c} \in [\vec{a}, \vec{b}]$. Also, $\vec{c}t = \vec{a} \cdot c_1t + \vec{b} \cdot c_2t$ so that $\vec{c}t \in [\vec{a}, \vec{b}]$.

   (b) In order to say that $[\vec{a}, \vec{b}]$ is a 2-dimensional vector space (over the reals), one would have to know that (a) there are two linearly independent members of $[\vec{a}, \vec{b}]$ and (b) there are not three linearly independent members of $[\vec{a}, \vec{b}]$.

3. (a) $(\frac{10}{3}, -\frac{11}{3}, \frac{17}{3})$

   (b) $x_1 = 2 - 3t$
   $x_2 = -1 + 5t$
   $x_3 = 3 - 5t$

   [There are many answers possible here. We gave the more likely one.]

   (c) $x_1 = 4 - 7s$
   $x_2 = -5 + 12s$
   $x_3 = 7 - 12s$

   (d) $P \in \overleftrightarrow{AC} \cap \overleftrightarrow{BD}$ if and only if there are numbers s and t such that

   $$\begin{aligned} 2 - 3t &= 4 - 7s \\ -1 + 5t &= -5 + 12s \\ 3 - 5t &= 7 - 12s \end{aligned}$$

   Solving this system yields the solution $t = 4$ and $s = 2$. So, (iii) is true. The coordinates of P are $(-10, 19, -17)$.

   (e) $x_1 = 2 + 2s - 5t$
   $x_2 = -1 - 4s + 8t$
   $x_3 = 3 + 4s - 8t$

   To tell whether $\overleftrightarrow{ABD}$ is a plane or not, we must check whether $\{A, B, C\}$ is collinear or not. The components of $B - A$ are $(2, -4, 4)$, and the components of $D - A$ are $(-5, 8, -8)$. Clearly, neither of $B - A$ or $D - A$ is a multiple of the other. So, $(B - A, D - A)$ is linearly independent. So, $\{A, B, D\}$ is noncollinear. Hence $\overleftrightarrow{ABD}$ is a plane.

(e) Write parametric equations for $\overleftrightarrow{ABD}$. Is $\overleftrightarrow{ABD}$ a plane or not? How do you know?

4. Suppose that $(\vec{a}, \vec{b})$ is linearly independent and that $S = R + \vec{a}2$, $T = R + \vec{b}2$, $U = R + (\vec{a} + \vec{b})$, $V = S + \vec{b}2$ and $W = U + \vec{b}$.
   (a) Are all the described points coplanar or not? Explain.
   (b) Determine whether or not $\{R, U, W\}$ is collinear.
   (c) Show that U is the midpoint of $\overline{ST}$.
   (d) Given that A is the centroid of $\triangle SVT$, express $A - R$ as a linear combination of $\vec{a}$ and $\vec{b}$.

5. Evaluate these determinants.

(a) $\begin{vmatrix} -3 & 5 \\ 6 & -7 \end{vmatrix}$ (b) $\begin{vmatrix} 5 & 4 & 0 \\ -3 & 5 & 5 \\ 6 & -7 & -7 \end{vmatrix}$ (c) $\begin{vmatrix} 16 \cdot 9 & 5 \cdot 9 \\ 16 \cdot 7 & 5 \cdot 7 \end{vmatrix}$

6. Use determinants to help solve this system of equations.

$$\begin{cases} 5x + 7y = 12 \\ -3x + 5y = 2 \end{cases}$$

7. Here are equations for planes π_1, π_2, and π_3:

$$\begin{aligned} \pi_1&: 4x + 2y - 3z = 9 \\ \pi_2&: 6x + 3y + 4z = 5 \\ \pi_3&: 2x + y - 6z = 9 \end{aligned}$$

   (a) Use determinants to help tell whether or not π_1 is parallel to π_3.
   (b) Give the components of a non-$\vec{0}$ vector in the direction of the line of intersection of π_1 and π_2.
   (c) Give parametric equations for the line of intersection of π_2 and π_3.
   (d) Use your results in parts (b) and (c) to tell whether or not $\pi_1 \cap \pi_2$ is parallel to $\pi_2 \cap \pi_3$.

8. Here are the components of vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ with respect to a given basis:

$$\begin{aligned} \vec{a}&: (2, 3, -1) \\ \vec{b}&: (6, 9, 4) \\ \vec{c}&: (-4, -6, 9) \end{aligned}$$

   (a) Use determinants to help show that $(\vec{a}, \vec{b})$ is linearly independent.
   (b) Write parametric equations for the plane which contains the point with coordinates $(-7, -12, -19)$ and has the bidirection $[\vec{a}, \vec{b}]$.
   (c) Determine whether or not $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent.

---

4. (a) Yes. They are all in the plane $\overline{R(R + \vec{a})(R + \vec{b})}$.

   (b) $\{R, U, W\}$ is noncollinear since $U - R = \vec{a} + \vec{b}$, $W - R = \vec{a} + \vec{b}2$ and $(\vec{a} + \vec{b}, \vec{a} + \vec{b}2)$ is linearly independent. [To show the latter is the case, let a and b be numbers such that $(\vec{a} + \vec{b})a + (\vec{a} + \vec{b}2)b = \vec{0}$. Then, $\vec{a}(a + b) + \vec{b}(a + 2b) = \vec{0}$ so that $a = -b$ and $a = -2b$. So, $a = 0$ and $b = 0$.]

   (c) $U - S = \vec{b} - \vec{a}$ and $T - U = \vec{b} - \vec{a}$. So, $U - S = T - U$. By definition, U is the midpoint of $\overline{ST}$.

   (d) $A - R = \vec{a}\frac{4}{3} + \vec{b}\frac{4}{3}$.

5. (a) -9 (b) 36 (c) 0

6. $$x = \frac{\begin{vmatrix} 12 & 7 \\ 2 & 5 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ -3 & 5 \end{vmatrix}} = \frac{60 - 14}{25 + 21} = \frac{46}{46} = 1, \quad y = \frac{\begin{vmatrix} 5 & 12 \\ -3 & 2 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ -3 & 5 \end{vmatrix}} = \frac{10 + 36}{25 + 21} = \frac{46}{46} = 1$$

   Answer: (1, 1)

7. (a) π_1 not parallel to π_3, since $\left(\begin{vmatrix} 2 & -3 \\ 1 & -6 \end{vmatrix}, \begin{vmatrix} -3 & 4 \\ -6 & 2 \end{vmatrix}, \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} \right) = (-9, 18, 0) \neq (0, 0, 0)$.

   (b) $(17, -34, 0)$, or any nonzero multiple of this component triple.

   (c) One point common to π_2 and π_3 has coordinates $(0, 3, -1)$. The line of intersection of π_2 and π_3 has the direction of the non-$\vec{0}$ vector with components $(-22, 44, 0)$. So, parametric equations for $\pi_2 \cap \pi_3$ are:

$$\begin{cases} x_1 = -22r \\ x_2 = 3 + 44r \\ x_3 = -1 \end{cases}$$

   Answers will, of course, vary among students. What must be checked is that the student has found the coordinates of a point in $\pi_2 \cap \pi_3$ and that the direction of the line described by his equations is that of the vector with components $(-22, 44, 0)$.

   (d) The vectors with components $(17, -34, 0)$ and $(-22, 44, 0)$ are clearly linearly dependent. So, the lines $\pi_1 \cap \pi_2$ and $\pi_2 \cap \pi_3$ are parallel.

8. (a) Since $\left(\begin{vmatrix} 3 & -1 \\ 9 & 4 \end{vmatrix}, \begin{vmatrix} -1 & 2 \\ 4 & 6 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} \right) = (21, -14, 0) \neq (0, 0, 0)$, the sequence $(\vec{a}, \vec{b})$ is linearly independent.

   (b) The "easiest" such parametric equations are:

$$\begin{cases} x_1 = -7 + 2s + 6t \\ x_2 = -12 + 3s + 9t \\ x_3 = -19 - s + 4t \end{cases}$$

   (c) $\begin{vmatrix} 2 & 3 & -1 \\ 6 & 9 & 4 \\ -4 & -6 & 9 \end{vmatrix} = 0$. So, $(\vec{a}, \vec{b}, \vec{c})$ is linearly dependent.

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